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FOURTEEN MOMENT THEORY FOR GRANULAR GASES

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Abstract. A fourteen moment theory for a granular gas is developed within the framework of the Boltzmann equation where the full contracted moment of fourth order is added to the thirteen moments of mass density, velocity, pressure tensor and heat flux vector. The spatially homogeneous solutions of the fourteen moment theory implied into a time decay of the temperature field which follows closely Haff's law, besides the more accentuated time decays of the pressure deviator, heat flux vector and fourth moment. The requirement that the fourth moment remains constant in time inferred into its identification with the coefficient a_2 in the Chapman-Enskog solution of the Boltzmann equation. The laws of Navier-Stokes and Fourier are obtained by restricting to a five field theory and using a method akin to the Maxwellian procedure. The dependence of the heat flux vector on the gradient of the particle number density was obtained thanks to the inclusion of the forth moment. The analysis of the dynamic behavior of small local disturbances from the spatially homogeneous solutions caused by spontaneous internal fluctuations is performed by considering a thirteen field theory and it is shown that for the longitudinal disturbances there exist one hydrodynamic and four kinetic modes, while for the transverse disturbances one hydrodynamic and two kinetic modes are present.

1. **Introduction.** One of the main feature of a rarefied gas – whose molecules undergo elastic collisions – is its relaxation to an equilibrium state which is characterized by a Maxwellian distribution function. Conversely, if the collisions are inelastic there exists a transformation of a part of the translational kinetic energy into heat and the mechanical energy lost implies a temperature decay of the gas. Gases whose molecules interact according to a hard-sphere potential and have a normal restitution coefficient less than unit – which characterizes inelastic collisions – are known in the literature as granular gases. The research of granular gases within the framework of the Boltzmann equation is a topic widely reported in the literature (see e.g. [1, 2, 3, 4, 5, 6, 7, 8] and the references quoted in these works) where some unusual properties include temperature decay, cluster formation, shock wave formation, anomalous diffusion, etc.

The aim of the present work is to develop a fourteen moment theory for a granular gas on the basis of the Boltzmann equation. It differs from the previous works [2, 3, 9] on moment method for granular gases, by the inclusion of a scalar field – the full contracted moment of fourth order – to the thirteen moments of mass density, velocity, pressure tensor and heat flux vector. The importance of the study of a fourteen moment theory for ideal gases was discussed in the paper [10] within the

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To the memory of Carlo Cercignani.

framework of extended thermodynamics. The inclusion of a fourth-order contracted moment for a bidimensional granular gas was investigated in the work [11], however the problems analyzed there are quite different from those proposed in the present work.

We show that spatially homogeneous solutions of the fourteen moment theory lead to a time decay of the temperature field which follows closely Haff's law, apart from the decays of the pressure deviator, heat flux vector and fourth moment. Furthermore, by imposing that the fourth moment remains constant in time it is possible to identify it with the coefficient a_2 in the Chapman-Enskog solution of the Boltzmann equation for a granular gas. By restricting to a five field theory – described by the fields of mass density, velocity and temperature – and using a method akin with the Maxwellian iteration procedure [12], the laws of Navier-Stokes and Fourier are recovered. The dependence of the heat flux vector on the gradient of particle number density is a direct consequence of the description of the granular gas by a fourteen moment theory.

Another analysis which is performed refers to the dynamic behavior of small local disturbances from the spatially homogeneous solutions caused by spontaneous internal fluctuations. This subject was investigated in the work [13] by using a five field theory. Here the analysis is carried out by considering a thirteen field theory and it is shown that there exist one hydrodynamic and four kinetic modes for the longitudinal disturbances and one hydrodynamic and two kinetic modes for the transverse one.

The work is structured as follows: the fourteen moment theory is presented in Section 2. Section 3 is devoted to the determination of the non-equilibrium distribution function in terms of the fourteen moments and of the constitutive quantities of the theory. The fourteen field equations are given in Section 4 and the spatially homogeneous solutions of the fourteen fields are analyzed in Section 5 where the decay of the temperature field is compared with Haff's law and the fourth moment is identified with the coefficient a_2 of the Chapman-Enskog solution of the Boltzmann equation. In Section 6 the Navier-Stokes and Fourier laws are obtained by restricting to a five field theory and using a method akin with the Maxwellian iteration procedure. The analysis of the dynamic behavior of small local disturbances from the spatially homogeneous solutions caused by spontaneous internal fluctuations is the subject of Section 7. The conclusions of the work are given in the last Section.

2. **Fourteen moment theory.** The gas which we are interested in refers to a rarefied granular gas free of external body forces where only binary encounters between the particles are considered. The mass and the diameter of a spherical gas particle will be denoted by m and d, respectively, and let $(\mathbf{c}, \mathbf{c}_1)$ and $(\mathbf{c}', \mathbf{c}_1')$ represent the velocities of two particles before and after collision. Furthermore, $\mathbf{g} = \mathbf{c}_1 - \mathbf{c}$ and $\mathbf{g}' = \mathbf{c}_1' - \mathbf{c}'$ denote the relative velocities before and after collision. The inelastic collisions are characterized by the relation $(\mathbf{g}' \cdot \mathbf{k}) = -\alpha(\mathbf{g} \cdot \mathbf{k})$ where $0 \leq \alpha \leq 1$ refers to a normal restitution coefficient and $\mathbf{k}$ is the unit collision vector which joins the centers of the two colliding spheres pointing from the center of the particle denoted by 1 to the center of the other particle without index.

The equations that connect the velocities before and after collision are given by

$$\mathbf{c}' = \mathbf{c} + \frac{1+\alpha}{2}(\mathbf{g} \cdot \mathbf{k})\mathbf{k}, \qquad \mathbf{c}_1' = \mathbf{c}_1 - \frac{1+\alpha}{2}(\mathbf{g} \cdot \mathbf{k})\mathbf{k}. \tag{1}$$

The relationships between the relative pre-collisional and post-collisional velocities and their modulus follow from the above equations and read

$$\mathbf{g}' = \mathbf{g} - (1+\alpha)(\mathbf{g}\cdot\mathbf{k})\mathbf{k}, \qquad g'^2 = g^2 - (1-\alpha^2)(\mathbf{g}\cdot\mathbf{k})^2, \tag{2}$$

so that the variation of the kinetic energy in terms of the pre- and post-collisional velocities becomes

$$\frac{m}{2}c'^2 + \frac{m}{2}c_1'^2 - \frac{m}{2}c^2 - \frac{m}{2}c_1^2 = -\frac{m}{4}(1-\alpha^2)(\mathbf{g}\cdot\mathbf{k})^2. \tag{3}$$

For $\alpha = 1$ the collisions are elastic and the conservation of the kinetic energy holds.

