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Problemas de vibraciones, acústica y disipación.

Problems of vibrations, acoustics and dissipation.

Tesis para optar al grado de
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Felipe Andrés Lepe Araya
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Profesor Guía: Rodolfo Rodríguez
CI^{2}MA y Departamento de Ingeniería Matemática
Universidad de Concepción Chile.

Vibration, acoustic and dissipation problems

Felipe Andrés Lepe Araya

Directores de Tesis: Prof. Rodolfo Rodríguez, Universidad de Concepción, Chile.
Prof. Salim Meddahi, Universidad de Oviedo, España.
Prof. David Mora, Universidad del Bío-Bío, Chile.

Director de Programa: Prof. Raimund Bürger, Universidad de Concepción, Chile

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COMISIÓN EXAMINADORA

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Firma: ______________________________
Prof. David Mora , Universidad del Bío-Bío, Chile.

Firma: ______________________________
Prof. Ricardo Oyarzúa , Universidad del Bío-Bío, Chile.

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Prof. Claudio Padra , Centro Atómico de Bariloche, Argentina.

Firma: ______________________________
Prof. Rodolfo Rodríguez , Universidad de Concepción, Chile.

Fecha Examen de Grado: ______________________________

Calificación: ______________________________

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Dedicado a la memoria
de Melania F. Lara Q.

Resumen

El objetivo de esta tesis es desarrollar y analizar herramientas numéricas eficientes para resolver problemas acoplados que involucran estructuras elásticas y fluidos acústicos disipativos. Con este fin, en particular consideraremos problemas de vibraciones para una viga de Timoshenko empotrada con sección transversal variable, el problema espectral para las ecuaciones de elasticidad lineal, un problema de interacción entre dos fluidos disipativos heterogéneos y un problema de elastoacústica disipativa. Para aproximar las soluciones de los problemas mencionados, utilizamos métodos numéricos basados en el clásico método de elementos finitos (FEM) y el método de Galerkin discontinuo (DG).

En la primera parte de este trabajo estudiaremos un método de elementos finitos de bajo orden para aproximar las frecuencias más bajas del problema de vibraciones de una viga modelada con las ecuaciones de Timoshenko. La formulación que estudiaremos incorpora el momento flector como incógnita adicional. Demostraremos orden óptimo de convergencia para el desplazamiento, la rotación, el momento flector y el esfuerzo de corte de los modos de vibración, como también orden doble de convergencia para las frecuencias de vibración. Las constantes de las estimaciones demostradas son independientes del parámetro de espesor de la viga, por lo que el modelo resulta libre de bloqueo numérico. Se mostrará que el desplazamiento y la rotación pueden ser eliminados, para obtener un problema matricial generalizado de valores propios con menor costo computacional, similar al de las formulaciones clásicas. Presentamos experimentos numéricos que confirman los resultados teóricos obtenidos.

En la segunda parte de esta tesis, abordamos el problema de interacción de dos fluidos heterogéneos viscosos. La viscocidad produce el efecto de disipación, el cual, nos llevará a estudiar un problema de valores propios cuadrático. En esta parte proponemos un riguroso análisis matemático del problema espectral para establecer una caracterización espectral del operador solución asociado al problema de valores propios, mostrando que el operador solución admite un espectro esencial. Para la aproximación numérica, implementaremos un método de elementos finitos basado en los elementos de Raviart-Thomas de primer orden, que usaremos para discretizar el problema todo el dominio. Notamos que la presencia de viscosidad en nuestra formulación conlleva a nuevas dificultades, debido a que el operador solución resulta ser no regularizante y por lo tanto no compacto. Por este motivo, y para demostrar las propiedades de aproximación espectral, debemos emplear nuevas técnicas. Mostraremos que nuestro método es convergente, que no introduce modos espurios, y que las correspondientes autofunciones y valores propios convergen con el orden teóricamente esperado. Corroboraremos nuestro análisis

con ejemplos numéricos.

En la tercera parte de este trabajo, consideramos una formulación dual mixta para el problema de elasticidad lineal, la cual está escrita en términos del tensor de esfuerzo y del tensor antisimétrico de rotaciones. La idea central de esta tercera parte es definir una discretización usando un método de Galerkin discontinuo (DG), para aproximar con polinomios de grado k el tensor de esfuerzos, y polinomios de grado $k-1$ el tensor de rotaciones. Adaptamos la teoría de operadores no compactos para estas nuevas normas dependientes de la malla, demostrando que las estimaciones son independientes de la malla y que el método numérico no introduce frecuencias de vibración espurias. Se realizarán diversos experimentos numéricos con el fin de estudiar el comportamiento del método en relación al parámetro de estabilización, el tamaño de las mallas y el grado polinomial.

Finalmente, presentamos un problema de elastoacústica donde consideramos un fluido viscoso contenido en una cavidad rígida. La presencia de viscosidad nos lleva en este caso particular a un problema de elastoacústica disipativa. El fluido será modelado con las ecuaciones de Stokes, considerando el término disipativo y el sólido con las ecuaciones de elasticidad lineal. Se presentará una formulación contínua escrita en términos de los desplazamientos del fluido y del sólido para el problema de valores propios, el cual resulta ser cuadrático debido a la presencia de disipación en el fluido. Se estudiará el buen planetamiento del problema continuo, y una caracterización espectral del operador solución asociado. Se introducirá un método de elementos finitos no conforme, donde el dominio del sólido es discretizado con elementos para H^1 y el fluido con elementos para $\mathrm{H}(\mathrm{div})$. Demostraremos que el método es convergente y no introduce modos espurios, utilizando las herramientas empleadas en la segunda parte de esta tesis. Presentaremos algunos experimentos numéricos que corroboran los resultados teóricos obtenidos.

Abstract

The goal of this dissertation is to develop and analyze efficient numerical tools to deal with vibration problems for coupled systems involving elastic structures and dissipative fluids.We will consider the vibration problem of a clamped Timoshenko beam with variable cross section, an elasticity eigenproblem, an interaction problem between two heterogeneous dissipative fluids and a dissipative elastoacustic problem. In order to approximate the solutions of these problems, we use numerical methods based on the classical finite element method (FEM) and the discontinuous Galerkin method (DG).

In the first part of this dissertation, we analyze a low-order finite element method to approximate the natural frequencies and the vibration modes of a non-homogeneous Timoshenko beam. We consider a formulation in which the bending moment is introduced as an additional unknown for the source problem. Optimal order error estimates are proved for displacements, rotations, shear stress and bending moment of the vibration modes, as well a double order of convergence for the vibration frequencies. These estimates are independent of the beam thickness, which leads us to the conclusion that the method is locking free with respect to this parameter. For the implementation of the numerical method, we show that the elimination of the displacements and rotations leads to a well posed generalized matrix eigenvalue problem whose solutions are comparable to the one obtained with other classical formulations in terms of computational cost. Some numerical experiments are presented to assess the performance of the method.

In the second chapter, we address the interaction problem between two dissipative fluids. The presence of dissipation leads us to the study of a quadratic eigenvalue problem. A rigorous mathematical analysis of the spectral problem is performed to establish a spectral characterization of the associated solution operator. We prove that the solution operator admits an essential spectrum that is well separated from the physical spectrum. We use the lowest order Raviart-Thomas elements to discretize the problem. We observe that the presence of viscosity leads to new difficulties, since the solution operator is non-regularizing and therefore non compact. We prove that our proposed method is convergent and spurious modes free and that the corresponding eigenfunctions and eigenvalues converge with the expected order. The theoretical results are validated with some numerical experiments.

In the third part we consider the dual mixed formulation for the elasticity equations written in terms of the stress and the rotation tensors. Our aim is to use a DG method to compute the lowest frequencies of the resulting mixed spectral problem. To this end, we approximate the stress tensor with polynomials of degree k and the rotations with polynomials of degree

$k-1$. We endow the DG spaces with their natural mesh dependent norms and adapt the non-compact operator theory to prove that our DG method does not introduce spurious modes for a small enough meshsize and a large enough stabilization parameter. We report some numerical examples to assess the performance of the method in relation with the stabilisation parameter, the meshsize and the polynomial degree.

Finally, we present an elastoacustic problem, where a dissipative fluid contained in a rigid cavity is considered. The presence of dissipation leads to a quadratic eigenvalue problem. As in the previous chapters, the fluid is modeled with the Stokes equations and the solid with the linear elasticity equations. A continuous spectral formulation written in terms of the displacements of the fluid and the solid is presented and analyzed. The solution operator associated to the eigenvalue problem is introduced and its spectrum is characterized. A finite element method is introduced, where the displacement of the fluid is approximated with H(div) elements and the solid displacement with H^1 elements. This particular choice of finite element spaces leads to a non-conforming method. The analysis of convergence, error estimates and spurious free results are obtained as in the second part of this dissertation. Some numerical experiments are presented to asses the performance of the method.

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Chapter 1

Introducción.

1.1 Vibraciones y acústica disipativa

El fenómeno de vibración ha sido ampliamente estudiado por los ingenieros desde hace muchos años. Las estrcuturas sufren vibraciones debido a fuerzas externas, así como también fluidos con propiedades acústicas y por supuesto, la interacción de fluidos y estructuras cuando ciertas fuerzas perturban dichos sistemas. Este último fenómeno se conoce como *elastoacústica*. Las propiedades físicas de los fluidos inciden en como se manifiestan los modos de vibración. Una de estas propiedades de gran interés es la viscosidad.

La presencia de viscosidad produce el fenómeno de disipación. Existen dos tipos de disipación: la disipación acústica interna dentro de una cavidad debido a la viscosidad y conducción térmica de un fluido, y la disipación generada dentro de la capa límite de una pared viscoelástica. Mayores detalles sobre estos fenómenos se pueden encontrar, por ejemplo, en [76]. En particular, a lo largo de nuestro trabajo, nos restringiremos al estudio de la disipación acústica interior.

Los fenómenos anteriormente descritos son de vital importancia para la construcción de puentes, edificios, automóviles, barcos, etc., ya que la estabilidad de ciertas estructuras o la interacción de éstas con ciertos fluidos involucran necesidades de diseño, costos, aspectos ambientales, etc.

Los modelos numéricos son una poderosa herramienta para tomar decisiones con respecto a dichas necesidades, ya que no es posible encontrar soluciones exactas a los sistemas de ecuaciones diferenciales que aparecen.

Los problemas de valores propios, en el contexto de las ecuaciones diferenciales parciales, y en particular en el análisis numérico de dichas ecuaciones, han motivado la formalización del estudio de los operadores asociados a estos problemas espectrales en el caso de ser compactos, no compactos y también en el caso de problemas espectrales en formulaciones primales, mixtas, etc. Una síntesis de cómo estas herramientas han ido desarrollándose a través de los años se puede encontrar en [20, 25].

Dentro de los métodos numéricos para resolver ecuaciones diferenciales parciales están el clásico método de elementos finitos (FEM) y el método de Galerkin discontinuo (DG), introducido en los años setenta para resolver numéricamente problemas hiperbólicos. En los años noventa, método DG se comenzó a aplicar para problemas elípticos, basándonse en el artículo de Arnold et. al. [5] donde se analiza un marco apropiado para la aplicación de éste método a estos problemas, siendo numerosas las aplicaciones a partir del año 2000 en adelante. Para revisar el estado del arte y aplicaciones se puede ver [38] y las referencias en él.

Nuesto objetivo principal es resolver numéricamente un problema de interacción entre un fluido disipativo y una estructura modelada por las clásicas ecuaciones de elasticidad lineal. Artículos como [12, 14, 16, 17, 70] estudian problemas de elastoacústica y analizan modelos numéricos empleando elementos finitos, demostrando convergencia y obteniendo estimaciones del error. Sin embargo, estos modelos elastoacústicos no consideran fluidos viscosos.

Un modelo de acústica disipativa que acopla las ecuaciones de Stokes con la ecuación del calor es el estudiado en [18]. En este artículo se asume que, para pequeños desplazamientos, el fluido es irrotacional. Esto implica que si $\boldsymbol{U}$ es el desplazamiento del fluido, $\mathbf{curl}\,\dot{\boldsymbol{U}} = 0$, donde el punto representa la derivada temporal de primer orden. Esto implica que en la siguiente identidad $\Delta\dot{\boldsymbol{U}} = \nabla(\operatorname{div}\dot{\boldsymbol{U}}) - \mathbf{curl}(\mathbf{curl}\,\dot{\boldsymbol{U}})$, la parte rotacional desaparece. Esta suposición de irrotacionalidad del fluido es útil para nosotros, ya que las ecuaciones de Stokes se simplifican.

El modelo de elastoacústica que queremos analizar, considera un fluido disipativo irrotacional modelado con las ecuaciones de Stokes, y una estructura elástica, donde las ecuaciones que modelan el fenómeno son las siguientes:

$$\rho_f\ddot{\boldsymbol{U}} - 2\nu_f\nabla(\operatorname{div}\dot{\boldsymbol{U}}) + \nabla P = \mathbf{0} \ \text{ in } \Omega_f \times (0,T), \tag{1.1.1}$$

$$P + \rho_f c^2 \operatorname{div}\boldsymbol{U} = 0 \ \text{ in } \Omega_f \times (0,T), \tag{1.1.2}$$

$$\rho_s\ddot{\boldsymbol{W}} - \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{W})) = \mathbf{0} \ \text{ in } \Omega_s \times (0,T), \tag{1.1.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{W}) - \lambda_S \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{W}))I - 2\mu_S\boldsymbol{\varepsilon}(\boldsymbol{W}) = 0 \ \text{ in } \Omega_s \times (0,T) \tag{1.1.4}$$

$$\boldsymbol{U}\cdot\boldsymbol{n} - \boldsymbol{W}\cdot\boldsymbol{n} = 0 \ \text{ in } \Gamma_{\mathrm{I}} \times [0,T], \tag{1.1.5}$$

$$\boldsymbol{\sigma}(\boldsymbol{W})\boldsymbol{n} - (P + 2\nu\operatorname{div}\boldsymbol{U})\boldsymbol{n} = 0 \ \text{ in } \Gamma_{\mathrm{I}} \times [0,T], \tag{1.1.6}$$

$$\boldsymbol{\sigma}(\boldsymbol{W}) = 0 \ \text{ in } \Gamma_N \times [0,T], \tag{1.1.7}$$

$$\boldsymbol{W} = 0 \ \text{ in } \Gamma_D \times [0,T], \tag{1.1.8}$$

donde $\boldsymbol{W}$ es el desplazamiento del sólido, P es la presión del fluido, $\boldsymbol{\sigma}$ es el tensor de esfuerzos de Cauchy, $\boldsymbol{\varepsilon}(\cdot)$ es el tensor de deformación lineal definido por $\boldsymbol{\varepsilon}(\boldsymbol{W}) := \frac{1}{2}[\nabla\boldsymbol{W} + (\nabla\boldsymbol{W})^{\mathtt{t}}]$, ρ_f y ρ_s son las densidades del fluido y del sólido, respectivamente, c es la velocidad acústica del fluido, ν_f es la viscosidad del fluido, λ_s y μ_s son las contantes de Lamé asociadas al material de la estructura, Ω_f y Ω_s son los dominios ocupados por el fluido y el sólido, respectivamente, Γ_{I} es la interfaz de contacto entre el sólido y el fluido, Γ_D y Γ_N son las fronteras Dirichlet y Neumann, respectivamente, y $\boldsymbol{n}$ es el vector normal unitario exterior a la frontera Γ_N. Si introducimos los siguientes espacios

$$\mathcal{H} := \mathrm{L}^2(\Omega_f)^n \times \mathrm{L}^2(\Omega_s)^2, \quad \mathcal{X} := \mathrm{H}(\mathrm{div};\Omega_f) \times \mathrm{H}^1_{\Gamma_D}(\Omega_s)^2,$$

$$\mathcal{V} := \{(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{X} \ : \ \boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{w}\cdot\boldsymbol{n}\},$$

donde $\mathrm{H}^1_{\Gamma_D}(\Omega_s)^2$ es el subespacio de funciones de $\mathrm{H}^1(\Omega_s)^2$ que se anulan sobre Γ_D y consideramos que las frecuencias de vibración del problema tienen la forma $\boldsymbol{U}(\boldsymbol{x},t) = e^{\lambda t}\boldsymbol{u}(\boldsymbol{x},t)$ y $\boldsymbol{W}(\boldsymbol{x},t) = e^{\lambda t}\boldsymbol{w}(\boldsymbol{x},t)$, una formulación variacional del problema elastoacústico disipativo, escrito en el dominio de la frecuencia y en términos de los desplazamientos del fluido y del solido es:

Problem 1.1.1 Hallar $\lambda \in \mathbb{C}$ y $(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{u},\boldsymbol{w}) \in \mathcal{V}$ tal que

$$\lambda^2\left(\int_{\Omega_f}\rho_f\boldsymbol{u}\cdot\overline{\boldsymbol{v}} + \int_{\Omega_s}\rho_s\boldsymbol{w}\cdot\overline{\boldsymbol{\tau}}\right) + 2\lambda\int_{\Omega_f}\nu\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\overline{\boldsymbol{v}} + \int_{\Omega_f}\rho_f c^2\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\overline{\boldsymbol{v}} + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{w}):\boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = 0 \quad \forall(\boldsymbol{v},\boldsymbol{\tau})\in\mathcal{V}. \tag{1.1.9}$$

donde la presión ha sido eliminada gracias a la condición de compresibilidad $P + \rho_f c^2\,\mathrm{div}\,\boldsymbol{u} = 0$.

La presencia de viscosidad en este modelo conlleva a nuevas dificultades. Por este motivo, para resolver numéricamente el problema de aproximar las frecuencias de vibración complejas de este problema de elastoacústica disipativa, primero resolveremos algunos problemas espectrales para fluidos y sólidos de forma separada.

Esta tesis tiene como objetivo desarrollar y analizar métodos numéricos para problemas espectrales en dos ejes principales. El primero será el problema de vibraciones de estructuras. Dentro de este tema consideraremos estructuras delgadas, particularmente vigas y las ecuaciones de elasticidad lineal. El segundo tema de interés es la acústica disipativa, donde estudiaremos los modos de vibración de fluidos con propiedades viscosas interactuando entre sí.

A continuación daremos una breve descripción del marco de trabajo.

Acústica disipativa

Los modelos descritos anteriormente consideran fluidos no viscosos, por lo tanto, no incorporan disipación en el fenómeno físico a analizar. La disipación debería ser un fenómeno considerado siempre al trabajar en acústica, ya que en estricto rigor, la disipación está presente en toda la naturaleza, y más aún en la ingeniería.

El estudio de la acústica disipativa permite, entre otras cosas, poder desarrollar técnicas de reducción de ruido. El ruido es una de las formas de contaminación ambiental y es importante mantenerlo controlado.

Si en el problema (1.1.1)–(1.1.8) consideramos $\nu_f = 0$ y eliminamos el desplazamiento $\boldsymbol{W}$ del sólido, las ecuaciones (1.1.3), (1.1.4), (1.1.7) y (1.1.8) son eliminadas y las condiciones de transmisión (1.1.5) y (1.1.6) son modificadas. Más aún, si reemplazamamos la condición (1.1.6) por una condición de amortiguamiento, obtenemos el siguiente problema modelo

$$\rho_f \frac{\partial^2 \boldsymbol{U}}{\partial t^2} + \nabla P = \boldsymbol{0} \quad \text{en } \Omega, \tag{1.1.10}$$

$$P = -\rho_f c^2 \operatorname{div} \boldsymbol{U} \quad \text{en } \Omega, \tag{1.1.11}$$

$$P = \left(\alpha \boldsymbol{U} \cdot \boldsymbol{n} + \beta \frac{\partial \boldsymbol{U}}{\partial t} \cdot \boldsymbol{n}\right) \quad \text{en } \Gamma_A, \tag{1.1.12}$$

$$\boldsymbol{U} \cdot \boldsymbol{n} = 0 \quad \text{en } \Gamma_R, \tag{1.1.13}$$

Las ecuaciones anteriores corresponden al problema disipativo de un fluido contenido en una cavidad rígida con paredes absorbentes, las cuales son capaces de disipar energía acústica. Este problema fue estudiado rigurosamente en [13], donde $\Omega \subset \mathbb{R}^d$, con $d = 2, 3$ es un dominio poligonal o poliédrico, $\boldsymbol{U}$ es el desplazamiento del fluido, P es la presión, α y β son los coeficientes asociados a la impedancia del fluido, $\boldsymbol{n}$ es el vector normal unitario exterior, Γ_A es la frontera asociada a la pared absorbente y Γ_R es la parte de la frontera rígida.

