Math 401: Sec 0501: Midterm # 2
May 1, 2015

Complete problems 1–5. For full credit you MUST write down clearly all steps and circle the final answer(s). Solve each problem in a separate sheet. If you use a *well known* theorem, make clear which theorem you are using and justify its use.

Problem 1 (25 points): Consider $\mathbb{V} = \mathbb{P}_2$, i.e., the space of polynomials of degree ≤ 2 with the inner product

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

for all $p, q \in \mathbb{P}_2$.

(1) Prove that the bilinear, symmetric function $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{V}$.

(2) Prove that $p(x) = 1$ and $q(x) = x$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$.

(3) Find b and c such that

$$r(x) = x^2 + bx + c,$$

is orthogonal to the set $\{p(x), q(x)\}$.

Problem 2 (10 points): Given the set of experimental data $(1, 10)$, $(2, 15)$, $(3, 17)$, $(4, 23)$, write the least squares problem $A\mathbf{x} \approx \mathbf{b}$ for fitting the data with the following function

$$q(x) = \frac{1}{a(3 + x) + b \sin(x - \pi)},$$

with unknowns coefficients a and b. Indicate how to find a and b, but **do not solve!**

Problem 3 (15 points):

a) Formulate the linear algebra problem of finding the closest polynomial $p \in \text{span}\{t, t^2\}$ to the function $f(t) = e^t \cos(t)$ with respect to the L^2 inner product (**do not solve!**).

b) Is the coefficient matrix of a) symmetric positive definite?.

Problem 4 (20 points):

(1) Let $A, B \in \mathbb{R}^{n \times n}$ be such that $A = BB^T$. Under what conditions on B is the matrix A symmetric and positive definite? Hint: Do not consider a particular case of B.

(2) Let $A \in \mathbb{R}^{3\times 3}$ be given by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$$

Prove that A is a symmetric and positive definite matrix and compute the Cholesky decomposition. Hint: As a first step, compute the LU decomposition of A.

Problem 5 (30 points): (a) Find a set of orthogonal vectors that spand the same subspace $S \subset \mathbb{R}^4$ as

$$\mathbf{a}_1 = (0, -1, 1, 0)^T, \qquad \mathbf{a}_2 = (2, 0, 2, 0)^T, \qquad \mathbf{a}_3 = (-1, 0, 0, 1)^T.$$

(b) Find the projection of $\mathbf{b} = (-1, 1, 1, 1)^T$ onto S.