The pre-collisional velocities $(\mathbf{c}, \mathbf{c}_1)$, the post-collisional velocities $(\mathbf{c}', \mathbf{c}_1')$ and the collision vector $\mathbf{k}$ characterize a direct collision, while the pre-collisional velocities $(\mathbf{c}^*, \mathbf{c}_1^*)$, the post-collisional velocities $(\mathbf{c}, \mathbf{c}_1)$ and the collision vector $\mathbf{k}^* = -\mathbf{k}$ describe a restitution collision. For a restitution collision $\mathbf{g}\cdot\mathbf{k}^* = -\alpha(\mathbf{g}^*\cdot\mathbf{k}^*) = -(\mathbf{g}\cdot\mathbf{k})$ and the pre-collisional velocities $(\mathbf{c}^*, \mathbf{c}_1^*)$ are related to the post-collisional velocities $(\mathbf{c}, \mathbf{c}_1)$ by

$$\mathbf{c} = \mathbf{c}^* + \frac{1+\alpha}{2}(\mathbf{g}^*\cdot\mathbf{k}^*)\mathbf{k}^*, \qquad \mathbf{c}_1 = \mathbf{c}_1^* - \frac{1+\alpha}{2}(\mathbf{g}^*\cdot\mathbf{k}^*)\mathbf{k}^*. \tag{4}$$

The above equations can be written as

$$\mathbf{c}^* = \mathbf{c} + \frac{1+\alpha}{2\alpha}(\mathbf{g}\cdot\mathbf{k})\mathbf{k}, \qquad \mathbf{c}_1^* = \mathbf{c}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{g}\cdot\mathbf{k})\mathbf{k}, \tag{5}$$

thanks to $\mathbf{k}^* = -\mathbf{k}$ and $(\mathbf{g}\cdot\mathbf{k}) = -\alpha(\mathbf{g}^*\cdot\mathbf{k})$.

The transformation of the volume elements $d\mathbf{c}_1^*\, d\mathbf{c}^* = |J| d\mathbf{c}_1\, d\mathbf{c}$ implies that the modulus of the Jacobian is given by $|J| = 1/\alpha$ and the following relationship holds

$$(\mathbf{g}^*\cdot\mathbf{k}^*)\, d\mathbf{c}^*\, d\mathbf{c}_1^* = \frac{1}{\alpha^2}(\mathbf{g}\cdot\mathbf{k})\, d\mathbf{c}\, d\mathbf{c}_1. \tag{6}$$

The Boltzmann equation for granular gases without external forces reads [8]

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = \int \left(\frac{1}{\alpha^2} f_1^* f^* - f_1 f \right) \mathsf{d}^2\, (\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1, \tag{7}$$

due to the relationship (6).

The transfer equation is obtained from the multiplication of the Boltzmann equation (7) by an arbitrary function $\psi(\mathbf{x}, \mathbf{c}, t)$ and integration of the resulting equation over all values of the velocity $\mathbf{c}$. Hence, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int \psi f\, d\mathbf{c} + \frac{\partial}{\partial x_i}\int \psi c_i f\, d\mathbf{c} - \int \left[\frac{\partial \psi}{\partial t} + c_i \frac{\partial \psi}{\partial x_i}\right] f d\mathbf{c} \\
&= \int \psi \left(\frac{1}{\alpha^2} f_1^* f^* - f_1 f \right) \mathsf{d}^2\, (\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c} \\
&= \frac{1}{2}\int \left[\psi_1' + \psi' - \psi_1 - \psi\right] f_1 f\, \mathsf{d}^2\, (\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}.
\end{aligned} \tag{8}$$

In order to obtain the second equality above we have taken into account the relationship (6), renamed the pre-collisional velocities $(\mathbf{c}^*, \mathbf{c}_1^*)$ as $(\mathbf{c}, \mathbf{c}_1)$ and the post-collisional velocities $(\mathbf{c}, \mathbf{c}_1)$ as $(\mathbf{c}', \mathbf{c}_1')$ and used the symmetry properties of the collision term when the two molecules are interchanged.

A macroscopic state of the granular gas will be characterized by the fourteen fields of mass density ϱ, hydrodynamic velocity v_i, pressure tensor p_{ij}, heat flux vector q_i and contracted fourth moment p_{iijj} which are defined by

$$\varrho = \int mf\, d\mathbf{c}, \qquad \varrho v_i = \int mc_i f\, d\mathbf{c}, \qquad p_{ij} = \int mC_iC_j f\, d\mathbf{c}, \tag{9}$$

$$q_i = \int \frac{m}{2}C^2 C_i f\, d\mathbf{c}, \qquad p_{iijj} = \int mC^4 f\, d\mathbf{c}, \tag{10}$$

where $C_i = c_i - v_i$ denotes the peculiar velocity.

The balance equations for the fourteen fields are obtained by choosing ψ equal to m, mc_i, mC_iC_j, $mC^2C_i/2$ and mC^4 into the transfer equation (8), yielding

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \tag{11}$$

$$\frac{\partial \varrho v_i}{\partial t} + \frac{\partial}{\partial x_j}(\varrho v_i v_j + p_{ij}) = 0, \tag{12}$$

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_k}(p_{ijk} + p_{ij}v_k) + p_{ki}\frac{\partial v_j}{\partial x_k} + p_{kj}\frac{\partial v_i}{\partial x_k} = P_{ij}, \tag{13}$$

$$\frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_j}(q_{ij} + q_i v_j) + p_{ijk}\frac{\partial v_j}{\partial x_k} + q_j\frac{\partial v_i}{\partial x_j} - \frac{p_{ki}}{\varrho}\frac{\partial p_{kj}}{\partial x_j} - \frac{1}{2}\frac{p_{rr}}{\varrho}\frac{\partial p_{ij}}{\partial x_j} = Q_i, \tag{14}$$

$$\frac{\partial p_{iijj}}{\partial t} + \frac{\partial}{\partial x_j}(p_{iijjk} + p_{iijj}v_j) + 8q_{ik}\frac{\partial v_i}{\partial x_k} - \frac{8}{\varrho}q_i\frac{\partial p_{ik}}{\partial x_k} = P, \tag{15}$$

respectively. Above, the new moments of the distribution function are defined by

$$p_{ijk} = \int mC_iC_jC_k f d\mathbf{c}, \quad q_{ij} = \int \frac{m}{2}C^2C_iC_j f d\mathbf{c}, \quad p_{iijjk} = \int mC^4C_k f d\mathbf{c}, \tag{16}$$

where $q_{ii} = p_{iijj}/2$. Moreover, the production terms are given by

$$P_{ij} = \frac{1}{2}\int m\left(C_i^{1'}C_j^{1'} + C_i'C_j' - C_i^1C_j^1 - C_iC_j\right) f_1 f\, \mathsf{d}^2\,(\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}, \tag{17}$$

$$Q_i = \frac{1}{2}\int \frac{m}{2}\left(C_1'^2C_i^{1'} + C'^2C_i' - C_1^2C_i^1 - C^2C_i\right) f_1 f\, \mathsf{d}^2\,(\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}, \tag{18}$$

$$P = \frac{1}{2}\int m\left(C_1'^4 + C'^4 - C_1^4 - C^4\right) f_1 f\, \mathsf{d}^2\,(\mathbf{g}\cdot\mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}. \tag{19}$$