Si las soluciones armónicas del sistema de ecuaciones anterior son de la forma $\boldsymbol{U}(\boldsymbol{x}, t) := e^{\lambda t}\boldsymbol{u}(\boldsymbol{x})$ y $P(\boldsymbol{x}, t) := e^{\lambda t}p(\boldsymbol{x})$, escribiendo (1.1.10)–(1.1.13) en el dominio de la frecuencia, tenemos el siguiente problema de valores propios cuadrático: Hallar $\lambda \in \mathbb{C}$, $\boldsymbol{u} : \Omega \to \mathbb{C}^n$ y $\psi : \Omega \to \mathbb{C}$, ambas funciones no nulas, tal que

$$\rho \lambda^2 \boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{en } \Omega, \tag{1.1.14}$$

$$p = -\rho c^2 \operatorname{div} \boldsymbol{u} \quad \text{en } \Omega, \tag{1.1.15}$$

$$p = (\alpha \boldsymbol{U} \cdot \boldsymbol{n} + \beta \frac{\partial \boldsymbol{U}}{\partial t} \boldsymbol{u} \cdot \boldsymbol{n}) \quad \text{en } \Gamma_A, \tag{1.1.16}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{en } \Gamma_R, . \tag{1.1.17}$$

Una formulación variacional para este problema espectral, estudiada en [13], escrita en términos del desplazamiento del fluido, donde la presión es eliminada convenientemente usando la condición de compresibilidad, es la siguiente:

Hallar $\lambda \in \mathbb{C}$ y $\boldsymbol{0} \neq \boldsymbol{u} \in \mathrm{H}_0(\operatorname{div}, \Omega)$ tal que

$$\lambda^2 \int_\Omega \rho_f \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \lambda \int_{\gamma_A} \beta \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \boldsymbol{n} + \int_{\Gamma_A} \alpha \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \overline{\boldsymbol{v}} + \int_\Omega \rho_f c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \quad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}; \Omega).$$

Se observa la no linealidad del problema en λ. Este problema se linearizó incorporando una variable auxiliar $\widehat{\boldsymbol{u}} := \lambda \boldsymbol{u}$. De esta forma se obtiene el siguiente sistema de doble tamaño: Hallar $\lambda \in \mathbb{C}$ y $(\boldsymbol{0}, \boldsymbol{0}) \neq (\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathrm{H}_0(\operatorname{div}; \Omega) \times \mathrm{L}^2(\Omega)^d$ tal que

$$\begin{cases} \displaystyle\int_\Omega \rho_f c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Gamma_A} \alpha \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \boldsymbol{n} = \lambda \left(- \int_{\Gamma_A} \beta \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \boldsymbol{n} - \int_\Omega \rho_f \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} \right) & \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}; \Omega), \\ \displaystyle\int_\Omega \rho_f \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}} = \lambda \int_\Omega \rho_f \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}} \qquad \forall \widehat{\boldsymbol{v}} \in \mathrm{L}^2(\Omega)^d. \end{cases}$$

Vibraciones elásticas

En nuestros problemas de vibraciones consideraremos, por un lado. las clásicas ecuaciones de elasticidad y por otro estructuras delgadas. Dentro de las estructuras moderadamente delgadas se encuentran las placas y las vigas, siendo estas últimas nuestro objeto de interés. En particular, consideraremos el modelo de vigas de Timoshenko. Este modelo, introducido a principios del siglo XX, permitió dar más realismo a la modelación de una viga comparado con el modelo de Bernard Euler, donde la rotación de las fibras no se mantienen perpendiculares al centroide de la viga, como ocurre con el modelo de Euler-Bernoulli. Por otro lado, el espesor de la viga era despreciable con respecto a las demás dimensiones de la estructura en el modelo de Euler. El modelo de Timoshenko en cambio considera el espesor dentro de la modelación, por lo que la estructura ya depende de este parámetro. Desde el punto de vista del análisis numérico, la consideración del espesor dentro del modelo llevó a que los modelos numéricos experimentaran el fenómeno conocido como *bloqueo numérico*, donde las técnicas de integración reducida pudieron evitar esta dificultad. Sin embargo, en 1981, Arnold en el artículo [4] demuestra que una formulación mixta que incorpora el esfuerzo de corte de la viga como nueva variable también evita el efecto de bloqueo. Esto llevó a estudiar nuevas formulaciones mixtas para el modelo de Timoshenko, con el objetivo de introducir métodos numéricos libres de bloqueo.

Por otro lado, las ecuaciones de elasticidad lineal describen la relación lineal que existe entre las tensiones y las deformaciones de sólidos generalmente isotrópicos, las cuales son abordadas ampliamente por la mecánica de medios continuos. La ecuación constitutiva que relaciona el desplazamiento de la estructura con el esfuezo es

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}) \quad \text{en } \Omega,$$

donde $\boldsymbol{w}$ representa el desplazamiento y $\Omega \subset \mathbb{R}^d$ algún dominio acotado, con $d = 2, 3$. Finalmente, $\mathcal{C}$ es el operador de elasticidad dado por la ley de Hooke definido como

$$\mathcal{C}\boldsymbol{\tau} := \lambda_s(\operatorname{tr}\boldsymbol{\tau})\boldsymbol{I} + 2\mu_s\boldsymbol{\tau},$$

donde λ_s y μ_s son los coeficientes de Lamé asociados al material.

Si en el problema (1.1.1)–(1.1.8) ahora consideramos el desplazamiento $\boldsymbol{U}$ del fluido igual a cero y consideramos que el desplazamiento $\boldsymbol{W}$ del sólido es de la forma $\boldsymbol{W}(\boldsymbol{x}, t) = e^{\lambda t}\boldsymbol{w}(\boldsymbol{x}, t)$, obtenemos el siguiente problema espectral de elasticidad

$$\begin{aligned} \boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}) \quad & \text{in } \Omega, \\ \operatorname{\mathbf{div}}\boldsymbol{\sigma} + \omega^2\rho_s\boldsymbol{w} = \mathbf{0} \quad & \text{in } \Omega, \\ \boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} \quad & \text{on } \Gamma_F, \\ \boldsymbol{w} = \mathbf{0} \quad & \text{on } \Gamma_R, \end{aligned}$$

donde $\sqrt{\lambda} = \omega \geq 0$ son las frecuencias, Γ_R es la parte de la frontera que está fija y Γ_F la parte libre de la frontera. Introduciendo la rotación como $\boldsymbol{r} := \frac{1}{2}[\nabla\boldsymbol{w} - (\nabla\boldsymbol{w})^{\mathtt{t}}]$ de modo que $\mathcal{C}^{-1} = \boldsymbol{\varepsilon}(\boldsymbol{w}) = \nabla\boldsymbol{w} - \boldsymbol{r}$, y eliminando el desplazamiento con la condición $\operatorname{\mathbf{div}}\boldsymbol{\sigma} + \omega^2\rho_s\boldsymbol{w} = \mathbf{0}$, podemos llegar al siguiente problema espectral variacional, escrito en una formulación dual mixta estudiada rigurosamente en [69] : Hallar $\lambda \in \mathbb{R}$ y $\mathbf{0} \neq (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ tal que

$$\int_\Omega \rho_s^{-1}\, \mathbf{div} \cdot \mathbf{div}\, \boldsymbol{\tau} = \lambda \left(\int_\Omega \mathcal{C}^{-1} \boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{r} \right) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$
$$\lambda \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} = 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}},$$

donde

$$\boldsymbol{\mathcal{W}} := \{ \boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}; \Omega) : \quad \boldsymbol{\tau n} = \mathbf{0} \quad \text{en } \Gamma_F \} \quad \text{y} \quad \boldsymbol{\mathcal{Q}} := \{ \boldsymbol{s} \in \mathrm{L}^2(\Omega)^{n \times n} : \quad \boldsymbol{s}^{\mathtt{t}} = -\boldsymbol{s} \}.$$

A continuación describiremos los aportes de este trabajo.

1.1.1 Vibración de una viga de Timoshenko empotrada y de sección transversal variable.

En ingeniería estructural, para la construcción y diseño de puentes, aeronaves, automóviles etc., es muy frecuente considerar estructuras delgadas tales como vigas o placas que sean tanto de geometría continua, como también geometrías que consideren discontinuidades. También podemos encontrar estructuras que posean propiedades físicas diferentes en su forma, es decir, podemos considerar por ejemplo una viga que este hecha de cobre y de acero, lo que implica que cantidades físicas como el módulo de Young, el radio de Poisson o el esfuerzo de corte, ya no serán continuas en la viga. Debido a esto, es importante conocer las vibraciones de estructuras bajo estas condiciones.

La teoría de Timoshenko es una de las más usadas para estudiar la deformación de una viga elástica moderadamente delgada. En algunos métodos de elementos finitos ocurre que, cuando el parámetro de grosor disminuye, se produce el efecto de bloqueo o *locking* en los métodos numéricos. Uno de los métodos clásicos para evitar este bloqueo es usar técnicas de integración reducida como la presentada en [4], o también considerar formulaciones mixtas. En particular, nos centraremos en la formulación mixta unidimensional estudiada en [60] que introduce como variables auxiliares el esfuerzo de corte y el momento flector. Introducimos un operador solución cuyos valores propios son los recíprocos del cuadrado de las frecuencias de vibración escaladas con respecto al parámetro de espesor. Los elementos finitos que usaremos para aproximar el momento flector y el esfuerzo de corte serán funciones lineales a trozos y para la rotación y desplazamiento de la viga usaremos constantes a trozos. El operador solución resultará ser compacto en virtud de la inclusión compacta de $\mathrm{H}^1(\Omega)$ en $\mathrm{L}^2(\Omega)$. Sin embargo, usaremos la teoría de aproximación para operadores no compactos de [36, 37] para estudiar la aproximación entre los operadores solución continuo y discreto, aproximación de espacios propios y para demostrar estimaciones del error, donde las constantes que se obtienen son independientes del espesor. Por otro lado, estudiamos el problema de vibraciones para el caso en que el espesor tiende a cero, demostrando que la solución del problema de vibraciones de Timoshenko converge al problema de vibraciones de Euler-Bernoulli. Mostramos experimentos numéricos que confirman los resultados teóricos.

Los resultados obtenidos contenidos en este capítulo se pueden encontrar en el siguiente artículo:

- F. Lepe, D. Mora and R. Rodríguez: *Finite element analysis of a bending moment formulation for the vibration problem of a non-homogeneous Timoshenko beam*, Journal of Scientific Computing, 66 (2016) 825-848.

1.1.2 Vibración acústica entre fluidos disipativos.

La disipación acústica es un tema de gran relevancia en la ingeniería. En los últimos años se han desarrollado diversos trabajos en este ámbito como [12, 13, 14, 16, 17, 70, 76] en los que se consideran discretizaciones con elementos de Raviart-Thomas para los fluidos y funciones lineales a trozos y continuas en el sólido. Sin embargo, la disipación no es una propiedad considerada frecuentemente en los estudios de acústica. En [12] estudiaron un problema de acústica disipativa para un fluido dentro de una cavidad con paredes absorbentes. Conocer los modos de vibración acústica de un sistema disipativo como éste permite, por ejemplo, desarrollar técnicas de reducción de ruido. En el capítulo 3 de esta tesis consideraremos el problema de aproximación numérica de los modos complejos de vibración de un sistema de dos fluidos disipativos dentro de una cavidad rígida. La presencia de disipación, hace que nuestro sistema se escriba como un problema de valores propios cuadrático, transformado en lineal considerando un sistema ampliado obtenido por la introducción de una incógnita auxiliar apropiada. Asumiremos que los fluidos del sistema son irrotacionales, como fue propuesto en [18] para un modelo disipativo similar. Para analizar el problema espectral, usaremos la teoría desarrollada en [55]. El método de elementos finitos que proponemos se basa en elementos de Raviart-Thomas de priomer orden y adaptamos la teoría de operadoes no compactos desarrollada en [36, 37] para demostrar que el espectro es aproximado correctamente y para obtener estimaciones del error. Finalmente mostramos experimentos numéricos para una geometría bidimensional que permite obtener una solución analítica y comparar los resultados obtenidos por el método de elementos finitos propuesto.

Los resultados obtenidos de capítulo se pueden encontrar en el siguiente artículo:

- F. Lepe, S. Meddahi, D. Mora and R. Rodríguez: *Acoustic vibration problem for dissipative fluids*, Mathematics of Computation (en prensa).

1.1.3 Problema espectral de elasticidad.

El problema de elasticidad es un problema clásico en ingeniería, de gran interés para el desarrollo de distintos métodos numéricos, no solo para estudiar el comportamiento de un cierto sólido cuando una fuerza externa lo perturba, sino también en distintos problemas de interacción de una estructura elástica con un fluido, por ejemplo. En este contexto, las formulaciones mixtas han sido relevantes, ya que evitan el fenómeno de bloqueo para el caso de materiales casi incompresibles.

Existen diversos trabajos que estudian el problema espectral de elasticidad. Artículos recientes como [69] hacen estudio de una formulación dual mixta del problema espectral, donde introducen el tensor de Cauchy y la rotación como incógnitas, eliminando así el desplazamiento, que puede ser recuperado mediante un post proceso, e imponiendo la simetría débilmente, como un multiplicador de Lagrange. Dentro de los métodos numéricos existentes para resolver numéricamente este tipo de problemas, el método de Galerkin discontinuo basado en una penalización interior (IPDG) ha tomado relevancia. Trabajos como [1] para el problema espectral de Laplace dan el primer paso en la forma de analizar los métodos DG para problemas de valores propios, donde las principales herramientas del análisis se basan en una desigualdad de Poincaré y estimaciones a priori del error en normas dependientes de la malla, de modo que la convergencia espectral, la no polución del espectro y el orden de convergencia se pueden demostrar, adaptando los resultados de [36, 37]

Recientemente en [68] se ha considerado el problema de elasticidad lineal con régimen armónico, considerando una formulación dual-mixta como la estudiada en [69], proponiendo un método DG para aproximar sus soluciones, demostrando la convergencia del método, y verificando mediante ensayos numéricos la flexibilidad el método para distintos grados polinomiales en mallas uniformes.

En este trabajo consideraremos el problema espectral de elasticidad asociado a la formulación discreta estudiada en [68], para caracterizar el espectro del operador solución. Introducimos espacios de polinomios discontinuos, de modo de aproximar con polinomios de grado k el tensor de esfuerzo, y polinomios de grado $k-1$ las rotaciones, con $k \geq 1$. Analizaremos de qué modo afectan el grado polinomial, la malla y el parámetro de estabilización del método en la aparición de modos espurios. Mostramos experimentos numéricos para observar el desempeño del método.

Los resultados obtenidos en este capítulo se pueden encontrar en la siguiente pre-publicación:

- F. Lepe, S. Meddahi, D. Mora and R. Rodríguez: *Mixed Discontinuous Galerkin approximation of the elasticity eigenproblem*, Preprint DIM 2017, Universidad de Concepción, Concepción, 2017.

1.1.4 Problema disipativo de elastoacústica.

El problema de interacción entre un fluido y una estructura es uno de los fenómenos mas estudiados por los ingenieros, debido al gran número de aplicaciones que existen en la industria y en la construcción y diseño de puentes, barcos, motores, etc., donde conocer los modos de vibración elastoacústicos de esta interacción son muy importantes.

Desde el enfoque del análisis numérico, el problema de elastoacústica ha sido muy estudiado, donde los métodos numéricos son una herramienta relevante para la aproximación de los modos de vibración físicos de éstos problemas de interacción. Artículos como [12, 14, 16, 17, 18] han estudiado rigurosamente el problema de elastoacústica, tanto a nivel continuo, caracterizando el espectro del operador solución asociado al problema espectral y también a nivel discreto, donde la principal herramienta han sido los métodos de elementos finitos. Por ejemplo, para discretizar el fluido han sido implementados elementos de Raviart-Thomas y para el sólido funciones lineales a trozos y continuas, los cuales son convergentes en las normas correspondientes y además no introducen modos de vibración espurios. Sin embargo, la presencia de disipación, tanto en el fluido como la estructura, no ha sido considerada de forma recurrente al estudiar el problema de elastoacústica. Recientemente en [62] se ha estudiado un método de elementos finitos para aproximar los modos de vibración de un problema de interacción entre dos fluidos disipativos. La disipación lleva al estudio de un problema de valores propios cuadrático, el cual se linealiza introduciendo una incógnita auxiliar adecuada.

Nuestro objetivo es estudiar el problema elastoacústico considerando como punto de partida solo disipación en el fluido. Introducimos el problema modelo, del cual deducimos una formulación variacional para el problema espectral. Dado que el problema de valores propios es cuadrático, introducimos incógnitas adicionales en el fluido y en el sólido de modo de obtener un problema de valores propios lineal de doble tamaño. Demostramos que el problema continuo está bien planteado e introducimos el operador solución correspondiente, cuya caracterización espectral es estudiada rigurosamente. Introducimos un método de elementos finitos donde discretizamos el fluido con elementos de Raviart-Thomas de bajo orden y funciones lineales a trozos para la estructura. Esto nos conduce a un método no conforme, para el cual adaptamos la teoría de operadores no compactos de [36, 37] para estudiar la convergencia del especto y para obtener estimaciones de error. Mostramos algunos experimentos numéricos que muestran el buen comportamiento del método.

Estos resultados preliminares están contenidos en el siguiente trabajo en desarrollo

- ▶ F. Lepe, S. Meddahi, D. Mora and R. Rodríguez: *Quadratic eigenvalue problem for a dissipative fluid-structure system*, (En desarrollo).

Introduction

1.2 Vibrations and dissipative acoustics

The vibration phenomenon is an important subject of study for engineers. Many structures, from a guitar string to plates or beams, experiment vibrations due to external forces. Another example could be fluids with acoustic properties and naturally, the interaction between fluids and structures. This last phenomenon is called *elastoacustic*. The physical properties of fluids are relevant for the analysis of the vibration modes of a system. One of the most interesting properties is viscosity.

Viscosity produces a phenomenon known as dissipation. There exists two types of dissipation: the internal acoustic dissipation inside a cavity due to the presence of viscosity and internal heat conduction of a fluid and boundary by wall boundary layers of acoustical isolating material. Details of these definitions can be found in [76]. Through this dissertation, we only consider the first type of dissipation.

The phenomenons previously described are of vital importance to build bridges, buildings, cars, ships etc., since the stability of certain structures or their interaction with fluids involve specific requirements of design, economical costs or environmental aspects among others. For these reasons, numerical methods are a powerful tool to make decisions, since in practice it is not possible to compute exactly the solutions of some systems of partial differential equations (PDEs).

Eigenvalue problems, in the context of PDEs and particularly the numerical analysis of this type of equations, motivated mathematicians to formalize the study of the operators associated to this spectral problems for primal and mixed formulations. For a view of the state of art of these tools, see [20, 25].

For the numerical resolution of PDEs there exist several methods. In this dissertation we will implement two of them. On one hand, we have the classic finite element method (FEM) which has been used for elliptic problems; on the other we have the Discontinuous Galerkin method (DG) introduced in the seventies by engineers as a tool to solve numerically hyperbolic problems. In the ninetys, the DG method starts to be used for elliptic problems. This leads to the article by Arnold et. al. [5] which gives an appropriate mathematical framework for the application of the DG method to elliptic problems; see [38] and references therein for more details about the applications of the DG method.

Our main goal is to solve numerically an interaction problem between a dissipative fluid and a structure modelled with the classical elasticity equations. Articles like [12, 14, 16, 17, 70] study the elastoacustic problem and analyse numerical methods implementing finite elements, proving convergence of the method and error estimates. However, these elastoacustic models neglect the presence of viscosity.

A dissipative acoustic model that couples the Stokes equtions with the heat equation is the one studied in [18]. In this paper is assumed that for small displacements, the fluid is irrotational. This fact implies that, if $\boldsymbol{U}$ is the displacement of the fluid, then $\mathbf{curl}\,\dot{\boldsymbol{U}} = 0$, where the dot represents the firs order temporal derivative. This implies that, in the following identity $\Delta\dot{\boldsymbol{U}} = \nabla(\operatorname{div}\dot{\boldsymbol{U}}) - \mathbf{curl}(\mathbf{curl}\,\dot{\boldsymbol{U}})$ the rotational part vanishes. This assumption is useful for us, since the Stokes equations are simplified.

The elastoacustic model that we will analyze, consider the interaction between a dissipative irrotational fluid modelled with the Stokes equations and the classical linear elasticity equations. The model problem is the following:

$$\rho_f\ddot{\boldsymbol{U}} - 2\nu_f\Delta\dot{\boldsymbol{U}} + \nabla P = \mathbf{0} \text{ in } \Omega_f \times (0,T), \tag{1.2.18}$$

$$P + \rho_f c^2 \operatorname{div}\boldsymbol{U} = 0 \text{ in } \Omega_f \times (0,T), \tag{1.2.19}$$

$$\rho_s\ddot{\boldsymbol{W}} - \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{W})) = \mathbf{0} \text{ in } \Omega_s \times (0,T), \tag{1.2.20}$$

$$\boldsymbol{\sigma}(\boldsymbol{W}) - \lambda_S \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{W}))I - 2\mu_S\boldsymbol{\varepsilon}(\boldsymbol{W}) = 0 \text{ in } \Omega_s \times (0,T), \tag{1.2.21}$$

$$\boldsymbol{U}\cdot\boldsymbol{n} - \boldsymbol{W}\cdot\boldsymbol{n} = 0 \text{ in } \Gamma_{\mathrm{I}} \times [0,T], \tag{1.2.22}$$

$$\boldsymbol{\sigma}(\boldsymbol{W})\boldsymbol{n} - (P + 2\nu\operatorname{div}\boldsymbol{U})\boldsymbol{n} = 0 \text{ in } \Gamma_{\mathrm{I}} \times [0,T], \tag{1.2.23}$$

$$\boldsymbol{\sigma}(\boldsymbol{W}) = 0 \text{ in } \Gamma_N \times [0,T], \tag{1.2.24}$$

$$\boldsymbol{W} = 0 \text{ in } \Gamma_D \times [0,T], \tag{1.2.25}$$

where $\boldsymbol{U}$ is the displacement of the fluid, $\boldsymbol{W}$ is the displacement of the solid, P is the pressure of the fluid, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}(\cdot)$ is the linear strain tensor defined by $\boldsymbol{\varepsilon}(\boldsymbol{W}) := \frac{1}{2}[\nabla\boldsymbol{W} + (\nabla\boldsymbol{W})^{\mathtt{t}}]$, ρ_f and ρ_s are the densities of the fluid and the solid, respectively, c is the acoustic speed of the fluid, ν_f is the viscosity of the fluid, λ_s and μ_s are the Lammé coefficients associated to the material of the structure, Ω_f and Ω_s are the domains occupied by the fluid and the solid, respectively, Γ_{I} is the interface between the solid and the fluid, Γ_D and Γ_N are the Dirichlet and Neumann parts of the boundary of the solid, respectively, $\boldsymbol{n}$ is the outward unit vector to Γ_N.

Introducing the following spaces

$$\mathcal{H} := \mathrm{L}^2(\Omega_f)^n \times \mathrm{L}^2(\Omega_s)^2, \quad \mathcal{X} := \mathrm{H}(\mathrm{div};\Omega_f) \times \mathrm{H}^1_{\Gamma_D}(\Omega_s)^2,$$
$$\mathcal{V} := \{(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{X} \ : \ \boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{w}\cdot\boldsymbol{n}\},$$

where $\mathrm{H}^1_{\Gamma_D}(\Omega_s)^2$ is the subspace of $\mathrm{H}^1(\Omega_s)^2$ of functions that vanishes in Γ_D and assuming that the solutions are of the form $\boldsymbol{U}(\boldsymbol{x},t) = e^{\lambda t}\boldsymbol{u}(\boldsymbol{x},t)$ and $\boldsymbol{W}(\boldsymbol{x},t) = e^{\lambda t}\boldsymbol{w}(\boldsymbol{x},t)$, $\lambda \in \mathbb{C}$, a variational formulation of the dissipative elastoacustic problem written in the frequency domain and in terms of the displacements of the fluid and the solid reads as follows:

Problem 1.2.1 Find $\lambda \in \mathbb{C}$ and $(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{u},\boldsymbol{w}) \in \mathcal{V}$ such that

$$\lambda^2 \left(\int_{\Omega_f} \rho_f \boldsymbol{u}\cdot\overline{\boldsymbol{v}} + \int_{\Omega_s} \rho_s \boldsymbol{w}\cdot\overline{\boldsymbol{\tau}} \right) + 2\lambda \int_{\Omega_f} \nu \,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\overline{\boldsymbol{v}} + \int_{\Omega_f} \rho_f c^2 \,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = 0 \qquad \forall (\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{V}. \tag{1.2.26}$$

where the pressure is eliminated by using the compressibility condition $P + \rho_f c^2 \,\mathrm{div}\,\boldsymbol{U} = 0$.

The presence of viscosity in this model leads to new difficulties. For this reason, first we will solve some subproblems for acoustics fluids and vibration of structures, separately.

In particular, we develop and analyze numerical methods for two kind of spectral problems: the dissipative acoustic problem, where we study the approximation of the vibration modes of dissipative fluids, and the analysis of vibration problems for structures, particularly beams and the linear elasticity equations.

1.2.1 Dissipative acoustics

Several models for acoustic fluids do not incorporate viscosity in their model equations, which means that the dissipation does not take part of the physical phenomena in the analysis. Nevertheless, dissipation is present in the whole nature, and of course in engineering, and should be considered for the acoustics problem in the context that corresponds.

For example, noise is one of the most frequent sources of contamination for the population and it is very important to keep it under control. The acoustic dissipative analysis allows developing noise reduction techniques for cars or public transport vehicles and for people who work in the construction of roads, buildings, etc. and need adequate equipment for their activities.

If in problem (1.2.18)–(1.2.25) we consider $\nu_f = 0$ and we neglect the displacement of the structure, this leads to eliminate equations (1.2.20), (1.2.21), (1.2.24) and (1.2.25). Moreover, if we replace condition (1.2.23) by a damping condition we obtain the following problem

$$\rho_f \frac{\partial^2 \boldsymbol{U}}{\partial t^2} + \nabla P = \boldsymbol{0} \quad \text{in } \Omega, \tag{1.2.27}$$

$$P = -\rho_f c^2 \operatorname{div} \boldsymbol{U} \quad \text{in } \Omega, \tag{1.2.28}$$

$$P = \left(\alpha \boldsymbol{U} \cdot \boldsymbol{n} + \beta \frac{\partial \boldsymbol{U}}{\partial t} \cdot \boldsymbol{n} \right) \quad \text{on } \Gamma_A, \tag{1.2.29}$$

$$\boldsymbol{U} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_R. \tag{1.2.30}$$

The equations above corresponds to a dissipative problem of a fluid contained in a rigid cavity with absorbing walls, which are able to dissipate acoustic energy. This problems was studied in [13], where $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$ is a polygonal/polyhedral domain, $\boldsymbol{U}$ represents the fluid displacement, P the fluid pressure, α and β are coefficients associated to the impedance of the fluid, Γ_A is the part of the boundary associated to the absorbing wall and Γ_R represents the rigid part of the boundary.

If the damped solutions of the previous system are of the form $\boldsymbol{U}(\boldsymbol{x}, t) := e^{\lambda t}\boldsymbol{u}(\boldsymbol{x})$ and $P(\boldsymbol{x}, t) := e^{\lambda t} p(\boldsymbol{x})$, by substituting these solutions into the system (1.2.27)–(1.2.30) to write the problem in the frequency domain, we arrive at the following quadratic eigenvalue problem: Find $\lambda \in \mathbb{C}$, $\boldsymbol{u} : \Omega \to \mathbb{C}^n$ and $p : \Omega \to \mathbb{C}$, both non-zero functions, such that

$$\rho_f \lambda^2 \boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{in } \Omega, \tag{1.2.31}$$

$$p = -\rho_f c^2 \operatorname{div} \boldsymbol{u} \quad \text{in } \Omega, \tag{1.2.32}$$

$$p = \left(\alpha \boldsymbol{u} \cdot \boldsymbol{n} + \beta \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{n} \right) \quad \text{on } \Gamma_A, \tag{1.2.33}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_R, . \tag{1.2.34}$$

A variational formulation of this non-linear eigenvalue problem, which has been studied in [13], is written in terms of the displacement of the fluid (eliminating the pressure using the compressibility condition) and reads as follows: Find $\lambda \in \mathbb{C}$ and $\boldsymbol{0} \neq \boldsymbol{u} \in \mathrm{H}_0(\operatorname{div}, \Omega)$ such that

$$\lambda^2 \int_O \rho_f \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \lambda \int_{\gamma_A} \beta \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \boldsymbol{n} + \int_{\Gamma_A} \alpha \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \overline{\boldsymbol{v}} + \int_\Omega \rho_f c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \quad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}; \Omega). \tag{1.2.35}$$

Since the problem is quadratic in λ, the analysis is not direct, and we need to linearize equation (1.2.35). With this purpose, we introduce the auxiliary unknown $\widehat{\boldsymbol{u}} := \lambda \boldsymbol{u}$ in order to obtain a linear double size problem which reads as follows: Find $\lambda \in \mathbb{C}$ and $(\boldsymbol{0}, \boldsymbol{0}) \neq (\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathrm{H}_0(\operatorname{div}; \Omega) \times \mathrm{L}^2(\Omega)^d$ such that

$$\begin{cases} \displaystyle\int_\Omega \rho_f c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Gamma_A} \alpha \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \boldsymbol{n} = \lambda \left(-\int_{\Gamma_A} \beta \boldsymbol{u} \cdot \boldsymbol{n} \overline{\boldsymbol{v}} \cdot \boldsymbol{n} - \int_\Omega \rho_f \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} \right) & \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}; \Omega), \\ \displaystyle\int_\Omega \rho_f \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}} = \lambda \int_\Omega \rho_f \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}} \qquad \forall \widehat{\boldsymbol{v}} \in \mathrm{L}^2(\Omega)^d. \end{cases}$$

Elastic vibrations

The vibration problems considered in this dissertation are focused in two type of problems. On one hand, we consider the classic linear elasticity equations, and on the other we study the vibration problem for thin structures, in particular, a beam modelled with the well known Timoshenko equations. The Timoshenko beam model (TBM) was introduced in the beginning of the XX century, and the main differences with the Euler-Bernoulli model are, on one hand, the incorporation of the rotation of the bending effects, this means that the plane sections no longer remain plane and perpendicular to the neutral axis during bending as the Euler-Bernoulli model describes. On the other hand, the TBM considers the thickness of the beam, and how the other dimension of the structure depends on the measure of the thickness. With this fact, the TBM becomes a parameter dependent problem. From the numerical analysis point of view, the thickness parameter of the TBM carry the numerical models to the so-called *locking* effect, where the reduced integration techniques avoid this difficulty. Nevertheless, in 1981, Arnold proves in [4] that a mixed-formulation of the TBM introducing the shear stress as an additional unknown, also avoid the numerical locking.

The linear elasticity equations describe the relation between the stress and deformation of isotropic solids. The constitutive equation that relates the displacement field and the stress is given by

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a certain bounded domain with $d = 2, 3$. Moreover, the elasticity operator $\mathcal{C}^{-1}$ is given by

$$\mathcal{C}\boldsymbol{\tau} := \lambda_s(\operatorname{tr}\boldsymbol{\tau})\boldsymbol{I} + 2\mu_s\boldsymbol{\tau},$$

where λ_s and μ_s are the Lamé coefficients associated to the material.

If in Problem (1.2.18)–(1.2.25) once again we consider $\nu_f = 0$ but now eliminating the displacement of the fluid $\boldsymbol{U}$ and considering that $\boldsymbol{W}$ is of the form $\boldsymbol{W}(\boldsymbol{x}, t) = e^{\lambda t}\boldsymbol{w}(\boldsymbol{x}, t)$, we obtain the following spectral elasticity problem:

$$\begin{aligned} \boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}) \quad & \text{in } \Omega, \\ \mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho_s\boldsymbol{w} = \mathbf{0} \quad & \text{in } \Omega, \\ \boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} \quad & \text{on } \Gamma_F, \\ \boldsymbol{w} = \mathbf{0} \quad & \text{on } \Gamma_R, \end{aligned}$$

where $\sqrt{\lambda} = \omega \geq 0$ are the frequencies, Γ_R the fixed part of the boundary and Γ_F the free part. Introducing the rotation $\boldsymbol{r} := \frac{1}{2}[\nabla\boldsymbol{w} - (\nabla\boldsymbol{w})^{\mathtt{t}}]$ and eliminating the displacement with the condition $\mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho_s\boldsymbol{w} = \mathbf{0}$, we obtain the following dual-mixed formulation, analyzed rigorously in [69]: Find $\lambda \in \mathbb{R}$ and $\mathbf{0} \neq (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ such that

$$\int_\Omega \rho_s^{-1}\,\mathbf{div}\cdot\mathbf{div}\,\boldsymbol{\tau} = \lambda\left(\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{r}\right) \qquad \forall\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$

$$\lambda\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} = 0 \qquad \forall\boldsymbol{s} \in \boldsymbol{\mathcal{Q}},$$

where

$$\mathcal{W} := \{\boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}; \Omega) : \quad \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma_F\} \quad \text{and} \quad \mathcal{Q} := \{\boldsymbol{s} \in \mathrm{L}^2(\Omega)^{n\times n} : \quad \boldsymbol{s}^{\mathrm{t}} = -\boldsymbol{s}\}.$$

In what follows, we will describe the contributions of the present dissertation.

1.2.2 Vibration of a Timoshenko beam.

In structural engineering, for the design and construction of bridges, aircrafts, cars, etc., it is frequent to consider thin structures as beams or plates with continuous or discontinuous geometries. It is also very typical to consider structures with discontinuous physical properties, for instance, a beam made of steel and iron along the axis for which some physical quantities like Young's modulus, Poisson ratio or shear stress among others, would not be continuous along the axis. Taking all that in consideration , it is important to know the vibration modes of these structures In particular, we will focus on the study on beams.

Timoshenko beam theory is one of the most popular models to study the deformation of a moderately thin elastic beam. When we reduce the thickness parameter, it produces the so called *locking* phenomenon is obtained, when we use standard finite elements. One of the technics to avoid this problem, is using reduced integration, studied by Arnold (see [4]), or considering mixed formulations. In this work, we will consider a unidimesional mixed formulation, as the one studied in [60], where the shear stress and the bending moment are introduced as additional unknowns. The eigenvalues of the solution operator correspond to the reciprocals of the square of the scaled vibration frequencies with respect to the thickness parameter. Moreover, the solution operator is compact due the compact embedding of $\mathrm{H}^1(\Omega)$ in $\mathrm{L}^2(\Omega)$. Piecewise linear elements are implemented to approximate both the bending moments and the shear stress, and piecewise constants for the rotation and the displacement. Despite the compactness of the solution operator, the approximation properties and error estimates are analyzed by using the non-compact theory on [36] and [37] respectively, where the constants of the estimates are independent of the thickness. We also study the vibration problem when the thickness t vanishes, proving that the solution of the Timoshenko vibration problem converge to the Euler-Bernoulli solution. Some numerical experiments will be presented to reinforce the theoretical results presented.

The results of this chapter are contained in the following published article

- F. Lepe, D. Mora, R. Rodríguez: *Finite element analysis of a bending moment formulation for the vibration problem of a non-homogeneous Timoshenko beam*, Journal of Scientific Computing, 66 (2016) 825-848.

1.2.3 Acoustic vibration for dissipative fluids.

The acoustic vibration problem is a classical engineering problem. Several articles studying numerical methods to approximate the vibration modes of acoustic inviscid fluids contained in rigid containers and their interaction with solid structures are, for instance, [12, 13, 14, 16, 17, 70, 76] of particular interest is [13] where the authors study the problem of an acoustic inviscid fluid contained in a cavity with absorbing walls, which dissipates acoustic energy. This problem leads to a quadratic eigenvalue problem, numerically analyzed with Raviart-Thomas elements. However, the previously mentioned articles consider in all cases inviscid fluid in the models.

In Chapter 3 we study the acoustic vibration problem of the interaction between two viscous fluids contained in a rigid cavity, which is the first step to analyze more complex systems, as the viscous fluid-structure problems. The main difficulty here is to formulate a well posed discrete problem. For this reason, we consider in our model that the fluid is irrotational, as in [18]. To analyze the spectral problem we use the theory of [55]. We characterize rigorously the spectrum of the solution operator according to Weyl's theorem (see [83]). The numerical method consider Raviart-Thomas elements for the discretization of the domain for both fluids. For the approximation properties we adapt the theory of [36] and for the error estimates the results of [37] as it was studied in [13]. Numerical results to show the performance of the method are presented. The results of this chapter can be seen in the following pre-print:

- F. Lepe, S. Meddahi, D. Mora, R. Rodríguez: *Acoustic vibration problem for dissipative fluids*, Mathematics of Computation (in press).

1.2.4 Elasticity spectral problem.

The elasticity problem is one of the classic problems in engineering, and has been of great interest for the development of different numerical methods to study the behaviour of certain solids when external forces are applied, and also when on interaction problems between fluids an elastic structures. In this context, mixed formulations have become really important, since they avoid the locking phenomenon for nearly incompressible materials.

Many papers in the literature deal with the elasticity eigenvalue problem. Recent papers, like [69], consider a dual-mixed formulation where the unknowns are the Cauchy stress tensor and the rotation tensor, thus eliminating the displacement which can be recovered through a post-process. Moreover, in this formulation the symmetry is imposed weakly as a Lagrange multiplier.

The interior penalty method for discontinuous Galerkin methods (IPDG) has been extensively used in the past years to solve spectral problem, due to the flexibility on the choice of the polynomial degrees for the approximation. Articles like [1] deal with the Laplace eigenvalue problem, and give the first steps for the approximation of eigenvalues and eigenfunctions with the DG method. The main tools for the analysis are a Poincaré inequality for the broken H^1 space and a priori error estimates with mesh dependent norms. Moreover, the spectral convergence, the non-pollution of the spectrum and the convergence orders can be proven by adapting the results of [36, 37].

Recently, in [68] a DG method for the elasticity equation for the harmonic regime has been analyzed, considering a dual-mixed formulation, like the studied in [69]. Error estimates are obtained for the method, and different numerical experiments are reported to assess the flexibility of the method for different polynomial degrees.

Our contribution is the application of the DG method for the spectral elasticity problem, considering the dual-mixed formulation studied in [69] and a IPDG method like the one used in [68]. We introduce the discontinuous polynomial spaces in order to approximate the Cauchy tensor with polynomials of degree k, and the rotation with polynomials of degree $k-1$, where $k \geq 1$. In order to obtain spectral correctness, approximation results and error estimates, we adapt the theory of [36, 37], proving that the constants are independent of the size of the mesh.