In order to decompose the balance equation for the pressure tensor (13) in its trace and traceless parts we introduce the traceless tensors $p_{\langle ij\rangle}$ and $p_{\langle ijk\rangle}$ defined through

$$p_{ij} = p_{\langle ij\rangle} + p\delta_{ij} \qquad p_{ijk} = p_{\langle ijk\rangle} + \frac{2}{5}\left(q_i\delta_{jk} + q_j\delta_{ik} + q_k\delta_{ij}\right), \tag{20}$$

where $p = nkT$ is the hydrostatic pressure. Hence, we may decompose (13) as

$$\frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i} + \frac{2}{3nk}\left(\frac{\partial q_i}{\partial x_i} + p\frac{\partial v_i}{\partial x_i} + p_{\langle ij\rangle}\frac{\partial v_i}{\partial x_j}\right) + T\zeta = 0, \tag{21}$$

$$\frac{\partial p_{\langle ij\rangle}}{\partial t} + \frac{\partial}{\partial x_k}\left(p_{\langle ijk\rangle} + p_{\langle ij\rangle}v_k\right) + p_{\langle ki\rangle}\frac{\partial v_j}{\partial x_k} + p_{\langle kj\rangle}\frac{\partial v_i}{\partial x_k} - \frac{2}{3}p_{\langle kr\rangle}\frac{\partial v_r}{\partial x_k}\delta_{ij}$$
$$+\frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = P_{\langle ij\rangle}. \tag{22}$$

Equation (21) is the balance equation for the temperature with ζ denoting the cooling rate

$$\zeta = \frac{\mathsf{d}^2 m(1-\alpha^2)}{12nkT}\int f_1 f(\mathbf{g}\cdot\mathbf{k})^3 d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}, \tag{23}$$

thanks to (3). The balance equation for the pressure deviator is given by (22) where we have introduced the notation for the traceless part of the velocity gradient

$$\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}, \tag{24}$$

with a similar notation for the traceless part of the heat flux vector gradient.

3. **Non-equilibrium distribution function.** The non-equilibrium distribution function for the fourteen moments follows from the representation

$$f = f^{(0)}\left(a + a_iC_i + a_{ij}C_iC_j + b_iC^2C_i + bC^4\right), \tag{25}$$

where the fourteen coefficients a, a_i, a_{ij}, b_i and b do not depend on the peculiar velocity C_i. In the above equation

$$f^{(0)} = n\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta C^2}. \tag{26}$$

is the Maxwellian distribution function with $\beta = m/2kT$ and $n = \varrho/m$ denoting the particle number density.

Since at equilibrium the contracted fourth moment is given by

$$p^{(0)}_{iijj} = \int mC^4 f^{(0)}\, d\mathbf{c} = 15\varrho\left(\frac{kT}{m}\right)^2, \tag{27}$$

it is convenient to introduce a dimensionless non-equilibrium part of p_{iijj} denoted by Δ and defined by

$$\Delta = \frac{1}{15\varrho}\left(\frac{m}{kT}\right)^2\left[p_{iijj} - p^{(0)}_{iijj}\right] = \frac{1}{15\varrho}\left(\frac{m}{kT}\right)^2\int mC^4(f - f^{(0)})d\mathbf{c}. \tag{28}$$

From the definition of the basic fields (9) and (10) together with the non-equilibrium distribution function (25) it is straightforward to determine the fourteen coefficients a, a_i, a_{ij}, b_i, b and it follows that

$$\begin{aligned} f = f^{(0)}\Bigg\{&1 + \frac{2\beta^2}{\varrho}p_{\langle ij\rangle}C_iC_j + \frac{8\beta^2}{5\varrho}q_iC_i\left(\beta C^2 - \frac{5}{2}\right) \\ &+ \left(\frac{15}{8} - \frac{5\beta C^2}{2} + \frac{\beta^2C^4}{2}\right)\Delta\Bigg\}. \end{aligned} \tag{29}$$

From the knowledge of the non-equilibrium distribution function we can determine the constitutive equations for the moments of the distribution function by inserting (29) into the expressions (16) and subsequent integration of the resulting equations, yielding

$$p_{\langle ijk\rangle} = 0, \qquad p_{iijjk} = 28\frac{kT}{m}q_k, \tag{30}$$

$$q_{ij} = \frac{5}{2}\varrho\left(\frac{kT}{m}\right)^2[1+\Delta]\,\delta_{ij} + \frac{7}{2}\frac{kT}{m}p_{\langle ij\rangle}. \tag{31}$$

The constitutive equations for the cooling rate ζ and for the production terms $P_{\langle ij\rangle}, Q_i, P$ are obtained in the same manner and it follows from (23), (17), (18) and (19) through integration that

$$\zeta = \frac{1}{3\tau}\sqrt{\frac{T}{T_0}}(1-\alpha^2)\left[1+\frac{3\Delta}{16}\right], \tag{32}$$

$$P_{\langle ij\rangle} = -\frac{1}{5\tau}\sqrt{\frac{T}{T_0}}(1+\alpha)(3-\alpha)\left[1-\underline{\frac{\Delta}{32}}\right]p_{\langle ij\rangle}, \tag{33}$$

$$Q_i = -\frac{1}{60\tau}\sqrt{\frac{T}{T_0}}(1+\alpha)\left\{49-33\alpha+\underline{(19-3\alpha)\frac{\Delta}{32}}\right\}q_i, \tag{34}$$

$$P = -\frac{\varrho}{\tau}\left(\frac{kT}{m}\right)^2(1+\alpha)\sqrt{\frac{T}{T_0}}\Big\{(2\alpha^2+9)(1-\alpha)$$
$$+\left[30\alpha^2(1-\alpha)+271-207\alpha\right]\frac{\Delta}{16}\Big\}. \tag{35}$$

In the above equations we have introduced a mean free time in terms of a reference temperature T_0, namely, $\tau = \sqrt{\frac{m}{\pi k T_0}}/4n\mathsf{d}^2$. Furthermore, we have considered in the expressions for ζ and P only linear terms in Δ, however the products of Δ with $p_{\langle ij\rangle}$ and q_i were not neglected in the production terms $P_{\langle ij\rangle}$ and Q_i. These terms are underlined in (33) and (34) and the justification of such procedure will be explained in Section 5.