The results of this chapter are available in the following pre-print:

- F. Lepe, S. Meddahi, D. Mora and R. Rodríguez: *Mixed Discontinuous Galerkin approximation of the elasticity eigenproblem*, Preprint DIM 2017, Universidad de Concepción, Concepción, 2017.

1.2.5 Dissipative elastoacustic problem.

The interaction problem between fluids and structures is one of the most studied subject for engineers, due the several applications in industry, construction and design of bridges, ships, cars, motors, etc., where knowing the vibration modes of this type of interaction is very important.

The elastoacustic problem has been plenty studied from the numerical analysis point of view, since numerical methods allow to approximate the vibration modes of these interaction problem. Articles like [12, 14, 16, 17, 18] studied rigorously the elastoacustic problem in the continuous level, proving the well posedness of the problem and characterizing the spectrum of the corresponding solution operator. For the discrete problem, the finite element method was the main tool, implementing lowest order Raviart-Thomas element in the fluid, and continuous piecewise linear functions in the solid, proving convergence, error estimates and that the analyzed methods does not introduce spurious modes. Nevertheless, the previous articles neglect the presence of dissipation, both the in fluid and the solid for the analysis of the elastoacustic problem.

Recently in [62] a finite element method was proposed to approximate the vibration modes of the interaction problem between two dissipative fluids. The presence of viscosity in the fluids leads to a quadratic eigenvalue problem, which is linearised by the introduction of an additional unknown, obtaining a double-size linear eigenvalue problem.

Our goal is to study the elastoacustic problem incorporating the dissipation in the model. For instance, we will consider dissipation only in the fluid. We introduce the model problem, deducing a variational formulation for the spectral problem. Since the problem is quadratic, we introduce additional unknowns for the fluid and the solid, obtaining a double-size linear eigenvalue problem. We prove that the continuous problem is well posed, and we introduce the corresponding solution operator. We give a rigorous characterization of the spectrum of the solution operator. We introduce a finite element method based in the implementation of Raviart-Thomas elements in the fluid and piecewise linear functions in the solid, leading to a non-conforming method. We adapt the non-compact spectral theory of [36] to prove convergence. Moreover, we prove that the method does not introduce spurious modes and error estimates for the eigenfunctions and eigenvalues by adapting the theory of [37].

These preliminary results are contained in the following ongoing paper

- F. Lepe, S. Meddahi, D. Mora and R. Rodríguez: *Quadratic eigenvalue problem for a dissipative fluid-structure system*, (In process).

Chapter 2

Finite element analysis of a bending moment formulation for the vibration problem of a non-homogeneous Timoshenko beam

2.1 Introduction

This paper deals with the analysis of a finite element method to compute the vibration modes of an elastic non-homogeneous beam modeled by Timoshenko equations. Structural components with continuous and discontinuous variations of the geometry and of the physical parameters are common in buildings and bridges as well as in aircraft, cars, ships, etc. For that reason, it is important to know the vibration frequencies and modes of this kind of structures. This problem can be formulated as a spectral problem whose eigenvalues and eigenfunctions are related with the vibration frequencies and modes, respectively.

The Timoshenko theory to date is one of the most used models to approximate the deformation of a thin or moderately thick elastic beam [10, 26, 33, 44, 49, 75, 89]). It is well understood that standard finite elements applied to this model lead to wrong results when the thickness of the beam is small due to the so called *locking* phenomenon. To avoid locking, the most used techniques since long ago are based on reduced integration or mixed formulations (see [4]).

In this paper, we present a rigorous analysis of a low-order finite element method to compute the vibration frequencies and modes of a non-homogeneous Timoshenko beam, by means of a mixed bending moment formulation. A similar method was recently introduced and analyzed for load problems in [60].

One advantage of such a formulation is that the bending moment and the shear stress are computed directly and not by means of a post-process, which might produce loss of accuracy. Moreover, the fact that these two quantities appear explicitly in the formulation could be useful to apply it to coupled problems in which the coupling involve these quantities. Another motivation for considering this one-dimensional problem is that it constitutes a stepping stone towards the more challenging goal of devising finite element spectral approximations for Reissner–Mindlin plates based on bending moments formulations. Let us remark that this kind of formulations have been recently proposed and analyzed in different frameworks for instance in the following references [2, 9, 11, 30, 32].

Numerical analysis of eigenvalue problems arising from the computation of the vibration modes for thin structures are not too many; among them we mention [39, 40, 41, 64, 65], where MITC-like methods for computing the vibration and buckling modes of beams and plates were analyzed. One reason for this is that the extension of mathematical results from load to vibration problems is not quite straightforward for mixed methods. In fact, Boffi et al. [23, 24] showed that eigenvalue problems for mixed formulations show peculiar features that make them substantially different from the same methods applied to the corresponding source problems. In particular, they showed that the standard inf-sup and ellipticity in the kernel conditions, which ensure convergence for the mixed formulation of source problems, are not enough to attain the same goal in the corresponding eigenvalue problem. Among the existing techniques to solve the vibration problem of Timoshenko beams, we can mention [50] where a mixed formulation in terms of displacement, rotation and shear stress has been proposed and analyzed for Timoshenko rods (which are of course applicable to Timoshenko beams).

In this paper, we consider the vibration problem for an elastic beam. We follow the approach proposed in [60] for the load problem. We introduce the bending moment together with the shear stress as new unknowns in the model (we note that the former usually represents a quantity of major interest in engineering applications), which together with the rotation and the transverse displacement lead us to a mixed variational formulation. Then, we introduce a solution operator whose eigenvalues are the reciprocals of the scaled squares of the vibration frequencies of the beam. For the numerical approximation, we use piecewise linear and continuous finite elements for the bending moments and shear stress and piecewise constants for the transverse displacement and the rotations. To study the convergence of the proposed method and obtain error estimates, we adapt the classical theory developed for non-compact operators in [36, 37]. We obtain optimal order error estimates in terms of the mesh size h for the approximation of the vibration modes and a double order for the vibration frequencies. These estimates are fully independent of the beam thickness, which allows us to conclude that the method is locking-free. Moreover, it is shown that the corresponding limit discrete problem that results from taking the thickness parameter $t = 0$ is well posed and that its solutions converge to those of the Euler-Bernoulli beam vibration problem with optimal order in terms of h, too.

Since we have included as additional variables the bending moment and the shear stress, one could think at first sight that the resulting method will be significantly more expensive than the classical ones, which are based only in displacement and rotation variables. However, we show that these last two variables can be eliminated in the resulting discrete problem without additional cost, which leads to an eigenvalue problem of the same size and sparseness as those of the classical methods.

The outline of the paper is as follows. In Section 2.2, we recall the vibration problem for a Timoshenko beam. In Section 2.3 we develop the mathematical analysis of the vibration problem. With this aim, we introduce a linear operator whose spectrum is related with the solution of the vibration problem. The resulting spectral problem is shown to be well posed. Its eigenvalues and eigenfunctions are proved to converge to the corresponding ones of the limit problem as the thickness of the plate goes to zero, which corresponds to a an Euler-Bernoulli beam model. We also prove in this section a regularity result for the eigenfunctions. In Section 2.4 we introduce the finite element discretization of the spectral problem and the discrete solution operator and prove some auxiliary results. In Section 2.5 we prove that the proposed numerical scheme provides a correct spectral approximation. We also establish error estimates for the eigenvalues and eigenfunctions. In Section 2.6 we show how the analysis can be adapted to the Euler-Bernoulli beam vibration problem. Finally, in Section 2.7, we discuss some implementation details and present a set of numerical experiments to assess the performance of the method, in order to confirm that the experimental rates of convergence are in accordance with the theory and to show that the method is completely locking-free. We also show in this section how the displacement and the rotation variables can be eliminated from the discrete eigenvalue problem, reducing its dimension to one half without affecting the sparseness, symmetry and positive definiteness of the matrices.

We use standard notations for Sobolev spaces, norms and seminorms. For $l \geq 0$ and I an open interval, $\|\cdot\|_{l,\mathrm{I}}$ stands for the norm of the Hilbertian Sobolev space $\mathrm{H}^l(\mathrm{I})$, with the convention $\mathrm{H}^0(\mathrm{I}) := \mathrm{L}^2(\mathrm{I})$. Moreover, $\mathcal{D}(\mathrm{I})$ denotes the space of infinitely differentiable functions with compact support contained in I. Additionally, we will denote with C a generic positive constant, possibly different at different occurrences, but always independent of the beam thickness t and the mesh parameter h which will be introduced in the next sections.

Finally, given a linear bounded operator $T : X \to X$, defined on a Hilbert space X, we denote its spectrum by $\mathrm{sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$ and by $\boldsymbol{\rho}(T) := \mathbb{C} \setminus \mathrm{sp}(T)$ the resolvent set of T. Moreover, for any $z \in \boldsymbol{\rho}(T)$, $R_z(T) := (zI - T)^{-1} : X \to X$ denotes the resolvent operator of T corresponding to z.

2.2 Timoshenko beam model

Let us consider an elastic beam which satisfies the Timoshenko hypotheses for the admissible displacements. We assume that the geometry and the physical parameters of the beam may change along the axial direction. The deformation of the beam is described in terms of the transverse displacement w and the rotation of the transverse fibers β.

The equations for the vibration problem of a clamped Timoshenko beam reads as follows (see [82, 86, 87]):

Find $\omega > 0$ *and* $0 \neq (\beta, w) \in \mathrm{H}_0^1(\mathrm{I}) \times \mathrm{H}_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} E\mathbb{I}\beta'\eta' + \int_{\mathrm{I}} GAk_c(\beta - w')(\eta - v') = \omega^2 \left(\int_{\mathrm{I}} \rho A w v + \int_{\mathrm{I}} \rho \mathbb{I} \beta \eta \right) \qquad \forall (\eta, v) \in \mathrm{H}_0^1(\mathrm{I}) \times \mathrm{H}_0^1(\mathrm{I}), \tag{2.2.1}$$

where $\mathrm{I} := (0, L)$, L being the length of the beam, ω is the angular vibration frequency, E is the Young modulus, $\mathbb{I}$ the moment of inertia of the cross-section, A the area of the cross-section, ρ the mass density, $G := E/(2(1 + \nu))$ the shear modulus, with ν being the Poisson ratio, and

k_c a correction factor. We consider that E, $\mathbb{I}$, A, ρ, k_c and ν are piecewise smooth functions of the axial coordinate $x \in \mathrm{I}$, the most usual case being when all those coefficients are piecewise constant. Moreover, primes denote derivatives with respect to the axial coordinate x.

It is well known that standard finite element procedures, used in formulations such as (2.2.1) for very thin structures, are subject to numerical locking, a phenomenon induced by the difference of magnitude between the coefficients in front of the different terms (see [4]). The appropriate framework for analyzing this is obtained by rescaling formulation (2.2.1) so as to identify a family of problems with a well-posed limit as the thickness becomes infinitely small. With this aim, we introduce the following non-dimensional parameter, characteristic of the thickness of the beam:

$$t^2 := \frac{1}{L} \int_{\mathrm{I}} \frac{\mathbb{I}}{AL^2}\, dx, \tag{2.2.2}$$

which we assume may take values in the range $(0, t_{\max}]$.

We define

$$\lambda := \frac{\omega^2}{t^2}, \quad \hat{\mathbb{I}} := \frac{\mathbb{I}}{t^3}, \quad \text{and} \quad \hat{A} := \frac{A}{t},$$

and assume that $\hat{\mathbb{I}}$ and $\hat{A}$ are bounded above and below far from zero by constants independent of the parameter t. Let us remark that, for instance, for a beam of rectangular section $b \times d$ with b being a fixed length and d the thickness of the beam, these values are constant and independent of d: $\hat{A} = 2\sqrt{3}bL$ and $\hat{\mathbb{I}} = 2\sqrt{3}bL^3$.

We also define

$$\mathrm{E} := E\hat{\mathbb{I}}, \qquad \kappa := G\hat{A}k_c, \qquad \mathrm{J} := \rho\hat{\mathbb{I}} \quad \text{and} \quad \mathrm{P} := \rho\hat{A},$$

so that provided the physical coefficients E, ν and ρ are bounded above and below far from zero, we immediately obtain that there exist strictly positive constants $\overline{\mathrm{E}}, \underline{\mathrm{E}}, \overline{\kappa}, \underline{\kappa}, \overline{\mathrm{P}}, \underline{\mathrm{P}}, \overline{\mathrm{J}}$ and $\underline{\mathrm{J}}$ independent of t such that

$$\begin{cases} \overline{\mathrm{E}} \geq \mathrm{E} \geq \underline{\mathrm{E}} > 0 & \forall x \in \mathrm{I}, \\ \overline{\kappa} \geq \kappa \geq \underline{\kappa} > 0 & \forall x \in \mathrm{I}, \\ \overline{\mathrm{P}} \geq \mathrm{P} \geq \underline{\mathrm{P}} > 0 & \forall x \in \mathrm{I}, \\ \overline{\mathrm{J}} \geq \mathrm{J} \geq \underline{\mathrm{J}} > 0 & \forall x \in \mathrm{I}. \end{cases} \tag{2.2.3}$$

Then, problem (2.2.1) can be equivalently written as follows:

Find $\lambda > 0$ *and* $0 \neq (\beta, w) \in \mathrm{H}_0^1(\mathrm{I}) \times \mathrm{H}_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} \mathrm{E}\beta'\eta' + \frac{1}{t^2}\int_{\mathrm{I}} \kappa(\beta - w')(\eta - v') = \lambda\left(\int_{\mathrm{I}} \mathrm{P}wv + t^2\int_{\mathrm{I}} \mathrm{J}\beta\eta\right) \qquad \forall (\eta, v) \in \mathrm{H}_0^1(\mathrm{I}) \times \mathrm{H}_0^1(\mathrm{I}). \tag{2.2.4}$$

It is easy to check that, as a consequence of (2.2.3), for each $t > 0$, the bilinear form on the left hand side of (2.2.4) is elliptic with an ellipticity constant independent of t.

Furthermore, because of the assumption on the physical and geometrical parameters, we have that E, κ, P and J are piecewise smooth. More precisely, we assume that there exists a partition $0 = s_0 < \cdots < s_n = L$ of the interval I, with $s_1, \ldots, s_{n-1}$ being the points of discontinuity of E, κ, P or J, such that if we denote by $S_i := (s_{i-1}, s_i)$, then, $\mathrm{E}_i := \mathrm{E}|_{S_i} \in W^{1,\infty}(S_i)$, $\kappa_i := \kappa|_{S_i} \in W^{1,\infty}(S_i)$, $\mathrm{P}_i := \mathrm{P}|_{S_i} \in W^{1,\infty}(S_i)$ and $\mathrm{J}_i := \mathrm{J}|_{S_i} \in W^{1,\infty}(S_i)$, $i = 1, \ldots, n$.

In this paper we will consider a bending moment formulation of the spectral problem (2.2.4). With this end, we introduce the scaled bending moment $\sigma := \mathrm{E}\beta'$ and shear stress $\gamma := t^{-2}\kappa(\beta - w')$ as new unknowns in the model and test (2.2.4) with $\eta, v \in \mathcal{D}(\mathrm{I})$ to obtain that $-\sigma' + \gamma = \lambda t^2 \mathrm{J}\beta$ and $\gamma' = \lambda \mathrm{P} w$.

Thus, problem (2.2.4) can be equivalently written as follows:

$$\begin{cases} \sigma = \mathrm{E}\beta' & \text{in} \quad \mathrm{I}, \\ -\sigma' + \gamma = \lambda t^2 \mathrm{J}\beta & \text{in} \quad \mathrm{I}, \\ \gamma = t^{-2}\kappa(\beta - w') & \text{in} \quad \mathrm{I}, \\ \gamma' = \lambda \mathrm{P} w & \text{in} \quad \mathrm{I}, \\ w(0) = \beta(0) = w(L) = \beta(L) = 0. \end{cases} \tag{2.2.5}$$

We introduce the following spaces that will be used in the sequel:

$$\mathbb{H} := \mathrm{H}^1(\mathrm{I}) \times \mathrm{H}^1(\mathrm{I}) \quad \text{and} \quad \mathbb{Q} := \mathrm{L}^2(\mathrm{I}) \times \mathrm{L}^2(\mathrm{I}).$$

We endow each space as well as $\mathbb{H} \times \mathbb{Q}$ with the corresponding product norm.

Testing the equations in (2.2.5) with adequate functions and integrating by parts, we obtain the following variational formulation of this problem:

Find $\lambda > 0$ and $0 \neq ((\sigma, \gamma), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ such that

$$\int_{\mathrm{I}} \frac{\sigma\tau}{\mathrm{E}} + t^2 \int_{\mathrm{I}} \frac{\gamma\xi}{\kappa} + \int_{\mathrm{I}} \beta(\tau' - \xi) - \int_{\mathrm{I}} w\xi' = 0 \quad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.2.6}$$

$$\int_{\mathrm{I}} \eta(\sigma' - \gamma) - \int_{\mathrm{I}} v\gamma' = -\lambda \left(t^2 \int_{\mathrm{I}} \mathrm{J}\beta\eta + \int_{\mathrm{I}} \mathrm{P} w v \right) \quad \forall (\eta, v) \in \mathbb{Q}. \tag{2.2.7}$$

We write this mixed problem in a more compact form as follows:

Find $\lambda > 0$ and $0 \neq ((\sigma, \gamma), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ such that

$$a((\sigma, \gamma), (\tau, \xi)) + b((\tau, \xi), (\beta, w)) = 0 \qquad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.2.8}$$

$$b((\sigma, \gamma), (\eta, v)) = -\lambda r((\beta, w), (\eta, v)) \qquad \forall (\eta, v) \in \mathbb{Q}, \tag{2.2.9}$$

where the bilinear forms $a : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, $b : \mathbb{H} \times \mathbb{Q} \to \mathbb{R}$ and $r : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ are defined by

$$a((\sigma, \gamma), (\tau, \xi)) := \int_{\mathrm{I}} \frac{\sigma\tau}{\mathrm{E}} + t^2 \int_{\mathrm{I}} \frac{\gamma\xi}{\kappa}, \tag{2.2.10}$$

$$b((\tau, \xi), (\eta, v)) := \int_{\mathrm{I}} \eta(\tau' - \xi) - \int_{\mathrm{I}} v\xi', \tag{2.2.11}$$

and

$$r((\beta, w), (\eta, v)) := \left(t^2 \int_{\mathrm{I}} \mathrm{J}\beta\eta + \int_{\mathrm{I}} \mathrm{P} w v \right), \tag{2.2.12}$$

for all $(\sigma, \gamma), (\tau, \xi) \in \mathbb{H}$ and $(\beta, w), (\eta, v) \in \mathbb{Q}$.

It is easy to check that the so called *continuous kernel*

$$\mathbb{K} := \{(\tau,\xi) \in \mathbb{H} : \ b((\tau,\xi),(\eta,v)) = 0 \quad \forall(\eta,v) \in \mathbb{Q}\},$$

is given in this case by

$$\mathbb{K} = \left\{(\tau,\tau') : \tau \in \mathbb{P}_1(\mathrm{I})\right\}.$$

The following lemmas, which have been proved in [60, Lemmas 2.1 and 2.2] show that the *ellipticity in the kernel* and *inf-sup* classical conditions of mixed problems holds true for (2.2.8)–(2.2.9).

Lemma 2.2.1 *There exists* $\alpha > 0$ *independent of* t *such that*

$$a((\tau,\xi),(\tau,\xi)) \geq \alpha \|(\tau,\xi)\|_{\mathbb{H}}^2 \qquad \forall(\tau,\xi) \in \mathbb{K}.$$

Lemma 2.2.2 *There exists* $C > 0$ *independent of* t *such that*

$$\sup_{0\neq(\tau,\xi)\in\mathbb{H}} \frac{b((\tau,\xi),(\eta,v))}{\|(\tau,\xi)\|_{\mathbb{H}}} \geq C\|(\eta,v)\|_{\mathbb{Q}} \qquad \forall(\eta,v) \in \mathbb{Q}.$$

Remark 2.2.1 *We note that the eigenvalues of problem* (2.2.8)–(2.2.9) *are strictly positive. Indeed, it is easy to check that*

$$\lambda = \frac{a((\sigma,\gamma),(\sigma,\gamma))}{r((\beta,w),(\beta,w))} \geq 0;$$

moreover $\lambda = 0$ *implies* $(\sigma,\gamma) = 0$*, so that from* (2.2.8) *and Lemma 2.2.2, we have that* $(\beta,w) = 0$.

The goal of this paper is to propose and analyze a finite element method to solve the spectral problem (2.2.8)–(2.2.9) and to obtain accurate approximations of the eigenvalues λ (from which we obtain the angular vibration frequencies ω of the beam) and the associated eigenfunctions.

2.3 Analysis of the spectral problem

Before introducing the numerical method, we define the linear operator corresponding to the source problem associated with the spectral problem (2.2.8)–(2.2.9) and prove some properties that will be useful for the subsequent convergence analysis:

Given $(g,f) \in \mathbb{Q}$*, find* $((\hat{\sigma},\hat{\gamma}),(\hat{\beta},\hat{w})) \in \mathbb{H}\times\mathbb{Q}$ *such that*

$$a((\hat{\sigma},\hat{\gamma}),(\tau,\xi)) + b((\tau,\xi),(\hat{\beta},\hat{w})) = 0 \qquad \forall(\tau,\xi) \in \mathbb{H}, \tag{2.3.13}$$

$$b((\hat{\sigma},\hat{\gamma}),(\eta,v)) = -r((g,f),(\eta,v)) \qquad \forall(\eta,v) \in \mathbb{Q}. \tag{2.3.14}$$

As a consequence of Lemmas 2.2.1 and 2.2.2, this problem is well posed (see, for instance, [46, Section II.1.1]) and there exists a constant $C > 0$, independent of t, such that

$$\|\hat{w}\|_{0,\mathrm{I}} + \|\hat{\beta}\|_{0,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \|\hat{\gamma}\|_{1,\mathrm{I}} \leq C(t^2\|g\|_{0,\mathrm{I}} + \|f\|_{0,\mathrm{I}}) \leq C\|(g,f)\|_{\mathbb{Q}}.$$

Thus, we are able to introduce the following bounded linear operator, which is called the *solution operator*:

$$\begin{aligned} T_t : \mathbb{Q} &\to \mathbb{Q}, \\ (g, f) &\mapsto (\hat{\beta}, \hat{w}). \end{aligned}$$

It is easy to check that $(\mu, (\beta, w))$, with $\mu \neq 0$, is an eigenpair of T_t (i.e., $T_t(\beta, w) = \mu(\beta, w)$) if and only if there exist $(\sigma, \gamma) \in \mathbb{H}$ such that, for $\lambda = 1/\mu$, $(\lambda, (\sigma, \gamma), (\beta, w))$ is a solution of problem (2.2.8)–(2.2.9). We recall that these eigenvalues are strictly positive (cf. Remark 2.2.1). Our aim is to approximate the smallest eigenvalues of problem (2.2.8)–(2.2.9), which correspond to the largest eigenvalues of the operator T_t.

This operator is self-adjoint with respect to the inner product $r(\cdot, \cdot)$ in $\mathbb{Q}$. In fact, given $(g, f), (\tilde{g}, \tilde{f}) \in \mathbb{Q}$, let $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})), ((\tilde{\sigma}, \tilde{\gamma}), (\tilde{\beta}, \tilde{w})) \in \mathbb{H} \times \mathbb{Q}$ be the solutions to problem (2.3.13)–(2.3.14), with right hand side (g, f) and $(\tilde{g}, \tilde{f})$, respectively, so that $T_t(g, f) = (\hat{\beta}, \hat{w})$ and $T_t(\tilde{g}, \tilde{f}) = (\tilde{\beta}, \tilde{w})$. Then, using the symmetry of the bilinear forms $a(\cdot, \cdot)$ and $r(\cdot, \cdot)$, we have

$$\begin{aligned} r((g, f), T_t(\tilde{g}, \tilde{f})) =& r((g, f), (\tilde{\beta}, \tilde{w})) \\ =& - \Big(a((\hat{\sigma}, \hat{\gamma}), (\tilde{\sigma}, \tilde{\gamma})) + b((\tilde{\sigma}, \tilde{\gamma}), (\hat{\beta}, \hat{w})) + b((\hat{\sigma}, \hat{\gamma}), (\tilde{\beta}, \tilde{w})) \Big) \\ =& r((\tilde{g}, \tilde{f}), (\hat{\beta}, \hat{w})) \\ =& r(T_t(g, f), (\tilde{g}, \tilde{f})). \end{aligned}$$

The operator T_t is also compact. To prove this we resort to the following additional regularity result, which has been proved in [60, Proposition 2.1].

Proposition 2.3.1 *Given* $(g, f) \in \mathbb{Q}$*. Let* $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ *be the unique solution to problem* (2.3.13)–(2.3.14)*. Then, there exists a constant* $C > 0$ *independent of* t*,* g *and* f *such that*

$$\|\hat{w}\|_{1,\mathrm{I}} + \|\hat{\beta}\|_{1,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \|\hat{\gamma}\|_{1,\mathrm{I}} \leq C \|(g, f)\|_{\mathbb{Q}}.$$

Hence, as a consequence of the compact inclusion $\mathrm{H}^1(\mathrm{I}) \hookrightarrow \mathrm{L}^2(\mathrm{I})$, T_t is a compact operator. Then, we know that the spectrum of T_t satisfies $\mathrm{sp}(T_t) = \{0\} \cup \{\mu_n \ : \ n \in \mathbb{N}\}$, where $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of positive eigenvalues which converges to zero, the multiplicity of each non-zero eigenvalue being finite. Moreover, additional regularity of the eigenfunctions holds as a consequence of the following improved form of Proposition 2.3.1, which has been proved in [60, Remark 2.1].

Proposition 2.3.2 *Let* $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ *be the solution of problem* (2.3.13)–(2.3.14)*. If* $g|_{S_i}, f|_{S_i} \in \mathrm{H}^1(S_i), i = 1, \ldots, n$*, then, there exists* $C > 0$ *independent of* t *such that*

$$\begin{aligned} \|\hat{w}\|_{1,\mathrm{I}} + \|\hat{\beta}\|_{1,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \left(\sum_{i=1}^{n} \|\hat{\sigma}''\|_{0,S_i}^2 \right)^{1/2} + \|\hat{\gamma}\|_{1,\mathrm{I}} + \left(\sum_{i=1}^{n} \|\hat{\gamma}''\|_{0,S_i}^2 \right)^{1/2} \\ \leq C \left(\|g\|_{0,\mathrm{I}}^2 + \|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \left(\|g'\|_{0,S_i}^2 + \|f'\|_{0,S_i}^2 \right) \right)^{1/2}. \end{aligned}$$

As a consequence of this result and Proposition 2.3.1, we easily obtain the following additional regularity for the eigenfunctions of problem (2.2.8)–(2.2.9).

Corollary 2.3.1 *Let* $(\lambda, (\sigma, \gamma, \beta, w))$ *be a solution of problem* (2.2.8)–(2.2.9)*. Then, there exists* $C > 0$ *independent of* t *such that*

$$\|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|\sigma\|_{1,\mathrm{I}} + \left(\sum_{i=1}^{n} \|\sigma''\|_{0,S_i}^2\right)^{1/2} + \|\gamma\|_{1,\mathrm{I}} + \left(\sum_{i=1}^{n} \|\gamma''\|_{0,S_i}^2\right)^{1/2} \leq C\lambda \|(\beta, w)\|_{\mathbb{Q}}.$$

The remainder of this section is devoted to prove the convergence of the operator T_t as t goes to zero to the analogous operator T_0 corresponding to the Euler-Bernoulli beam. For this purpose, we consider problem (2.3.13)–(2.3.14) with the thickness parameter $t = 0$, which reads as follows:

Given $f \in \mathrm{L}^2(\mathrm{I})$*, find* $((\sigma_0, \gamma_0), (\beta_0, w_0)) \in \mathbb{H} \times \mathbb{Q}$ *such that*

$$\int_{\mathrm{I}} \frac{\sigma_0 \tau}{\mathrm{E}} + \int_{\mathrm{I}} \beta_0 (\tau' - \xi) - \int_{\mathrm{I}} w_0 \xi' = 0 \qquad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.3.15}$$

$$\int_{\mathrm{I}} \eta (\sigma_0' - \gamma_0) - \int_{\mathrm{I}} v \gamma_0' = - \int_{\mathrm{I}} \mathrm{P} f v \qquad \forall (\eta, v) \in \mathbb{Q}. \tag{2.3.16}$$

By arguments similar to those that will be used below to derive (2.3.21), it can be seen that this is a mixed formulation of the load problem for an Euler-Bernoulli clamped beam. Repeating the arguments used in the proof of [60, Theorem 2.3], we have that problem (2.3.15)–(2.3.16) is well posed. Moreover, the proof of Proposition 2.3.1 holds for $t = 0$, too. Thus, the solution of problem (2.3.15)–(2.3.16) satisfies the following regularity result: There exists a constant $C > 0$ independent of f such that

$$\|w_0\|_{1,\mathrm{I}} + \|\beta_0\|_{1,\mathrm{I}} + \|\gamma_0\|_{1,\mathrm{I}} + \|\sigma_0\|_{1,\mathrm{I}} \leq C \|f\|_{0,\mathrm{I}}. \tag{2.3.17}$$

Now, let T_0 be the bounded linear operator defined by

$$\begin{aligned} T_0 &: \mathbb{Q} \to \mathbb{Q}, \\ &(g, f) \mapsto (\beta_0, w_0). \end{aligned} \tag{2.3.18}$$

Notice that T_0 actually does not depend on g but only on f. However, we define it in this way so that T_0 be a map from one space into itself, which is necessary for its spectral analysis. In fact, T_0 can be seen as the solution operator of the following mixed eigenvalue problem:

Find $\lambda_0 > 0$ *and* $0 \neq ((\sigma_0, \gamma_0), (\beta_0, w_0)) \in \mathbb{H} \times \mathbb{Q}$ *such that*

$$\int_{\mathrm{I}} \frac{\sigma_0 \tau}{\mathrm{E}} + \int_{\mathrm{I}} \beta_0 (\tau' - \xi) - \int_{\mathrm{I}} w_0 \xi' = 0 \qquad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.3.19}$$

$$\int_{\mathrm{I}} \eta (\sigma_0' - \gamma_0) - \int_{\mathrm{I}} v \gamma_0' = -\lambda_0 \int_{\mathrm{I}} \mathrm{P} w_0 v \qquad \forall (\eta, v) \in \mathbb{Q}. \tag{2.3.20}$$

As usual $(\mu^0, (\beta_0, w_0))$, with $\mu^0 \neq 0$, is an eigenpair of T_0 (i.e., $T_0(\beta_0, w_0) = \mu^0(\beta_0, w_0)$) if and only if there exist $(\sigma_0, \gamma_0) \in \mathbb{H}$ such that, for $\lambda_0 = 1/\mu^0$, $(\lambda_0, (\sigma_0, \gamma_0), (\beta_0, w_0))$ is a solution of problem (2.3.19)–(2.3.20). Moreover, λ_0 is positive. In fact, by taking appropriate test functions, it is easy to check that any solution of (2.3.19)–(2.3.20) satisfies

$$\begin{cases} \sigma_0 = \mathrm{E}\beta_0' & \text{in} \quad \mathrm{I}, \\ \sigma_0' = \gamma_0 & \text{in} \quad \mathrm{I}, \\ \beta_0 = w_0' & \text{in} \quad \mathrm{I}, \\ \gamma_0' = \lambda_0 \mathrm{P} w_0 & \text{in} \quad \mathrm{I}, \\ w_0(0) = \beta_0(0) = w_0(L) = \beta_0(L) = 0. \end{cases}$$

From these, we derive the classical fourth order differential equation of the Euler-Bernoulli clamped beam vibration problem,

$$\begin{cases} (\mathrm{E}w_0'')'' = \lambda_0 \mathrm{P} w_0 & \text{in} \quad \mathrm{I}, \\ w_0(0) = w_0(L) = w_0'(0) = w_0'(L) = 0, \end{cases} \tag{2.3.21}$$

whose corresponding variational formulation reads

$$w_0 \in H_0^2(\mathrm{I}): \quad \int_{\mathrm{I}} \mathrm{E} w_0'' v'' = \lambda_0 \int_{\mathrm{I}} \mathrm{P} w_0 v \qquad \forall v \in H_0^2(\mathrm{I}).$$

Therefore,

$$\lambda_0 = \frac{\int_{\mathrm{I}} \mathrm{E}(w_0'')^2}{\int_{\mathrm{I}} \mathrm{P} w_0^2} > 0.$$

We note that, because of (2.3.17), T_0 is a compact operator. So, its spectrum is given by $\mathrm{sp}(T_0) = \{0\} \cup \{\mu_n^0 : n \in \mathbb{N}\}$, where $\{\mu_n^0\}_{n \in \mathbb{N}}$ is a sequence of positive eigenvalues which converges to zero, the multiplicity of each non-zero eigenvalue being finite.

The following lemma states the convergence in norm of T_t to T_0.

Lemma 2.3.1 *There exists a positive constant C independent of t such that*

$$\|(T_t - T_0)(g, f)\|_{\mathbb{Q}} \leq Ct\|(g, f)\|_{\mathbb{Q}}.$$

Proof. Subtracting (2.3.15)–(2.3.16) from (2.3.13)–(2.3.14), we obtain

$$\int_{\mathrm{I}} \frac{(\hat{\sigma} - \sigma_0)\tau}{\mathrm{E}} + \int_{\mathrm{I}} (\hat{\beta} - \beta_0)(\tau' - \xi) - \int_{\mathrm{I}} (\hat{w} - w_0)\xi' = -t^2 \int_{\mathrm{I}} \frac{\hat{\gamma}\xi}{\kappa} \quad \forall (\tau, \xi) \in \mathbb{H},$$
$$\int_{\mathrm{I}} \eta((\hat{\sigma}' - \sigma_0') - (\hat{\gamma} - \gamma_0)) - \int_{\mathrm{I}} v(\hat{\gamma}' - \gamma_0') = -t^2 \int_{\mathrm{I}} \mathrm{J} g \eta \quad \forall (\eta, v) \in \mathbb{Q}.$$

Testing the system above with $\tau = \hat{\sigma} - \sigma_0$, $\xi = \hat{\gamma} - \gamma_0$, $\eta = \hat{\beta} - \beta_0$ and $v = \hat{w} - w_0$ and subtracting the resulting equations, we obtain

$$\int_{\mathrm{I}} \frac{(\hat{\sigma} - \sigma_0)^2}{\mathrm{E}} = t^2 \int_{\mathrm{I}} \mathrm{J} g(\hat{\beta} - \beta_0) - t^2 \int_{\mathrm{I}} \frac{\hat{\gamma}(\hat{\gamma} - \gamma_0)}{\kappa}.$$

Thus, by using (2.2.3), Proposition 2.3.1 and (2.3.17), we have

$$\begin{aligned}\|\hat{\sigma}-\sigma_0\|_{0,\mathrm{I}}^2 &\le Ct^2(\|g\|_{0,\mathrm{I}}\|\hat{\beta}-\beta_0\|_{0,\mathrm{I}}+\|\hat{\gamma}\|_{0,\mathrm{I}}\|\hat{\gamma}-\gamma_0\|_{0,\mathrm{I}})\\ &\le Ct^2\Big(\|g\|_{0,\mathrm{I}}(\|\hat{\beta}\|_{0,\mathrm{I}}+\|\beta_0\|_{0,\mathrm{I}})+\|\hat{\gamma}\|_{0,\mathrm{I}}(\|\hat{\gamma}\|_{0,\mathrm{I}}+\|\gamma_0\|_{\mathrm{I}})\Big)\\ &\le Ct^2\|(g,f)\|_{\mathbb{Q}}^2.\end{aligned}$$

Now, we use Lemma 2.2.2, (2.3.13), (2.3.15), (2.2.3) and the above inequality, to derive

$$\begin{aligned}\|(\hat{\beta},\hat{w})-(\beta_0,w_0)\|_{\mathbb{Q}} &\le C\sup_{0\neq(\tau,\xi)\in\mathbb{H}}\frac{b((\tau,\xi),(\hat{\beta}-\beta_0,\hat{w}-w_0))}{\|(\tau,\xi)\|_{\mathbb{H}}}\\ &= C\sup_{0\neq(\tau,\xi)\in\mathbb{H}}\frac{-\displaystyle\int_{\mathrm{I}}\frac{(\hat{\sigma}-\sigma_0)\tau}{\mathrm{E}}-t^2\int_{\mathrm{I}}\frac{\hat{\gamma}\xi}{\kappa}}{\|(\tau,\xi)\|_{\mathbb{H}}}\\ &\le Ct\|(g,f)\|_{\mathbb{Q}},\end{aligned}$$

which allows us to complete the proof. □

As a consequence of this lemma, standard properties of separation of isolated parts of the spectrum (see, for instance [53]) yield the following result.

Lemma 2.3.2 *Let $\mu^0>0$ be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ^0 and containing no other element of the spectrum of T_0. Then, for t small enough, D contains exactly m eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu^0>0$ of T_0 is a limit of eigenvalues μ of T_t, as t goes to zero.*

2.4 Spectral approximation

We will study in this section, the numerical approximation of the eigenvalue problem (2.2.8)–(2.2.9). With this aim, first we consider a family of partitions of I,

$$\mathcal{T}_h:\ 0=x_0<\cdots<x_N=L,$$

which are all refinements of the initial partition $0=s_0<\cdots<s_n=L$. Recall that $s_1,\ldots,s_{n-1}$ are the points of discontinuity of any of the coefficients, E, κ, P or J. We denote $\mathrm{I}_j:=(x_{j-1},x_j)$, $j=1,\ldots,N$, and the largest subinterval length is denoted $h:=\max_{1\le j\le N}(x_j-x_{j-1})$. Notice that for any mesh $\mathcal{T}_h$, each I_j is contained in one of the subinterval S_i, $i=1,\ldots,n$, where the physical coefficients are smooth.