For the elastic case $\alpha = 1$ and the production terms (32) – (35) reduce to

$$\zeta = 0, \qquad P_{\langle ij\rangle} = -\frac{4}{5\tau}\sqrt{\frac{T}{T_0}}p_{\langle ij\rangle}, \tag{36}$$

$$Q_i = -\frac{8}{15\tau}\sqrt{\frac{T}{T_0}}q_i, \qquad P = -\frac{8\varrho}{\tau}\left(\frac{kT}{m}\right)^2\sqrt{\frac{T}{T_0}}\Delta, \tag{37}$$

by neglecting now the products of Δ with $p_{\langle ij\rangle}$ and q_i.

4. **Fourteen field equations.** Once the constitutive equations are known functions of the basic fields, the insertion of (30) – (35) into the balance equations (11), (12), (14), (15), (21) and (22), leads to the following system of fourteen field equations for ϱ, v_i, T, $p_{\langle ij\rangle}$, q_i and Δ:

$$\mathcal{D}\varrho + \varrho\frac{\partial v_i}{\partial x_i} = 0, \tag{38}$$

$$\varrho\mathcal{D}v_i + kT\frac{\partial n}{\partial x_i} + kn\frac{\partial T}{\partial x_i} + \frac{\partial p_{\langle ij\rangle}}{\partial x_j} = 0, \tag{39}$$

$$\mathcal{D}T + \frac{2}{3nk}\left(\frac{\partial q_i}{\partial x_i} + p\frac{\partial v_i}{\partial x_i} + p_{\langle ij\rangle}\frac{\partial v_i}{\partial x_j}\right) + \frac{T}{3\tau}\sqrt{\frac{T}{T_0}}(1-\alpha^2)\left[1+\frac{3\Delta}{16}\right] = 0, \tag{40}$$

$$\mathcal{D}p_{\langle ij\rangle} + p_{\langle ij\rangle}\frac{\partial v_k}{\partial x_k} + p_{\langle ki\rangle}\frac{\partial v_j}{\partial x_k} + p_{\langle kj\rangle}\frac{\partial v_i}{\partial x_k} - \frac{2}{3}p_{\langle kr\rangle}\frac{\partial v_r}{\partial x_k}\delta_{ij} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}$$
$$+2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -\frac{1}{5\tau}\sqrt{\frac{T}{T_0}}(1+\alpha)(3-\alpha)\left[1-\frac{\Delta}{32}\right]p_{\langle ij\rangle}, \tag{41}$$

$$\mathcal{D}q_i + \frac{7}{5}q_i\frac{\partial v_j}{\partial x_j} + \frac{7}{5}q_j\frac{\partial v_i}{\partial x_j} + \frac{2}{5}q_j\frac{\partial v_j}{\partial x_i} + \frac{kT}{m}\frac{\partial p_{\langle ij\rangle}}{\partial x_j} + \left(\frac{5k}{2m}\frac{\partial T}{\partial x_j} - \frac{kT}{\varrho}\frac{\partial n}{\partial x_j}\right)p_{\langle ij\rangle}$$
$$-\frac{p_{\langle ki\rangle}}{\varrho}\frac{\partial p_{\langle kj\rangle}}{\partial x_j} + \underline{\frac{5(kT)^2}{2m}\left(n\frac{\partial \Delta}{\partial x_i} + \Delta\frac{\partial n}{\partial x_i}\right)} + \frac{5k^2Tn}{2m}(1+2\Delta)\frac{\partial T}{\partial x_i}$$
$$= -\frac{1}{60\tau}\sqrt{\frac{T}{T_0}}(1+\alpha)\left\{49 - 33\alpha + (19-3\alpha)\frac{\Delta}{32}\right\}q_i, \tag{42}$$

$$15\varrho\left(\frac{kT}{m}\right)^2\mathcal{D}\Delta + \frac{8kT}{m}\left(1 - \frac{5}{2}\Delta\right)\left[\frac{\partial q_j}{\partial x_j} + p_{\langle ij\rangle}\frac{\partial v_i}{\partial x_j}\right] + 20\frac{k}{m}q_i\frac{\partial T}{\partial x_i}$$
$$-\frac{8q_i}{\varrho}\left(\frac{\partial p_{\langle ij\rangle}}{\partial x_j} + kT\frac{\partial n}{\partial x_i}\right) = \frac{\varrho}{\tau}\left(\frac{kT}{m}\right)^2(1+\alpha)\sqrt{\frac{T}{T_0}}\bigg\{(1-2\alpha^2)(1-\alpha)$$
$$-\left[81 - 17\alpha + 30\alpha^2(1-\alpha)\right]\frac{\Delta}{16}\bigg\}. \tag{43}$$

In (43) we have used (38) and (40) in order to eliminate the time derivatives of ϱ and T and neglected products of Δ. Furthermore, we have introduced in the above equations the material time derivative $\mathcal{D} = \partial/\partial t + v_i\partial/\partial x_i$.

5. **Spatially homogeneous solutions.** Let us search for spatially homogeneous solutions of the fourteen field equations where the fields depend only on time. In this case the field equations (38) – (43) reduce to

$$\frac{d\varrho}{dt_*} = 0, \qquad \frac{dv_i}{dt_*} = 0, \tag{44}$$
$$\frac{dp^*_{\langle ij\rangle}}{dt_*} = -\frac{(1+\alpha)(3-\alpha)}{5}\sqrt{T_*}\left[1 - \frac{\Delta}{32}\right]p^*_{\langle ij\rangle}, \tag{45}$$
$$\frac{dq^*_i}{dt_*} = -\frac{(1+\alpha)}{60}\sqrt{T_*}\left\{49 - 33\alpha + (19-3\alpha)\frac{\Delta}{32}\right\}q^*_i, \tag{46}$$
$$\frac{dT_*}{dt_*} + \frac{T_*^{\frac{3}{2}}}{3}(1-\alpha^2)\left[1 + \frac{3\Delta}{16}\right] = 0, \tag{47}$$
$$\frac{d\Delta}{dt_*} = \frac{1+\alpha}{15}\sqrt{T_*}\left\{(1-2\alpha^2)(1-\alpha) - [81 - 17\alpha + 30\alpha^2(1-\alpha)]\frac{\Delta}{16}\right\}. \tag{48}$$

In the above equations we have introduced the dimensionless quantities: time $t_* = t/\tau$, temperature $T_* = T/T_0$, pressure deviator $p^*_{\langle ij\rangle}$ and heat flux vector q^*_i.

From (44) we may infer that the mass density and the velocity fields remain constant in time, while (45) – (47) compose a system of coupled differential equations for the determination of the temperature T_*, pressure deviator $p^*_{\langle ij\rangle}$, heat flux vector q^*_i and fourth moment Δ.

Because the system of differential equations (45) – (47) is non-linear, it was solved numerically by considering the initial conditions $T_*(0) = 1$, $p^*_{\langle ij\rangle}(0) = 1$, $q^*_i(0) = 1$ and $\Delta(0) = 1$.