We consider the space of piecewise linear continuous finite elements:

$$W_h:=\{\xi_h\in\mathrm{H}^1(\mathrm{I})\ :\ \xi_h|_{\mathrm{I}_j}\in\mathbb{P}_1(\mathrm{I}_j),\ j=1,\ldots,N\}.$$

For $\xi\in\mathrm{H}^1(\mathrm{I})$ let $\mathcal{L}_h\xi\in W_h$ be its Lagrange interpolant. We recall that

$$\|\xi-\mathcal{L}_h\xi\|_{1,\mathrm{I}}\le Ch\left(\sum_{j=1}^N\|\xi''\|_{0,\mathrm{I}_j}^2\right)^{1/2}\qquad\forall\xi|_{\mathrm{I}_j}\in\mathrm{H}^2(\mathrm{I}_j),\ j=1,\ldots,N. \tag{2.4.22}$$

We will also consider the space of piecewise constant functions:

$$Z_h := \{v_h \in \mathrm{L}^2(\mathrm{I}) \ : \ v_h|_{\mathrm{I}_j} \in \mathbb{P}_0(\mathrm{I}_j), \ j = 1, \dots, N\},$$

and the L^2-projector onto Z_h:

$$\begin{aligned} \mathcal{P}_h : L^2(\mathrm{I}) &\to Z_h, \\ v \mapsto \mathcal{P}_h v \in Z_h \ &: \ \int_{\mathrm{I}} (v - \mathcal{P}_h v) q_h = 0 \quad \forall q_h \in Z_h. \end{aligned}$$

It is well known that

$$\|v - \mathcal{P}_h v\|_{0,\mathrm{I}} \le C h |v|_{1,\mathrm{I}} \qquad \forall v \in \mathrm{H}^1(\mathrm{I}). \tag{2.4.23}$$

Defining $\mathbb{H}_h := W_h \times W_h$ and $\mathbb{Q}_h := Z_h \times Z_h$, the discretization of problem (2.2.8)–(2.2.9) reads as follows:

Find $\lambda_h > 0$ and $0 \neq ((\sigma_h, \gamma_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$a((\sigma_h, \gamma_h), (\tau_h, \xi_h)) + b((\tau_h, \xi_h), (\beta_h, w_h)) = 0 \qquad \forall (\tau_h, \xi_h) \in \mathbb{H}_h, \tag{2.4.24}$$

$$b((\sigma_h, \gamma_h), (\eta_h, v_h)) = -\lambda_h r((\beta_h, w_h), (\eta_h, v_h)) \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h. \tag{2.4.25}$$

As in the continuous case, we introduce for the analysis the *discrete solution operator*

$$\begin{aligned} T_{th} : \mathbb{Q} &\to \mathbb{Q} \\ (g, f) &\mapsto (\hat{\beta}_h, \hat{w}_h), \end{aligned}$$

where $((\hat{\sigma}_h, \hat{\gamma}_h), (\hat{\beta}_h, \hat{w}_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ is the solution of the corresponding discrete source problem:

$$a((\hat{\sigma}_h, \hat{\gamma}_h), (\tau_h, \xi_h)) + b((\tau_h, \xi_h), (\hat{\beta}_h, \hat{w}_h)) = 0 \qquad \forall (\tau_h, \xi_h) \in \mathbb{H}_h, \tag{2.4.26}$$

$$b((\hat{\sigma}_h, \hat{\gamma}_h), (\eta_h, v_h)) = -r((g, f), (\eta_h, v_h)) \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h. \tag{2.4.27}$$

It is easy to check that the *discrete kernel*

$$\mathbb{K}_h := \{(\tau_h, \xi_h) \in \mathbb{H}_h : \ b((\tau_h, \xi_h), (\eta_h, v_h)) = 0 \quad \forall (\eta_h, v_h) \in \mathbb{Q}_h\},$$

coincides with the continuous one $\mathbb{K} = \{(\tau, \tau') : \tau \in \mathbb{P}_1(\mathrm{I})\}$. Therefore, the ellipticity estimate from Lemma 2.2.1 holds true for $(\tau_h, \xi_h) \in \mathbb{K}_h$ with the same constant α independent of t and h. Moreover, the discrete *inf-sup* condition

$$\sup_{0 \neq (\tau_h, \xi_h) \in \mathbb{H}_h} \frac{b((\tau_h, \xi_h), (\eta_h, v_h))}{\|(\tau_h, \xi_h)\|_{\mathbb{H}}} \ge C \|(\eta_h, v_h)\|_{\mathbb{Q}} \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h \tag{2.4.28}$$

holds true with a positive constant C independent of t and h (see [60, Lemma 3.2]). Consequently, the discrete mixed problem (2.4.26)–(2.4.27) has a unique solution and there holds

$$\|(\hat{\sigma}_h, \hat{\gamma}_h)\|_{\mathbb{H}} + \|(\hat{\beta}_h, \hat{w}_h)\|_{\mathbb{Q}} \le C \|(g, f)\|_{\mathbb{Q}}, \tag{2.4.29}$$

once more with a positive constant C independent of t and h. Hence, T_{th} is a well defined bounded linear operator.

Remark 2.4.1 *The above estimate can be improved as follows:*

$$\|(\hat{\sigma}_h,\hat{\gamma}_h)\|_{\mathbb{H}}^2+\|(\hat{\beta}_h,\hat{w}_h)\|_{\mathbb{Q}}^2\le C\left(t^2\int_{\mathrm{I}}\mathrm{J}|g|^2+\int_{\mathrm{I}}\mathrm{P}|f|^2\right),\tag{2.4.30}$$

always with a positive constant C independent of t and h. In fact, this follows easily from taking into account the particular form of the right hand side of problem (2.4.26)–(2.4.27) *and using, for instance, [46, Remark II.1.3].*

As in the continuous case, $(\mu_h,(\beta_h,w_h))$, with $\mu_h\neq 0$, is an eigenpair of T_{th} if and only if there exists $(\sigma_h,\gamma_h)\in\mathbb{H}_h$ such that, for $\lambda_h=1/\mu_h$, $(\lambda_h,(\sigma_h,\gamma_h,\beta_h,w_h))$ is a solution of problem (2.4.24)–(2.4.25). Moreover, the same arguments used for T_t allow us to show that the operator T_{th} is self-adjoint with respect to the inner product $r(\cdot,\cdot)$.

Our next goal is to obtain a spectral characterization for problem (2.4.24)–(2.4.25):

Lemma 2.4.1 *The variational problem* (2.4.24)–(2.4.25) *has exactly* $\dim\mathbb{Q}_h$ *eigenvalues, repeated according to their respective multiplicities. All of them are real and positive.*

Proof. Taking particular bases of the discrete spaces, problem (2.4.24)–(2.4.25) can be written in matrix form as follows:

$$\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{E} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{C} & -\boldsymbol{D} & -\boldsymbol{E}\\ \boldsymbol{E}^{\mathrm{t}} & -\boldsymbol{D}^{\mathrm{t}} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{E}^{\mathrm{t}} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{\sigma}}_h\\ \vec{\boldsymbol{\gamma}}_h\\ \vec{\boldsymbol{\beta}}_h\\ \vec{\boldsymbol{w}}_h\end{bmatrix}=-\lambda_h\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q}\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{\sigma}}_h\\ \vec{\boldsymbol{\gamma}}_h\\ \vec{\boldsymbol{\beta}}_h\\ \vec{\boldsymbol{w}}_h\end{bmatrix},\tag{2.4.31}$$

where $\vec{\boldsymbol{\sigma}}_h$, $\vec{\boldsymbol{\gamma}}_h$, $\vec{\boldsymbol{\beta}}_h$ and $\vec{\boldsymbol{w}}_h$ denote the vectors whose entries are the components in those basis of σ_h, γ_h, β_h and w_h, respectively.

Now we define the following matrices

$$\boldsymbol{R}:=\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{C}\end{bmatrix},\quad \boldsymbol{S}:=\begin{bmatrix}\boldsymbol{E} & \boldsymbol{0}\\ -\boldsymbol{D} & -\boldsymbol{E}\end{bmatrix},\quad \boldsymbol{T}:=\begin{bmatrix}\boldsymbol{P} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q}\end{bmatrix},$$

and vectors

$$\vec{\boldsymbol{u}}_h:=\begin{bmatrix}\vec{\boldsymbol{\sigma}}_h\\ \vec{\boldsymbol{\gamma}}_h\end{bmatrix},\quad \vec{\boldsymbol{v}}_h:=\begin{bmatrix}\vec{\boldsymbol{\beta}}_h\\ \vec{\boldsymbol{w}}_h\end{bmatrix},$$

to rewrite (3.5) as follows:

$$\begin{bmatrix}\boldsymbol{R} & \boldsymbol{S}\\ \boldsymbol{S}^{\mathrm{t}} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\boldsymbol{v}}_h\end{bmatrix}=-\lambda_h\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{T}\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{u}}_h\\ \vec{\boldsymbol{v}}_h\end{bmatrix}.\tag{2.4.32}$$

The system above is equivalent to solve

$$\begin{aligned}\boldsymbol{R}\vec{\boldsymbol{u}}_h+\boldsymbol{S}\vec{\boldsymbol{v}}_h&=\boldsymbol{0}\\ \boldsymbol{S}^{\mathrm{t}}\vec{\boldsymbol{u}}_h&=-\lambda_h\boldsymbol{T}\vec{\boldsymbol{v}}_h.\end{aligned}$$

Since $\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{P}$ and $\boldsymbol{Q}$ are scaled mass matrices, it is easy to check that all of them, as well as $\boldsymbol{R}$ and $\boldsymbol{T}$, are symmetric and positive definite (although not uniformly in t) and hence invertible. Thus, from the first equation above we have that $\vec{\boldsymbol{u}}_h = -\boldsymbol{R}^{-1}\boldsymbol{S}\vec{\boldsymbol{v}}_h$ and substituting this into the second equation we obtain:

$$(\boldsymbol{S}^{\mathrm{t}}\boldsymbol{R}^{-1}\boldsymbol{S})\vec{\boldsymbol{v}}_h = \lambda_h \boldsymbol{T}\vec{\boldsymbol{v}}_h. \tag{2.4.33}$$

Conversely, if $(\lambda_h, \vec{\boldsymbol{v}}_h)$ is an eigenpair of the above problem, by defining $\vec{\boldsymbol{u}}_h := -\boldsymbol{R}^{-1}\boldsymbol{S}\vec{\boldsymbol{v}}_h$, we have that $(\lambda_h, (\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h))$ is an eigenpair of (5.3.23).

The eigenvalue problem (2.4.33) is well posed because $\boldsymbol{T}$ is symmetric and positive definite. The same holds true for $\boldsymbol{S}^{\mathrm{t}}\boldsymbol{R}^{-1}\boldsymbol{S}$. In fact, this matrix is clearly symmetric and positive semi-definite. Moreover, it is positive definite because $\boldsymbol{S}\vec{\boldsymbol{v}}_h = \boldsymbol{0}$ implies $\vec{\boldsymbol{v}}_h = \boldsymbol{0}$, as a consequence of (2.4.28). Then, the generalized eigenvalue problem is well posed and all its eigenvalues are real and positive. Therefore, the number of eigenvalues of problem (3.5) equals the number of eigenvalues of this problem, namely $\dim \mathbb{Q}_h$, and we complete the proof. □

In order to prove that the solutions of the discrete problem (2.4.26)–(2.4.27) converge to those of the continuous problem (2.3.13)–(2.3.14), the standard procedure would be to show that T_{th} converges in norm to T_t as h goes to zero. However, such a proof does not seem straightforward in our case. In fact, $\|(T_t - T_{th})(g,f)\|_{\mathbb{Q}}$ is bounded for g and f piecewise smooth as follows:

$$\|(T_t - T_{th})(g,f)\|_{\mathbb{Q}} \leq Ch \left(\|g\|_{0,\mathrm{I}}^2 + \|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \left(\|g'\|_{0,S_i}^2 + \|f'\|_{0,S_i}^2 \right) \right)^{1/2},$$

but the last terms above are not bounded in general by $\|(g,f)\|_{\mathbb{Q}}$. To circumvent this drawback, we will resort instead to the spectral theory from [36] and [37]. In spite of the fact that the main use of this theory is when T_t is a non compact operator, it can also be applied to compact T_t and we will show that in our case it works.

The remainder of this section is devoted to prove the following properties which will be used in the next section:

P1. $\displaystyle \|T_t - T_{th}\|_h := \sup_{0 \neq (g_h, f_h) \in \mathbb{Q}_h} \frac{\|(T_t - T_{th})(g_h, f_h))\|_{\mathbb{Q}}}{\|(g_h, f_h)\|_{\mathbb{Q}}} \to 0 \quad \text{as} \quad h \to 0.$

P2. $\displaystyle \forall (\eta, v) \in \mathbb{Q}, \quad \inf_{(\eta_h, v_h) \in \mathbb{Q}_h} \|(\eta, v) - (\eta_h, v_h)\|_{\mathbb{Q}} \to 0 \quad \text{as} \quad h \to 0.$

Property P2 is a consequence of the fact that piecewise constant functions are dense in $\mathrm{L}^2(\mathrm{I})$. Regarding property P1, we have the following result.

Lemma 2.4.2 *Property P1 holds true; moreover, there exists a constant $C > 0$ independent of t and h such that*

$$\|T_t - T_{th}\|_h \leq Ch.$$

Proof. Given $(g_h, f_h) \in \mathbb{Q}_h$, let $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ and $((\hat{\sigma}_h, \hat{\gamma}_h), (\hat{\beta}_h, \hat{w}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ be the solutions of problems (2.3.13)–(2.3.14) and (2.4.26)–(2.4.27), respectively, in both cases with $(g,f) = (g_h, f_h)$. Therefore, $(\hat{\beta}, \hat{w}) = T_t(g_h, f_h)$ and $(\hat{\beta}_h, \hat{w}_h) = T_{th}(g_h, f_h)$.

The same arguments used in the proof of Proposition 2.3.2 (see [60, Remark 2.1]) allow us to show that there exists a constant $C > 0$, independent of t, g_h and f_h, such that

$$\|\hat{w}\|_{1,\mathrm{I}} + \|\hat{\beta}\|_{1,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \left(\sum_{j=1}^{N} \|\hat{\sigma}''\|_{0,\mathrm{I}_j}^2\right)^{1/2} + \|\hat{\gamma}\|_{1,\mathrm{I}} + \left(\sum_{j=1}^{N} \|\hat{\gamma}''\|_{0,\mathrm{I}_j}^2\right)^{1/2} \leq C\|(g_h, f_h)\|_{\mathbb{Q}}, \tag{2.4.34}$$

where we have also used that $g_h'|_{\mathrm{I}_j} = f_h'|_{\mathrm{I}_j} = 0$, because g_h and f_h are piecewise constant. On the other hand, since problem (2.4.26)–(2.4.27) is just the finite element discretization of problem (2.3.13)–(2.3.14), using again the results from [60] (in particular, Theorem 3.3), we have that

$$\begin{aligned}
\|(T_t - T_{th})(g_h, f_h)\|_{\mathbb{Q}} \leq & \|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\hat{\sigma}_h, \hat{\gamma}_h), (\hat{\beta}_h, \hat{w}_h))\|_{\mathbb{H}\times\mathbb{Q}} \\
\leq & C \inf_{((\tau_h,\xi_h),(\eta_h,v_h))\in\mathbb{H}_h\times\mathbb{Q}_h} \|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\tau_h, \xi_h), (\eta_h, v_h))\|_{\mathbb{H}\times\mathbb{Q}} \\
\leq & C\|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\mathcal{L}_h\hat{\sigma}, \mathcal{L}_h\hat{\gamma}), (\mathcal{P}_h\hat{\beta}, \mathcal{P}_h\hat{w}))\|_{\mathbb{H}\times\mathbb{Q}} \\
\leq & Ch\|(g_h, f_h)\|_{\mathbb{Q}},
\end{aligned}$$

where, for the last inequality we have used the error estimates (2.4.22) and (2.4.23) together with the additional regularity result (2.4.34). Thus, the proof follows from the definition of $\|T_t - T_{th}\|_h$ and the above estimate. □

2.5 Convergence and error estimates.

In this section we will adapt the arguments from [36, 37] to prove convergence of our spectral approximation as well as to obtain error estimates for the approximate eigenvalues and eigenfunctions. With this end, we will use the following results.

Lemma 2.5.1 *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{sp}(T_0) = \emptyset$. Then, there exist strictly positive constants t_0 and C such that, $\forall t < t_0$, $F \cap \mathrm{sp}(T_t) = \emptyset$ and*

$$\|R_z(T_t)\| := \sup_{0\neq(\eta,v)\in\mathbb{Q}} \frac{\|R_z(T_t)(\eta, v)\|_{\mathbb{Q}}}{\|(\eta, v)\|_{\mathbb{Q}}} \leq C \quad \forall z \in F.$$

Proof. We consider the mapping $z \to \|(zI - T_0)^{-1}\|$, which is continuous for all $z \in \boldsymbol{\rho}(T_0)$. It is clear that this mapping goes to zero as $|z| \to \infty$. Hence, if $F \subset \boldsymbol{\rho}(T_0)$ is a closed subset, then the mapping above attains its maximum. Let $\hat{C} := \max_{z\in F} \|(zI - T_0)^{-1}\|$; there holds

$$\|(zI - T_0)(\eta, v)\|_{\mathbb{Q}} \geq \frac{1}{\hat{C}}\|(\eta, v)\|_{\mathbb{Q}} \quad \forall(\eta, v) \in \mathbb{Q} \quad \forall z \in F.$$

Next, we observe that

$$\|(zI - T_0)(\eta, v)\|_{\mathbb{Q}} \leq \|(zI - T_t)(\eta, v)\|_{\mathbb{Q}} + \|(T_t - T_0)(\eta, v)\|_{\mathbb{Q}}.$$

Moreover, according to Lemma 2.3.1, there exists $t_0 > 0$ such that, for all $t < t_0$,

$$\|(T_t - T_0)(\eta, v)\|_{\mathbb{Q}} \leq \frac{1}{2\hat{C}} \|(\eta, v)\|_{\mathbb{Q}} \qquad \forall (\eta, v) \in \mathbb{Q}.$$

Therefore, for all $(\eta, v) \in \mathbb{Q}$, for all $z \in F$ and for all $t < t_0$,

$$\|(zI - T_t)(\eta, v)\|_{\mathbb{Q}} \geq \|(zI - T_0)(\eta, v)\|_{\mathbb{Q}} - \|(T_t - T_0)(\eta, v)\|_{\mathbb{Q}} \geq \frac{1}{2\hat{C}} \|(\eta, v)\|_{\mathbb{Q}}.$$

Consequently, z is not an eigenvalue of T_t. Moreover, $z \neq 0$, because $0 \notin \boldsymbol{\rho}(T_0)$. Hence, since the spectrum of T_t consists of eigenvalues and $\mu = 0$, we have that $z \notin \mathrm{sp}(T_t)$, so that $(zI - T_t)$ is invertible for all $t < t_0$ and for all $z \in F$. Moreover, from the above inequality, we have that

$$\|R_z(T_t)\| = \|(zI - T_t)^{-1}\| \leq 2\hat{C}$$

and we conclude the proof. □

The following result shows that $R_z(T_{th}|_{\mathbb{Q}_h})$ is bounded on any closed subset of the complex plane not intersecting $\mathrm{sp}(T_0)$, provided t and h are small enough. Here and thereafter, h_0 and t_0 denote small positive constants, not necessarily the same at each occurrence.

Lemma 2.5.2 *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{sp}(T_0) = \emptyset$. Then, there exist strictly positive constants h_0, t_0 and C such that, $\forall h < h_0$ and $\forall t < t_0$, $F \cap \mathrm{sp}(T_{th}) = \emptyset$ and*

$$\|R_z(T_{th})\|_h \leq C \quad \forall z \in F.$$

Proof. Let F be a closed set such that $F \cap \mathrm{sp}(T_0) = \emptyset$. As an immediate consequence of Lemma 2.5.1, we have that for all $(\eta, v) \in \mathbb{Q}$, for all $z \in F$ and for all $t < t_0$,

$$\|(\eta, v)\|_{\mathbb{Q}} \leq C \|(zI - T_t)(\eta, v)\|_{\mathbb{Q}}.$$

Now, from Lemma 2.4.2, we have that there exists $h_0 > 0$ such that for all $h < h_0$

$$\|(T_t - T_{th})(\eta_h, v_h)\|_{\mathbb{Q}} \leq \frac{1}{2C} \|(\eta_h, v_h)\|_{\mathbb{Q}} \quad \forall (\eta_h, v_h) \in \mathbb{Q}_h.$$

Then, for $(\eta_h, v_h) \in \mathbb{Q}_h$ and $z \in F$, we have

$$\|(zI - T_{th})(\eta_h, v_h)\|_{\mathbb{Q}} \geq \|(zI - T_t)(\eta_h, v_h)\|_{\mathbb{Q}} - \|(T_t - T_{th})(\eta_h, v_h)\|_{\mathbb{Q}} \geq \frac{1}{2C} \|(\eta_h, v_h)\|_{\mathbb{Q}}.$$

Since $\mathbb{Q}_h$ is finite dimensional, we deduce that $(zI - T_{th})$ is invertible and, hence, $z \notin \mathrm{sp}(T_{th})$. Moreover

$$\|R_z(T_{th})\|_h = \|(zI - T_{th})^{-1}\|_h \leq 2C \quad \forall z \in F$$

and we complete the proof. □

An equivalent form of the first assertion of this theorem is that any open set of the complex plane containing $\mathrm{sp}(T_0)$, also contains $\mathrm{sp}(T_{th})$ for h and t small enough.