In the left frame of Figure 1 it is plotted the time decay of the temperature (solid line) in comparison with Haff's law (dashed line) which is the solution of (46) when $\Delta =$ constant (see (50) below). We may observe that the temperature decay T_* follows closely Haff's law and by increasing the restitution coefficient the time decay of the temperature is less accentuated. From the right frame of the figure we observe that the pressure deviator, the heat flux vector and the dimensionless fourth

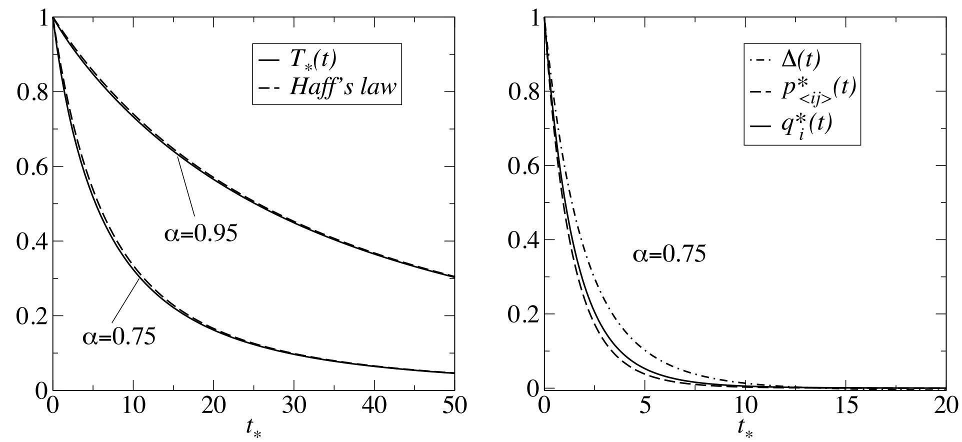

FIGURE 1. Left frame: time decay of the temperature with the solid line indicating the solution of the system and the dashed line showing Haff's law. Right frame: time decay of the pressure tensor, heat flux vector and dimensionless fourth moment.

moment decay also with time and the trend to equilibrium is more pronounced for the pressure deviator followed by the heat flux vector and the dimensionless fourth moment. The curves in the right frame were obtained for the restitution coefficient $\alpha = 0.75$. By increasing the value of the restitution coefficient the time decay of the pressure deviator and dimensionless fourth moment are more accentuated but the one of the heat flux vector is less pronounced, since it is connected with the transport of energy which is directly affected by the inelasticity.

If we analyze the differential equations (46) and (47) we may infer that the pressure deviator and the heat flux vector do not evolve with respect to time when the initial conditions for these fields vanish. However, this is not the case for the fourth moment, since a vanishing initial condition for Δ implies from (48) that it may evolve with time. There is an interesting case where the fourth moment remains constant in time and which is expressed by the condition that the right-hand side of (48) must vanish, yielding

$$\Delta = \frac{16(1-\alpha)(1-2\alpha^2)}{30\alpha^2(1-\alpha)+81-17\alpha} = a_2. \tag{49}$$

The above expression for Δ is the same as that found for the coefficient a_2 which follows from the Chapman-Enskog method (see e.g. [8]). The expression (49) for Δ indicates that it vanishes in the elastic case where $\alpha = 1$.

The substitution of Δ given by (49) into (47) and the integration of the resulting equation leads to Haff's law

$$T_*(t) = \frac{1}{\left\{1+\frac{(1-\alpha^2)}{6}\left[1+\frac{3(1-\alpha)(1-2\alpha^2)}{81-17\alpha+30\alpha^2(1-\alpha)}\right]t_*\right\}^2}. \tag{50}$$

In the elastic case $\alpha = 1$, the temperature T_* remains constant in time and

$$p^*_{\langle ij\rangle}(t) = p^*_{\langle ij\rangle}(0)e^{-\frac{4}{5}\sqrt{T_*}t_*}, \qquad q^*_i(t) = q^*_i(0)e^{-\frac{8}{15}\sqrt{T_*}t_*}, \tag{51}$$

i.e, both decay exponentially with time, which is a well known result.

6. **Five field theory.** Let us analyze the case of a five field theory described by the fields of mass density, velocity and temperature. The balance equations for these fields are given by (11), (12) and (21) and in this case the pressure deviator and the heat flux vector are no longer fields but constitutive quantities, since in this section we shall consider that the dimensionless fourth moment is given by (49). In order to get the constitutive equations for these quantities we base on the eight remaining field equations (41) and (42) and make use of a method akin with the Maxwellian iteration procedure [12]. According to this method the first iterated values of the pressure deviator and heat flux vector are obtained by inserting their equilibrium values $p_{\langle ij\rangle} = 0$ and $q_i = 0$ on the left-hand side of (41) and (42), yielding

$$p_{\langle ij\rangle} = -2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}, \qquad q_i = -\lambda\frac{\partial T}{\partial x_i} - \vartheta\frac{\partial n}{\partial x_i}. \tag{52}$$

The above equations represent the laws of Navier-Stokes and Fourier, respectively. The transport coefficients, namely, shear viscosity μ, thermal conductivity λ and the coefficient ϑ are given by

$$\mu = \frac{5}{4\mathsf{d}^2}\sqrt{\frac{mkT}{\pi}}\frac{1}{(1+\alpha)(3-\alpha)\left[1-\frac{a_2}{32}\right]}, \tag{53}$$

$$\lambda = \frac{75k}{2\mathsf{d}^2 m}\sqrt{\frac{mkT}{\pi}}\frac{1+2a_2}{(1+\alpha)\left\{49-33\alpha+(19-3\alpha)\frac{a_2}{32}\right\}}, \tag{54}$$

$$\vartheta = \frac{75kT}{2n\mathsf{d}^2 m}\sqrt{\frac{mkT}{\pi}}\frac{a_2}{(1+\alpha)\left\{49-33\alpha+(19-3\alpha)\frac{a_2}{32}\right\}}. \tag{55}$$

By neglecting the coefficient a_2 we obtain the expressions of Jenkins and Richman [2] for the transport coefficients of shear viscosity and thermal conductivity. Note that if we consider a thirteen moment theory the heat flux vector is proportional only to the temperature gradient and the dependence on the particle number gradient does not appear. This fact can be understood by noting that this dependence comes out from the underlined term in (42) which depends exclusively on the dimensionless fourth moment.

The transport coefficients (53) – (55) obtained by using the Maxwellian iteration procedure are not the same as those obtained from the Chapman-Enskog method (see e.g. [8]).

If we consider the elastic case ($\alpha = 1$) and the production term of the dimensionless fourth moment $(37)_2$, we obtain from (43) that the first iterated value of Δ vanishes by taking into account the Maxwellian iteration procedure. This result, which was also obtained in [10], is connected with the one given by (49), since Δ is a non-equilibrium part of the fourth moment which vanishes at equilibrium characterized by a Maxwellian distribution function.

7. **Eigenmodes in the thirteen field theory.** Let us now consider the dynamical behavior of a small local disturbance from the spatially homogeneous solution caused by a spontaneous internal fluctuation with wavenumber κ according to the thirteen field theory. We assume that the disturbance is sufficiently weak so that only linear deviations from the basic homogeneous solution need be taken into account. Hence,

we define the perturbations of the hydrodynamic fields as

$$\widetilde{n}(\mathbf{x},t)=\frac{n(\mathbf{x},t)}{n_0}-1,\qquad \widetilde{v}_i(\mathbf{x},t)=\frac{v_i(\mathbf{x},t)}{v(t)},\qquad \widetilde{T}(\mathbf{x},t)=\frac{T(\mathbf{x},t)}{T(t)}-1,\quad (56)$$

$$\widetilde{p}_{\langle ij\rangle}(\mathbf{x},t)=\frac{p_{\langle ij\rangle}(\mathbf{x},t)-p_{\langle ij\rangle}(t)}{n_0kT(t)},\qquad \widetilde{q}_i(\mathbf{x},t)=\frac{q_i(\mathbf{x},t)-q_i(t)}{n_0kT(t)v(t)},\tag{57}$$

where n_0 is the average number density of the system and $v(t)=\sqrt{5kT(t)/3m}$ is the adiabatic sound velocity in the homogeneous cooling state. Insertion of the perturbations (56) and (57) into the balance equations (38) – (42) leads, after linearization about the homogeneous solution, to a system of partial differential equations with time-dependent coefficients. However, it is possible to transform this system into a system of partial differential equations with constant coefficients by introducing the dimensionless time $\widetilde{t}=t/\tau(t)$ and the dimensionless position $\widetilde{\mathbf{x}}=\mathbf{x}/v(t)\tau(t)$, where

$$\tau(t)=\frac{1}{4n_0\mathsf{d}^2}\sqrt{\frac{m}{\pi kT(t)}},\tag{58}$$

is a mean free time in the homogeneous cooling state. Hence, we obtain

$$\frac{\partial\widetilde{n}}{\partial\widetilde{t}}+\frac{\partial\widetilde{v}_i}{\partial\widetilde{x}_i}=0,\tag{59}$$

$$\frac{\partial\widetilde{v}_i}{\partial\widetilde{t}}+\frac{3}{5}\left(\frac{\partial\widetilde{n}}{\partial\widetilde{x}_i}+\frac{\partial\widetilde{T}}{\partial\widetilde{x}_i}+\frac{\partial\widetilde{p}_{\langle ij\rangle}}{\partial\widetilde{x}_j}\right)-\frac{\xi_1}{2}\widetilde{v}_i=0,\tag{60}$$

$$\frac{\partial\widetilde{T}}{\partial\widetilde{t}}+\frac{2}{3}\left(\frac{\partial\widetilde{v}_i}{\partial\widetilde{x}_i}+\frac{\partial\widetilde{q}_i}{\partial\widetilde{x}_i}\right)+\xi_1\left(\widetilde{n}+\frac{\widetilde{T}}{2}\right)=0,\tag{61}$$

$$\frac{\partial\widetilde{p}_{\langle ij\rangle}}{\partial\widetilde{t}}+\frac{4}{5}\frac{\partial\widetilde{q}_{\langle i}}{\partial\widetilde{x}_{j\rangle}}+2\frac{\partial\widetilde{v}_{\langle i}}{\partial\widetilde{x}_{j\rangle}}+\xi_2\,\widetilde{p}_{\langle ij\rangle}=0,\tag{62}$$

$$\frac{\partial\widetilde{q}_i}{\partial\widetilde{t}}+\frac{3}{5}\frac{\partial\widetilde{p}_{\langle ij\rangle}}{\partial\widetilde{x}_j}+\frac{3}{2}\left[a_2\frac{\partial\widetilde{n}}{\partial\widetilde{x}_i}+(1+2a_2)\frac{\partial\widetilde{T}}{\partial\widetilde{x}_i}\right]+\xi_3\,\widetilde{q}_i=0,\tag{63}$$

where the dimensionless quantities

$$\xi_1=\frac{(1-\alpha^2)}{3}\left[1+\frac{3}{16}a_2\right],\tag{64}$$

$$\xi_2=\frac{2(1+\alpha)}{15}\left[2+\alpha-\frac{39-33\alpha}{64}a_2\right],\tag{65}$$

$$\xi_3=\frac{(1+\alpha)}{60}\left[19-3\alpha-\frac{161-177\alpha}{32}a_2\right],\tag{66}$$

depend only on the normal restitution coefficient. Note that in the derivation of above system of partial differential equations we have used the fact that in a homogeneous cooling state the pressure deviator and the heat flux vector decay in time faster than the temperature (see Figure 1).

Since the time evolution and the decay of the local disturbances are described by the so-called eigenmodes, let us look for solutions of the form

$$\begin{pmatrix}\widetilde{n} & \widetilde{v}_i & \widetilde{T} & \widetilde{p}_{\langle ij\rangle} & \widetilde{q}_i\end{pmatrix}^T=\begin{pmatrix}\bar{n} & \bar{v}_i & \bar{T} & \bar{p}_{\langle ij\rangle} & \bar{q}_i\end{pmatrix}^T\exp[\imath(\boldsymbol{\kappa}\cdot\mathbf{x}-\omega t)],\tag{67}$$

where ω and $\boldsymbol{\kappa}$ are the angular frequency and the wave-vector of the disturbance. If the wave-vector of the disturbance is taken parallel to the x-axis we obtain two independent systems of linear algebraic equations for the amplitudes of the perturbations, namely: the longitudinal system

$$\begin{pmatrix} \omega & -\kappa & 0 & 0 & 0 \\ -\frac{3}{5}\kappa & \omega - \imath\frac{\xi_1}{2} & -\frac{3}{5}\kappa & -\frac{3}{5}\kappa & 0 \\ \imath\xi_1 & -\frac{2}{3}\kappa & \omega + \imath\frac{\xi_1}{2} & 0 & -\frac{2}{3}\kappa \\ 0 & -\frac{4}{3}\kappa & 0 & \omega + \imath\xi_2 & -\frac{8}{15}\kappa \\ -\frac{3a_2}{2}\kappa & 0 & -\frac{3(1+2a_2)}{2}\kappa & -\frac{3}{5}\kappa & \omega + \imath\xi_3 \end{pmatrix} \begin{pmatrix} \bar{n} \\ \bar{v}_x \\ \bar{T} \\ \bar{p}_{\langle xx\rangle} \\ \bar{q}_x \end{pmatrix} = 0, \tag{68}$$

and the transverse system

$$\begin{pmatrix} \omega - \imath\frac{\xi_1}{2} & -\frac{3}{5}\kappa & 0 \\ -\kappa & \omega + \imath\xi_2 & -\frac{2}{5}\kappa \\ 0 & -\frac{3}{5}\kappa & \omega + \imath\xi_3 \end{pmatrix} \begin{pmatrix} \bar{v}_y \\ \bar{p}_{\langle yx\rangle} \\ \bar{q}_y \end{pmatrix} = 0. \tag{69}$$

Both systems of algebraic equations have non-trivial solutions if the determinant of each matrix of the coefficients vanish. This condition leads to a dispersion relation which can be used to determine the angular frequency as a function of the wavenumber or vice versa. For the determination of the eigenmodes, we take κ as a real function and solve the dispersion relation for ω leading to a relation of the form $\omega = \omega(\kappa)$. In this case, the real part of ω describes the frequency of oscillation of a small internal perturbation with wavenumber κ, while the imaginary part describes the decay (or growth) of its amplitude in time.