The eigenvalues μ of T_t are typically simple and converges to simple eigenvalues T_0 as t tends to zero. Because of this, we state our results only for eigenvalues of T_t converging to a simple eigenvalue of T_0 as t goes to zero.

Let $\mu^0 \neq 0$ be an eigenvalue of T_0 with multiplicity $m = 1$. Let D be a closed disk centered at μ^0 with boundary Γ such that $0 \notin D$ and $D \cap \operatorname{sp}(T_0) = \left\{\mu^0\right\}$. Let $t_0 > 0$ be small enough so that, for all $t < t_0$, D contains only one eigenvalue μ of T_t, which we already know is simple (cf. Lemma 2.3.2). Let $\mathcal{E}$ be the eigenspace of T_t corresponding to μ.

According to Lemma 2.5.2 there exist $t_0 > 0$ and $h_0 > 0$ such that $\forall t < t_0$ and $\forall h < h_0$, $\Gamma \subset \boldsymbol{\rho}(T_{th})$. Moreover, proceeding as in [36, Section 2], from properties P1 and P2 it follows that for h small enough T_{th} has exactly one eigenvalue $\mu_h \in D$. Let $\mathcal{E}_h$ be the eigenspace of T_{th} associated to μ_h. The theory in [37] could be adapted too, to prove error estimates for the eigenvalues and eigenfunctions of T_{th} to those of T_0 as h and t go to zero. However, our goal is not this one, but to prove that μ_h converges to μ as h goes to zero, with $t < t_0$ fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [37].

Let $\Pi_h : \mathbb{Q} \to \mathbb{Q}$ be defined for all $(\eta, v) \in \mathbb{Q}$ by $\Pi_h(\eta, v) = (\mathcal{P}_h\eta, \mathcal{P}_h v) \in \mathbb{Q}_h$, with $\mathcal{P}_h$ being the L^2-projector defined in the previous section. The properties of $\mathcal{P}_h$ lead to analogous properties for Π_h; for instance, Π_h is bounded uniformly on h, namely, $\|\Pi_h(\eta, v)\|_{\mathbb{Q}} \leq \|(\eta, v)\|_{\mathbb{Q}}$. Moreover, the error estimate (2.4.23) holds for Π_h too:

$$\|\Pi_h(\eta, v) - (\eta, v)\|_{\mathbb{Q}} \leq Ch(|\eta|_{1,\mathrm{I}} + |v|_{1,\mathrm{I}}) \qquad \forall(\eta, v) \in \mathbb{H}. \tag{2.5.35}$$

Next, we define

$$B_{th} := T_{th}\Pi_h : \mathbb{Q} \to \mathbb{Q},$$

We observe that B_{th} and T_{th} have the same non-zero eigenvalues and corresponding eigenfunctions. Furthermore, we have the following result analogous to [37, Lemma 1].

Lemma 2.5.3 *There exist strictly positive constants h_0, t_0 and C such that*

$$\|R_z(B_{th})\| \leq C \qquad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

Proof. It is essentially identical to that of Lemma 5.2 from [64]. □

Next, we introduce

- $E_t : \mathbb{Q} \to \mathbb{Q}$, the spectral projector of T_t corresponding to the isolated eigenvalue μ, namely,

$$E_t := \frac{1}{2\pi i}\int_\Gamma R_z(T_t)\,dz;$$

- $F_{th} : \mathbb{Q} \to \mathbb{Q}$, the spectral projector of B_{th} corresponding to the eigenvalue μ_h, namely,

$$F_{th} := \frac{1}{2\pi i}\int_\Gamma R_z(B_{th})\,dz.$$

As a consequence of Lemma 2.5.3, the spectral projectors F_{th} are bounded uniformly in h and t for h and t small enough. Notice that $E_t(\mathbb{Q})$ is the eigenspace of T_t associated to μ and $F_{th}(\mathbb{Q})$ is the eigenspace of B_{th} (and hence of T_{th}) associated to μ_h.

We recall the definition of the gap $\hat{\delta}$ between two closed subspaces Y and Z of $\mathbb{Q}$:

$$\hat{\delta}(Y,Z) := \max\left\{\delta(Y,Z),\delta(Z,Y)\right\},$$

where

$$\delta(Y,Z) := \sup_{\substack{y\in Y\\ \|y\|_{\mathbb{Q}}=1}} \left(\inf_{z\in Z}\|y-z\|_{\mathbb{Q}}\right).$$

The following results will be used to prove convergence of the eigenspaces.

Lemma 2.5.4 *There exist positive constants h_0, t_0 and C such that, for all $h<h_0$ and for all $t<t_0$,*

$$\|(E_t-F_{th})|_{E_t(\mathbb{Q})}\| \le \|(T_t-B_{th})|_{E_t(\mathbb{Q})}\| \le Ch.$$

Proof. The proof of the first inequality follows from Lemmas 2.5.1 and 2.5.3 and the same arguments as Lemma 3 from [37]. For the other inequality, let $(\beta,w)\in E_t(\mathbb{Q})$. We have

$$\begin{aligned}\|(T_t-B_{th})(\beta,w)\|_{\mathbb{Q}} &\le \|(T_t-T_t\Pi_h)(\beta,w)\|_{\mathbb{Q}} + \|(T_t\Pi_h-B_{th})(\beta,w)\|_{\mathbb{Q}}\\ &\le \|T_t\|\|(I-\Pi_h)(\beta,w)\|_{\mathbb{Q}} + \|(T_t-T_{th})\Pi_h(\beta,w)\|_{\mathbb{Q}}\\ &\le Ch(|\beta|_{1,\mathrm{I}}+|w|_{1,\mathrm{I}}) + Ch\|\Pi_h(\beta,w)\|_{\mathbb{Q}}\\ &\le Ch\|(\beta,w)\|_{\mathbb{Q}},\end{aligned}$$

where we have used (2.5.35), Lemma 2.4.2 and Corollary 2.3.1. □

Now, we are in position to prove an optimal order error estimate for the eigenspaces.

Theorem 2.5.1 *There exist positive constants h_0, t_0 and C such that, for all $h<h_0$ and for all $t<t_0$,*

$$\hat{\delta}\left(F_{th}(\mathbb{Q}),E_t(\mathbb{Q})\right) \le Ch.$$

Proof. The proof follows by using Lemma 2.5.4 and arguing exactly as in the proof of [37, Theorem 1]. □

In what follows, we state a preliminary suboptimal error estimate for $|\mu-\mu_h|$ that will be used in the sequel but which will be improved below (cf. Theorem 2.5.2).

Lemma 2.5.5 *There exist strictly positive constants h_0, t_0 and C such that, for $h<h_0$ and $t<t_0$,*

$$|\mu-\mu_h| \le Ch.$$

Proof. The proof follows by repeating the arguments used in [64] to derive Lemma 5.6 from this reference. □

Since the eigenvalue μ of T_t corresponds to an eigenvalue $\lambda=1/\mu$ of problem (2.2.8)-(2.2.9), Lemma 2.5.5 leads to an error estimate for the approximation of λ as well. However, the order of convergence $O(h)$ in this lemma is not optimal. The following lemma will be used to prove a double order of convergence for the corresponding eigenvalues, but it is interesting by itself, too. In fact, it shows optimal order convergence for the bending moment and shear stress of the vibration modes.

Lemma 2.5.6 *Let $(\lambda, (\sigma, \gamma, \beta, w))$ and $(\lambda_h, (\sigma_h, \gamma_h, \beta_h, w_h))$ be the solutions of problems* (2.2.8)–(2.2.9) *and* (2.4.24)–(2.4.25)*, respectively, with $\|(\beta, w)\|_{\mathbb{Q}} = \|(\beta_h, w_h)\|_{\mathbb{Q}} = 1$ and such that*

$$\|\beta - \beta_h\|_{0,\mathrm{I}} + \|w - w_h\|_{0,\mathrm{I}} \leq Ch. \tag{2.5.36}$$

Then, for h and t small enough,

$$\|\sigma - \sigma_h\|_{1,\mathrm{I}} + \|\gamma - \gamma_h\|_{1,\mathrm{I}} \leq Ch. \tag{2.5.37}$$

Proof. Let $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ be the solution of the following auxiliary problem:

$$a((\hat{\sigma}, \hat{\gamma}), (\tau, \xi)) + b((\tau, \xi), (\hat{\beta}, \hat{w})) = 0 \qquad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.5.38}$$

$$b((\hat{\sigma}, \hat{\gamma}), (\eta, v)) = -\lambda_h r((\beta_h, w_h), (\eta, v)) \qquad \forall (\eta, v) \in \mathbb{Q}. \tag{2.5.39}$$

Notice that problem (2.4.24)–(2.4.25) can be seen as a discretization of the load problem above. The arguments in the proof of Lemma 2.4.2 can be repeated, considering $g_h = \lambda_h \beta_h$ and $f_h = \lambda_h w_h$, to show that

$$\|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\sigma_h, \gamma_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \leq Ch\lambda_h \|(\beta_h, w_h)\|_{\mathbb{Q}} \leq Ch\lambda. \tag{2.5.40}$$

the last inequality because $\lambda_h \to \lambda$ as a consequence of Lemma 2.5.5.

On the other hand, subtracting (2.2.8)–(2.2.9) from (2.5.38)–(2.5.39), we obtain

$$a((\sigma - \hat{\sigma}, \gamma - \hat{\gamma}), (\tau, \xi)) + b((\tau, \xi), (\beta - \hat{\beta}, w - \hat{w})) = 0 \qquad \forall (\tau, \xi) \in \mathbb{H},$$

$$b((\sigma - \hat{\sigma}, \gamma - \hat{\gamma}), (\eta, v)) = -r((\lambda\beta - \lambda_h\beta_h, \lambda w - \lambda_h w_h), (\eta, v)) \qquad \forall (\eta, v) \in \mathbb{Q}.$$

As a consequence of Lemmas 2.2.1 and 2.2.2, the problem above has a unique solution (see, for instance, [46, Section II.1.1]) and there exists $C > 0$ such that

$$\begin{aligned}
\|\sigma - \hat{\sigma}\|_{1,\mathrm{I}} + \|\gamma - \hat{\gamma}\|_{1,\mathrm{I}} \leq\ & C(\|\lambda\beta - \lambda_h\beta_h\|_{0,\mathrm{I}} + \|\lambda w - \lambda_h w_h\|_{0,\mathrm{I}}) \\
\leq\ & C(\lambda\|\beta - \beta_h\|_{0,\mathrm{I}} + |\lambda - \lambda_h|\|\beta_h\|_{0,\mathrm{I}} \\
& \quad + \lambda\|w - w_h\|_{0,\mathrm{I}} + |\lambda - \lambda_h|\|w_h\|_{0,\mathrm{I}}) \\
\leq\ & Ch,
\end{aligned}$$

the last inequality because of (2.5.36) and Lemma 2.5.5.

Finally, from the above inequality and (2.5.40) we obtain (2.5.37) and the proof is complete.
□

Now we are in a position to prove a double order of convergence for the eigenvalues.

Theorem 2.5.2 *There exist strictly positive constants h_0, t_0 and C such that, for $h < h_0$ and $t < t_0$,*

$$|\lambda - \lambda_h| \leq Ch^2.$$

Proof. Let $(\lambda, (\sigma, \gamma, \beta, w))$ and $(\lambda_h, (\sigma_h, \gamma_h, \beta_h, w_h))$ be as in Lemma 2.5.6. Then, we write problem (2.2.8)–(2.2.9) and problem (2.4.24)–(2.4.25) as follows:

$$A((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)) = -\lambda B((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)),$$

$$A((\sigma_h, \gamma_h, \beta_h, w_h), (\tau_h, \xi_h, \eta_h, v_h)) = -\lambda_h B((\sigma_h, \gamma_h, \beta_h, w_h), (\tau_h, \xi_h, \eta_h, v_h)),$$

where the bilinear forms A and B are defined by

$$\begin{aligned} A((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)) &:= a((\sigma, \gamma), (\tau, \xi)) + b((\tau, \xi), (\beta, w)) + b((\sigma, \gamma), (\eta, v)), \\ B((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)) &:= r((\beta, w), (\eta, v)). \end{aligned}$$

Let $U := (\sigma, \gamma, \beta, w)$ and $U_h := (\sigma_h, \gamma_h, \beta_h, w_h)$. Then, it is easy to check the following identity (see, for instance, [8, Lemma 9.1]):

$$(\lambda - \lambda_h) B(U_h, U_h) = A(U - U_h, U - U_h) + \lambda B(U - U_h, U - U_h).$$

Now, since $B(U_h, U_h) = t^2 \int_{\mathrm{I}} \mathrm{J}\beta_h^2 + \int_{\mathrm{I}} \mathrm{P} w_h^2$ and $(\sigma_h, \gamma_h, \beta_h, w_h)$ can be seen as the solution of problem (2.4.26)–(2.4.27) with data $(g, f) = \lambda_h(\beta_h, w_h)$, as a consequence of Remark 2.4.1, we have that

$$B(U_h, U_h) \geq \frac{1}{C\lambda_h^2} \|(\beta_h, w_h)\|_{\mathbb{Q}}^2 = \frac{1}{C\lambda_h^2}.$$

Since $\lambda_h \to \lambda$ and $\lambda > 0$, for h small enough

$$B(U_h, U_h) \geq \frac{1}{2C\lambda^2},$$

the right hand side being a positive constant independent of h and t. Hence from Theorem 2.5.1 and Lemma 2.5.6, we obtain

$$|\lambda - \lambda_h| \leq Ch^2$$

and the proof is complete. □

2.6 The Euler-Bernoulli beam

The analysis above can be extended to the Euler-Bernoulli beam vibration problem (2.3.19)–(2.3.20). To simplify the notation, from now on we drop out the index 0 from eigenvalues and eigenfunctions of this problem.

The discretization of (2.3.19)–(2.3.20) reads:

Find $\lambda_h > 0$ and $0 \neq ((\sigma_h, \gamma_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\int_{\mathrm{I}} \frac{\sigma_h \tau_h}{\mathrm{E}} + \int_{\mathrm{I}} \beta_h(\tau_h' - \xi_h) - \int_{\mathrm{I}} w_h \xi_h' = 0 \qquad \forall (\tau_h, \xi_h) \in \mathbb{H}_h, \tag{2.6.41}$$

$$\int_{\mathrm{I}} \eta_h(\sigma_h' - \gamma_h) - \int_{\mathrm{I}} v_h \gamma_h' = -\lambda_h \int_{\mathrm{I}} \mathrm{P} w_h v_h \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h. \tag{2.6.42}$$

Using the same notation as in (3.5), this discrete problem can be written in matrix form as follows:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{D} & -\boldsymbol{E} \\ \boldsymbol{E}^{\mathrm{t}} & -\boldsymbol{D}^{\mathrm{t}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{E}^{\mathrm{t}} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\sigma}}_h \\ \vec{\gamma}_h \\ \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix} = -\lambda_h \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\sigma}}_h \\ \vec{\gamma}_h \\ \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix}. \tag{2.6.43}$$

To show the well-posedness of this generalized eigenvalue problem, we cannot proceed as we did to derive (2.4.33), because in this case matrix $\boldsymbol{S}$ is not positive definite and $\boldsymbol{R}$ is not invertible. However, we can use the following alternative. From (2.6.43), we know that $\boldsymbol{\sigma}_h = -\boldsymbol{A}^{-1}\boldsymbol{E}\boldsymbol{\beta}_h$. Using this in the third row we obtain $\boldsymbol{E}^t\boldsymbol{A}^{-1}\boldsymbol{E}\boldsymbol{\beta}_h + \boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h = 0$ and rewrite (2.6.43) as follows:

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{E} \\ \boldsymbol{D}^{\mathrm{t}} & \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{E}^{\mathrm{t}} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \vec{\gamma}_h \\ \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix} = \lambda_h \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \vec{\gamma}_h \\ \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix}, \tag{2.6.44}$$

where $\boldsymbol{G} := \boldsymbol{E}^{\mathrm{t}}\boldsymbol{A}^{-1}\boldsymbol{E}$. This matrix is symmetric and positive definite, because $\boldsymbol{A}$ so is and $\ker \boldsymbol{E} = \{0\}$.

For $\lambda_h \neq 0$, (2.6.44) is equivalent to

$$\begin{aligned} \boldsymbol{E}\boldsymbol{Q}^{-1}\boldsymbol{E}^{\mathrm{t}}\boldsymbol{\gamma}_h + \lambda_h\boldsymbol{D}\boldsymbol{\beta}_h &= 0, \\ \boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h + \boldsymbol{G}\boldsymbol{\beta}_h &= 0. \end{aligned}$$

In its turn, by substituting $\boldsymbol{\beta}_h = -\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h$ into the first equation, the problem above turns out equivalent to

$$\begin{aligned} \boldsymbol{E}\boldsymbol{Q}^{-1}\boldsymbol{E}^{\mathrm{t}}\boldsymbol{\gamma}_h = \lambda_h\boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h &\iff (\underbrace{\boldsymbol{E}\boldsymbol{Q}^{-1}\boldsymbol{E}^{\mathrm{t}} + \boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}}}_{\boldsymbol{H}})\boldsymbol{\gamma}_h = (\lambda_h + 1)\boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h \\ &\iff \boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h = \mu_h\boldsymbol{H}\boldsymbol{\gamma}_h, \end{aligned} \tag{2.6.45}$$

where $\mu_h = 1/(\lambda_h + 1)$.

Our next goal is to prove that $\boldsymbol{H}$ is symmetric and positive definite. With this aim, we observe that

$$\boldsymbol{E}^{\mathrm{t}}\boldsymbol{\gamma}_h = 0 \iff \int_{\mathrm{I}} \gamma_h' v_h = 0 \qquad \forall v_h \in Z_h. \tag{2.6.46}$$

Testing (2.6.46) with $v_h = \gamma_h'$, we have that $\int_{\mathrm{I}} |\gamma_h'|^2 = 0$, which implies that $\gamma_h \in \mathbb{P}_0$; namely, $\ker \boldsymbol{E}^{\mathrm{t}} = \mathbb{P}_0$.

On the other hand

$$\boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h = 0 \iff \int_{\mathrm{I}} \gamma_h \eta_h = 0 \ \ \forall \eta_h \in Z_h \iff \int_{\mathrm{I}_j} \gamma_h = 0, \ \ j = 1, \dots, N.$$

Consequently, $\boldsymbol{\gamma}_h$ has to be a multiple of the function $\gamma_h^0 \in Z_h$ defined by $\gamma_h^0(x_i) = (-1)^i$, $0 \leq i \leq N$; namely, $\ker \boldsymbol{D}^{\mathrm{t}} = \langle \gamma_h^0 \rangle$.

Therefore, since $\boldsymbol{Q}$ is positive definite, $\boldsymbol{\gamma}_h^{\mathrm{t}}(\boldsymbol{E}\boldsymbol{Q}^{-1}\boldsymbol{E}^{\mathrm{t}})\boldsymbol{\gamma}_h > 0$ if and only if $\boldsymbol{\gamma_h} \notin \mathbb{P}_0$, whereas, since $\boldsymbol{G}$ is positive definite, $\boldsymbol{\gamma}_h^{\mathrm{t}}(\boldsymbol{D}\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}})\boldsymbol{\gamma}_h > 0$ if and only if $\boldsymbol{\gamma}_h \notin \langle \gamma_h^0 \rangle$. Thus, since $\mathbb{P}_0 \cap \langle \gamma_h^0 \rangle = \{0\}$, we conclude that $\boldsymbol{H}$ is positive definite.