The solution of the dispersion relation for the longitudinal system (68) gives five eigenmodes, which can be divided into hydrodynamic modes, where $\omega(\kappa)$ tends to zero when κ goes to zero, and kinetic modes, where $\omega(\kappa)$ tends to a constant value when κ goes to zero. In the small wavenumber limit, it is possible to determine explicitly the value of eigenmodes by expanding the angular frequency as $\omega(\kappa) = \gamma_0 + \gamma_1\kappa + \gamma_2\kappa^2 + \dots$, where the expansion coefficients γ_i depend on the normal restitution coefficient. Hence, we have

$$\omega_1 = -\imath\frac{6}{5\xi_1}\kappa^2 + \dots, \qquad \omega_2 = \imath\frac{\xi_1}{2} - \imath\frac{2(5\xi_1 + 2\xi_2)}{5\xi_1(\xi_1 + 2\xi_2)}\kappa^2 + \dots, \tag{70}$$

$$\omega_3 = -\imath\frac{\xi_1}{2} - \imath\frac{2[\xi_1(9 + 10a_2) - 8\xi_3]}{5\xi_1(\xi_1 - 2\xi_3)}\kappa^2 + \dots, \tag{71}$$

$$\omega_4 = -\imath\xi_2 - \imath\frac{8(\xi_1 + 7\xi_2 - 5\xi_3)}{25(\xi_1 + 2\xi_2)(\xi_2 - \xi_3)}\kappa^2 + \dots, \tag{72}$$

$$\omega_5 = -\imath\xi_3 - \imath\frac{2[4\xi_1 + 25\xi_2(1 + 2a_2) - \xi_3(33 + 25a_2)]}{25(\xi_1 - 2\xi_3)(\xi_3 - \xi_2)}\kappa^2 + \dots. \tag{73}$$

Expressions (70) – (73) indicate that in the longitudinal case there are one hydrodynamic mode and four kinetic modes. Furthermore, for values of the wavenumber close to zero, these eigenmodes are purely diffusive, since their real parts vanish.

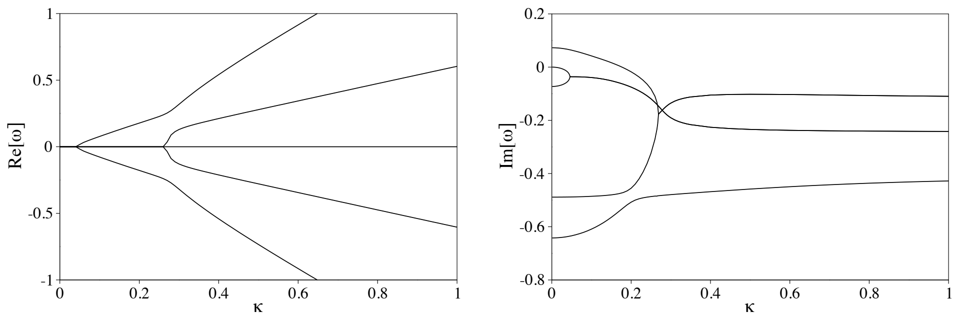

FIGURE 2. Longitudinal eigenmodes as a function of κ for $\alpha = 0.75$.

Concerning the imaginary part of the eigenmodes we notice that their values depend on the normal restitution coefficient through the dimensionless quantities ξ_1, ξ_2 and ξ_3. When $\mathrm{Im}[\omega] < 0$ the amplitude of the perturbations decays towards zero in time and the eigenmode is said to be stable. However, when $\mathrm{Im}[\omega] > 0$, the eigenmode is unstable, since the amplitude of the perturbations growths with time. Figure 2 shows the wavenumber dependence of the longitudinal eigenmodes according to the thirteen field theory for a normal restitution coefficient $\alpha = 0.75$. For values of κ close to zero, the five eigenmodes are purely diffusive, but one of them is unstable since its imaginary part is positive. A first kinetic sound mode starts to propagate at $\kappa \approx 0.04$, while a slower second kinetic sound mode appears at $\kappa \approx 0.26$. Moreover, we notice that the unstable diffuse mode becomes stable for $\kappa > \kappa_c \approx 0.18$, where the critical wavenumber κ_c is determined by imposing the condition $\mathrm{Im}[\omega(\kappa_c)] = 0$. Figure 3 shows the critical wavenumber for the unstable longitudinal kinetic eigenmode (solid line) as a function of the normal restitution coefficient in the interval $0.75 < \alpha < 1$. We may observe that unstable perturbations with large wavenumbers becomes more frequent in the system when collisions become more inelastic.

At this point, it is noteworthy to discuss the behavior of the longitudinal eigenmodes when the molecular collisions are elastic, i.e., when $\alpha = 1$. In the small wavenumber limit, the longitudinal elastic modes read

$$\omega_1 = -\imath\frac{9}{8}\kappa^2 + \dots, \quad \omega_2 = \kappa - \imath\frac{7}{8}\kappa^2 + \dots, \quad \omega_3 = -\kappa - \imath\frac{7}{8}\kappa^2 + \dots, \tag{74}$$

$$\omega_4 = -\imath\frac{4}{5} + \imath\frac{11}{5}\kappa^2 + \dots, \qquad \omega_5 = -\imath\frac{8}{15} + \imath\frac{27}{40}\kappa^2 + \dots. \tag{75}$$

In this case, we verify from the above expressions that there are three hydrodynamic eigenmodes and two kinetic eigenmodes. Two of the hydrodynamic modes are sound modes that describe sound propagation in opposite directions parallel to the x-axis, while the other hydrodynamic mode is purely diffusive. Furthermore, for κ close to zero, we observe from Figure 4 that the two kinetic modes do not propagate, since their real part vanish. However, a kinetic sound mode that propagates slower than the hydrodynamic sound appears at $\kappa \approx 0.32$.

Concerning the transverse system (69) we infer that the solution of the dispersion relation gives three kinetic eigenmodes when $\alpha \neq 1$. However, when $\alpha = 1$, the solution of the transversal dispersion relation yields one hydrodynamic eigenmode

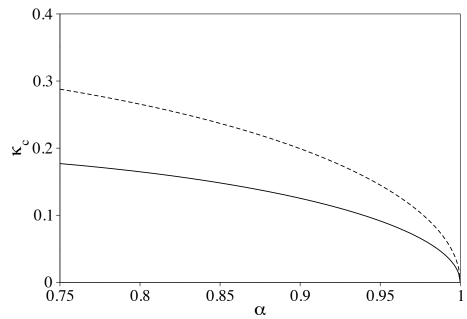

FIGURE 3. Longitudinal (solid) and transversal (dashed) critical wavenumbers as functions of α.

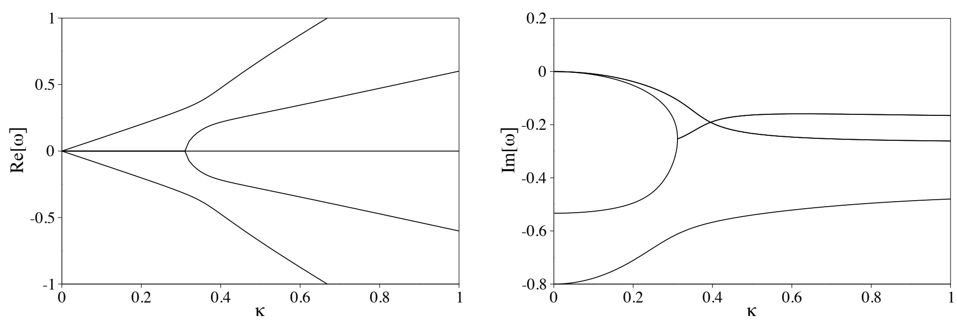

FIGURE 4. Longitudinal eigenmodes as a function of κ for $\alpha = 1$.

and two kinetic eigenmodes. In the small wavenumber limit, the elastic eigenmodes read

$$\omega_1 = -\imath\frac{3}{4}\kappa^2 + \dots, \quad \omega_2 = -\imath\frac{4}{15} - \imath\frac{3}{4}\kappa^2 + \dots, \quad \omega_3 = -\imath\frac{4}{5} + \imath\frac{3}{2}\kappa^2 + \dots, \tag{76}$$

while the inelastic eigenmodes are given by

$$\omega_1 = \imath\frac{\xi_1}{2} - \imath\frac{6}{5(\xi_1 + 2\xi_2)}\kappa^2 + \dots, \quad \omega_2 = -\imath\xi_3 - \imath\frac{6}{25(\xi_2 - \xi_3)}\kappa^2 + \dots, \tag{77}$$

$$\omega_3 = -\imath\xi_2 + \imath\frac{6(\xi_1 + 7\xi_2 - 5\xi_3)}{5(\xi_1 + 2\xi_2)(\xi_2 - \xi_3)}\kappa^2 + \dots. \tag{78}$$

The dependence of the transversal eigenmodes on the wavenumber is shown in Figures 5 and 6 for the elastic and inelastic cases, respectively. In the elastic case, all eigenmodes are stable and the kinetic sound modes starts to propagate at $\kappa \approx 0.23$. On the other hand, for the inelastic case, we verify from Figure 6 that one kinetic eigenmode is unstable for values of the wavenumber lower than 0.28, while the propagation of the kinetic sound modes starts at lower wavenumber values than that of the elastic case. Lastly, we would like to mention that the dependence of the critical transversal wavenumber on the normal restitution coefficient is also shown in Figure 3 and represented by a dashed line. Note that its behavior is very similar to the behavior of the critical longitudinal wavenumber.

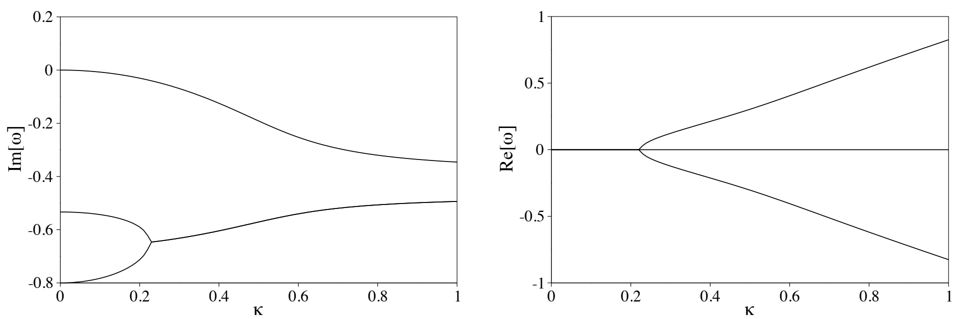

FIGURE 5. Transversal eigenmodes as a function of κ for $\alpha = 1$.

8. **Conclusions.** We have developed a fourteen moment theory for a granular gas where the inclusion of the full contracted fourth moment leads to some results that are quite different from a thirteen moment theory. The main achievements are related to the identification of the fourth moment with the coefficient a_2 in the Chapman-Enskog solution of the Boltzmann equation when it remains constant in time. Furthermore, the dependence of the heat flux vector on the gradient of the particle number density was obtained due to the inclusion of the fourth moment. We have also studied the dynamic behavior of small local disturbances from the spatially homogeneous solutions caused by spontaneous internal fluctuations in the case of a thirteen field theory and have shown that for the longitudinal disturbances there exist one hydrodynamic and four kinetic modes, while for the transverse disturbances one hydrodynamic and two kinetic modes are present.

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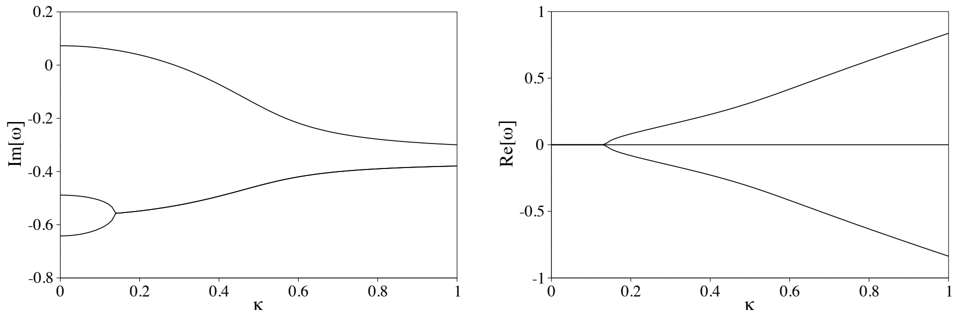

FIGURE 6. Transversal eigenmodes as a function of κ for $\alpha = 0.75$.

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