Note that $\mu_h^0 = 0$ is an eigenvalue of problem (2.6.45) with eigenfunction γ_h^0. The rest of the spectrum are eigenvalues $\mu_h^i \in (0,1]$ (which correspond to $\lambda_h = (1/\mu_h^i) - 1 \geq 0$) with eigenfunctions $\boldsymbol{\gamma}_h^i$, $i = 1, \ldots, N$. One of these eigenvalues is $\mu_h^i = 1$ (which correspond to $\lambda_h^N = 0$) with corresponding eigenfunction $\boldsymbol{\gamma}_h^N \in \mathbb{P}_0$.

For $1 \leq i \leq N-1$, defining $\boldsymbol{\beta}_h^i := -\boldsymbol{G}^{-1}\boldsymbol{D}^{\mathrm{t}}\boldsymbol{\gamma}_h^i$; $\boldsymbol{w}_h^i := (1/\lambda_h^i)\boldsymbol{Q}^{-1}\boldsymbol{E}^{\mathrm{t}}\boldsymbol{\gamma}_h^i$ and $\boldsymbol{\sigma}_h^i := -\boldsymbol{A}^{-1}\boldsymbol{E}\boldsymbol{\beta}_h^i$, we have that $(\lambda_h^i, (\boldsymbol{\sigma}_h^i, \boldsymbol{\gamma}_h^i, \boldsymbol{\beta}_h^i, \boldsymbol{w}_h^i))$ are eigenpairs of problem (2.6.43). The remaining solution of (2.6.45), $\lambda_h^N = 0$, with $\gamma_h^N \in \mathbb{P}_0$, does not lead a solution of problem (2.6.43). In fact, $\lambda_h^N = 0$ cannot be an eigenvalue of this problem, since the matrix on its left-hand side is invertible. Thus we are led to the following result.

Proposition 2.6.1 *Problem* (2.6.43) *has* $N-1$ *eigenvalues, repeated according to their respective multiplicities.*

Since all the results from [60] used for the theoretical analysis remain valid for $t = 0$, the same happens with the results of the present paper. In particular the $\mathcal{O}(h)$ estimates for the eigenspaces from Theorem 1 and the $\mathcal{O}(h^2)$ estimate for the eigenvalues from Theorem 2 hold true for the finite element approximation (2.6.41)–(2.6.42) of the Euler-Bernoulli beam vibration problem.

2.7 Numerical results

We report in this section the results of some numerical tests computed with a MATLAB code implementing the finite element method described above. For all the tests we have considered a clamped beam of length L and uniform meshes of N elements, with different values of N.

In all the tests, we have used the following physical parameters:

- Young modulus: $E = 2.1 \times 10^6$ Kgf/cm^2, (1 Kgf= 980 kg/cm^2),
- Poisson ratio: $\nu = 0.3$,
- Density: $\rho = 7.85 \times 10^{-3}$ kg/cm^3,
- Correction factor: $k_c = 1$.

2.7.1 Implementation

The generalized eigenvalue problem that has to be solved has been written in matrix form into the proof of Lemma 2.4.1 (cf. (3.5)). This is a degenerate matrix generalized eigenvalue problem since none of the matrices is positive definite. Therefore, its solution would need of some specialized software. Alternatively, problem (3.5) has been equivalently written as (2.4.33), where both matrices are symmetric and positive definite. However, on the left hand side there is a full matrix, because $\boldsymbol{R}^{-1}$ is full too. Therefore, (2.4.33) is not appropriate for the computer solution of the problem, either.

Instead, we proceed from (5.3.23) as follows: From the second equation $\boldsymbol{T}\vec{\boldsymbol{v}}_h = -\frac{1}{\lambda_h}\boldsymbol{S}^{\mathrm{t}}\vec{\boldsymbol{u}}_h$ and, since $\boldsymbol{T}$ is invertible, $\vec{\boldsymbol{v}}_h = -\frac{1}{\lambda_h}\boldsymbol{T}^{-1}\boldsymbol{S}^{\mathrm{t}}\vec{\boldsymbol{u}}_h$. Substituting this into the first equation of (5.3.23) we arrive at

$$(\boldsymbol{S}\boldsymbol{T}^{-1}\boldsymbol{S}^{\mathrm{t}})\vec{\boldsymbol{u}}_h = \lambda_h \boldsymbol{R}\vec{\boldsymbol{u}}_h. \tag{2.7.47}$$

Matrix $\boldsymbol{R}$ is symmetric and positive definite, whereas $(\boldsymbol{S}\boldsymbol{T}^{-1}\boldsymbol{S}^{\mathrm{t}})$ is symmetric and positive semi-definite. Thus, this generalized eigenvalue problem can be solved with standard software. Moreover, since $\boldsymbol{T}$ is formed by two mass matrices with piecewise constant elements, it is diagonal. Hence to compute $\boldsymbol{T}^{-1}$ is completely inexpensive and the matrix $(\boldsymbol{S}\boldsymbol{T}^{-1}\boldsymbol{S}^{\mathrm{t}})$ result as sparse as $\boldsymbol{R}$. The only minor drawback is that the eigenvalue problem (2.7.47) has the spurious eigenvalue $\lambda_h = 0$ with multiplicity 2. Since $\boldsymbol{T}^{-1}$ is positive definite, the eigenspace associated to $\lambda_h = 0$ is the kernel of $\boldsymbol{S}$. Using the standard basis of the finite element spaces W_h (piecewise linear and continuous elements) and Z_h (piecewise constant elements) it is possible to prove that if $\vec{\boldsymbol{u}}_h = (\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\gamma}}_h)^{\mathrm{t}}$ with $\vec{\boldsymbol{\sigma}}_h$ and $\vec{\boldsymbol{\gamma}}_h$ being the vector of nodal components of $\sigma_h \in W_h$ and $\gamma_h \in W_h$, respectively, thus $(\boldsymbol{S}\vec{\boldsymbol{u}}_h)_i = \int_{\mathrm{I}_j}(\sigma_h' - \gamma_h),\ i = 1, \ldots, N$. Therefore, $\vec{\boldsymbol{u}}_h \in \ker \boldsymbol{S}$ implies that either $\gamma_h = 0$ and σ_h is constant or γ_h is constant and $\sigma_h' = \gamma_h$. Thus, the eigenspace of $\lambda_h = 0$ in problem (2.7.47) is spanned by $(1, 0) \in \mathbb{H}_h$ and $(x, 1) \in \mathbb{H}_h$.

2.7.2 Test 1: Uniform beam with analytical solution

The aim of this first test is to validate the computer code by solving a problem with known analytical solution. With this purpose, we will compare the exact vibration frequencies of a uniform clamped beam as that shown in Figure 2.1 (undeformed beam) with those computed with the method analyzed in this paper.

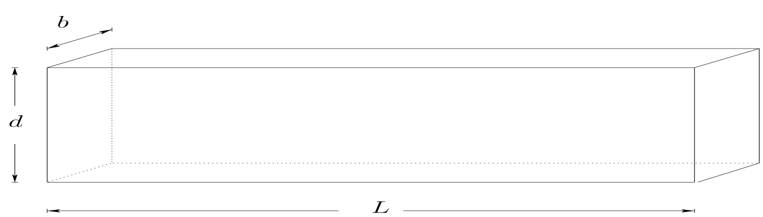

Figure 2.1: Undeformed uniform beam.

We note also that for this kind of beam, we have that $\mathbb{I} = \frac{bd^3}{12}$ and $A = bd$ are constant.

In Table 2.1 we report the three lowest vibration frequencies computed by our method with four different meshes ($N = 16, 32, 64, 128$). We have taken $L = 120$ cm and a square cross section of side-length $b = d = 20$ cm. The table includes computed orders of convergence and the exact vibration frequencies.

Table 2.1: Angular vibration frequencies of a uniform beam.

Mode	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Exact
ω_1^h	4017.49	4000.74	3996.84	3995.90	2.1	3995.61
ω_2^h	9778.27	9644.64	9613.68	9606.23	2.1	9603.80
ω_3^h	170614.73	16621.41	16520.22	16495.89	2.1	16487.94

It can be seen from Table 2.1 that the computed frequencies converge to the exact ones with an optimal quadratic order.

2.7.3 Test 2: Beam with a smoothly varying cross-section.

In this test we apply the method analyzed in this paper to a beam of rectangular section with smoothly varying thickness. With this purpose, we consider a beam as that shown in Figure 2.2.

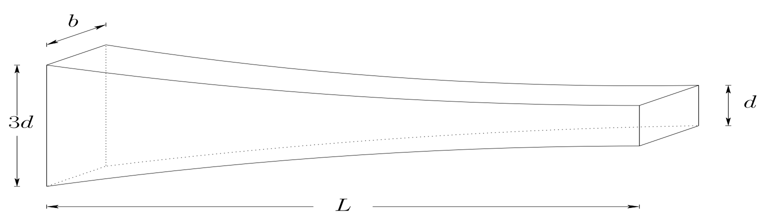

Figure 2.2: Smoothly varying cross-section beam.

Let b and d be as shown in Figure 2.2. We have taken $L = 100$, $b = 3$ and $d = 3$ cm. The equation of the top and bottom surfaces of the beam are

$$z = \pm \frac{150d}{2x + 100}, \quad 0 \leq x \leq 100.$$

Hence, the area of the cross section and the moment of inertia are given by

$$A(x) = \frac{900d}{2x + 100}, \qquad \mathbb{I}(x) = \frac{1}{4}\left(\frac{300d}{2x + 100}\right)^3, \qquad 0 \leq x \leq 100.$$

In Table 2.2 we report the four lowest vibration frequencies computed by our method with four different meshes ($N = 16, 32, 64, 128$). The table includes computed orders of convergence as well as more accurate values obtained by means of a least-squares fitting.

Table 2.2: Angular vibration frequencies of a beam with a smoothly varying cross-section.

Mode	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Extrap.
ω_1^h	1674.8167	1667.2007	1665.2819	1664.8012	2.03	1664.6419
ω_2^h	4382.5912	4308.8768	4290.4391	4285.8294	2.03	4284.3014
ω_3^h	8432.5758	8139.6797	8067.2309	8049.1697	2.03	8043.1848
ω_4^h	13875.8820	13078.9166	12884.6634	12836.4208	2.03	12820.4405

It can be seen from Table 2.2 that the computed vibration frequencies also converge with an optimal quadratic order as predicted by the theoretical results.

We show in Figure 2.3 the deformed beam for the four lowest vibration modes.

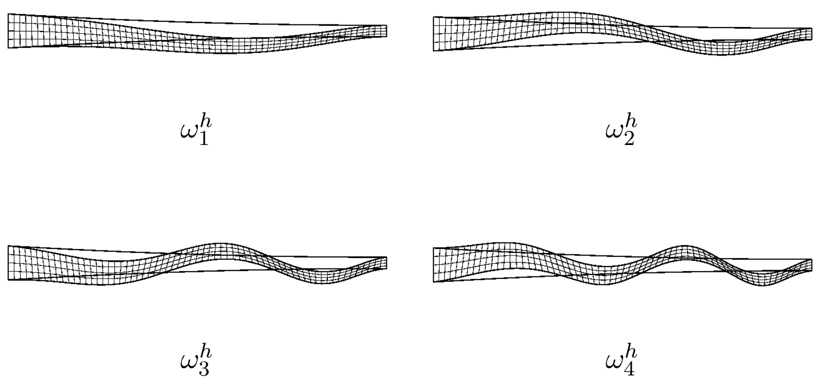

Figure 2.3: Smoothly varying cross-section beam; four vibration modes with lowest frequency.

2.7.4 Test 3: Rigidly joined beams.

The aim of this test is to apply the method analyzed in this paper to a beam with area varying discontinuously along its axis. With this purpose, we consider a composed beam formed by two rigidly joined beams as shown in Figure 2.4. Moreover, we will assess the performance of the method as the thickness d approaches to zero to check that the proposed method is thoroughly locking-free.

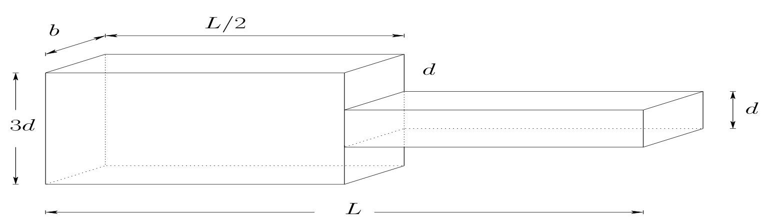

Figure 2.4: Rigidly joined beams.

Let b and d be as shown in Figure 2.4. We have taken $L = 100$ and $b = 3$, so that the area of the cross section and the moment of inertia are:

$$A(x) = \begin{cases} 9d, & 0 \leq x \leq 50, \\ 3d, & 50 < x \leq 100, \end{cases} \qquad \mathbb{I}(x) = \begin{cases} \frac{27d^3}{4}, & 0 \leq x \leq 50, \\ \frac{d^3}{4}, & 50 < x \leq 100. \end{cases}$$

We have used uniform meshes with an even number N of elements, so that the point $x = L/2$ is always a node of the mesh as required by the theory.

In Table 2.3 we present the results for the lowest computed rescaled eigenvalue $\lambda_1^h = \left(\omega_1^h/t\right)^2$, with varying thickness d and different meshes. According to (2.2.2), the non dimensional parameter t is given in this case by $t^2 = \frac{5d^2}{8L^2}$. Again, we have computed the orders of convergence and more accurate extrapolated values by means of a least-squares fitting.

The results from Table 2.3 show clearly that the method does not deteriorate when the thickness parameter becomes small, thus we may conclude that the method is locking-free.

Table 2.3: Lowest rescaled eigenvalue λ_h^1 (multiplied by 10^{-10}) of a composed beam with varying thickness d.

Thickness	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Extrap.
$d = 4$	4.72371	4.68871	4.67989	4.67768	2.03	4.67695
$d = 0.4$	5.00424	4.96518	4.95534	4.95288	2.03	4.95207
$d = 0.04$	5.00724	4.96813	4.95829	4.95582	2.03	4.95500
$d = 0.004$	5.00727	4.96816	4.95831	4.95585	2.03	4.95503

Finally, we show in Figure 2.5 the deformed beam for the two lowest frequency vibration modes.

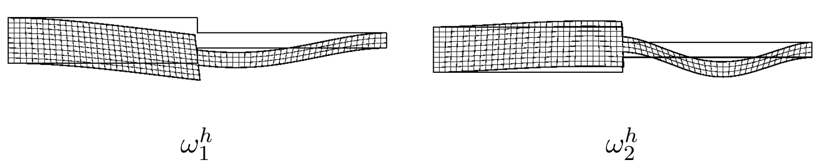

Figure 2.5: Rigidly joined beams; two lowest frequency vibration modes.

Chapter 3

Acoustic vibration problem for dissipative fluids

3.1 Introduction

This paper deals with the numerical approximation of an acoustic dissipative fluid system. This kind of problem has attracted much interest, since it is frequently encountered in engineering applications (see, for instance, [14, 54, 76]). One typical example is to achieve optimal designs that reduce noise and vibrations in fluid-structure systems like cars, aircraft or dams.

Although dissipation is usually neglected in standard acoustics, modeling this phenomenon is essential to study the effect of noise reduction techniques. Indeed, in most real situations, damping mechanisms that transform mechanical energy into heat do exist. Sometimes these mechanisms are based on surface damping arising from viscoelastic materials placed on the boundary of the propagation domain. In these cases, the dissipative effects are typically included in the model by means of a surface impedance in the boundary conditions (see, for instance, [13, 16, 17]). The present paper addresses damping when it arises in the propagation media itself due to friction and heat conduction. A general approach to this topic can be found in the books by Landau and Lifshitz [56], Morse [74], and Pierce [81], all of which include extensive bibliographic references on the subject.

This paper focus on computing the (complex) vibration frequencies and modes of an acoustic dissipative fluid system within a rigid cavity. One motivation for considering this problem is that it constitutes a stepping stone towards the more challenging goal of devising numerical approximations for coupled systems involving fluid-structure interaction between viscous fluids and solid structures. The natural model for the fluid system should be based on the Stokes equations for compressible fluids. However, since in real applications the viscosity is typically very small, the resulting problem turns out a singular perturbation of that for an inviscid fluid. This fact leads to a kind of dilemma, since appropriate finite elements for the Stokes equations introduce spurious modes in the limit case of a vanishing viscosity, whereas the finite elements that avoid such spectral pollution fail when applied to the Stokes equation.

To circumvent this drawback, we resort to an alternative model based on a curl-free displacement formulation (see [18] for the derivation of a similar model in the time domain from basic mechanical laws). Let us remark that in principle the fluid displacement does not need to be curl-free. However, since the viscosity term due to vorticity is typically very small, except perhaps near the walls of the enclosure, it may be neglected in the interior of the enclosure and eventually modeled as a wall impedance on its boundary (see [76] for a similar model).

This curl-free displacement formulation for a viscous fluid leads to a quadratic eigenvalue problem (QEP), as it happens in [13]. However, the resulting problem involves additional challenges related to the fact that the essential spectrum does not reduce to a single point as in [13]. In fact, in this case, we can only prove that the essential spectrum is well separated from the physically relevant eigenvalues when the viscosity is sufficiently small (as it happens in practice). On the other hand, the associated solution operator is not regularizing. Because of this, we need to split it into two terms for the numerical analysis. One of them is dealt with the techniques from [13], but the spectral approximation analysis for the other is new.

As is shown below (cf. Remark 2.1), the vibration frequencies and modes of a viscous homogeneous irrotational fluid within a rigid cavity can be obtained without actually solving a QEP. In fact, these frequencies can be algebraically computed from those of the analogous inviscid fluid, whose approximation is nowadays a well known subject (see, for instance, [14]). However, this is not the case for a heterogeneous fluid and this is the reason why we choose this as our model problem. In particular, we consider the QEP arising from the acoustic vibration problem for a dissipative fluid system that consists of two homogeneous viscous immiscible fluids contained in a rigid cavity.

QEP has many applications in the study of vibration for solid systems, acoustic fluids, electrical circuits etc., where the damping effects are involved. A state of art for the QEP up to the beginning of this century can be found in [88]. However, there are not many works with a rigorous mathematical framework in the context of the numerical approximation of the eigenvalue problem of a partial differential operator involving damping. The first article proving this type of results is [13], where the authors have considered a displacement formulation for a fluid in a rigid cavity with absorbing walls. The theory of non-compact operators of [37] is used to obtain error estimates with minor modifications due the non-conformity.

On the other hand, alternative formulations for the absorbing wall problem have been studied in the Engineering literature. For instance, a formulation for the QEP in terms of the fluid pressure has been proposed in [57]. This type of formulation leads to a rational eigenvalue problem, for which different algorithms to compute the vibration frequencies have been introduced. Alternatively, two formulations one based on the fuid pressure and the other on the fluid displacement, have been considered in [34], where an improved Arnoldi algorithm have been proposed to solve the discrete problem. On the other hand, an application of the damping effects in electromechanical-thermoelastic systems is presented in [3]. Moreover, an a-posteriori estimator for a QEP contextualized in the photonic crystal applications is proposed in [42]. Nevertheless, all the previous mentioned studies present different numerical technologies to solve the QEP, but without a rigorous mathematical analysis. Such a rigorous analysis is present instead in the recent paper [67], where an efficient multiscale technique based on localized orthogonal decompositions to solve discrete generic damped vibration problems has been proposed (see also

[66]).

In the present paper, we consider a displacement based variational formulation of the transmission eigenproblem resulting from our physical model. This approach leads to a QEP, which is transformed into an equivalent double-size linear eigenvalue problem that fits within the functional framework for nonselfadjoint and noncompact bounded operators. At the continuous level, we follow [55] to obtain an appropriate spectral characterization. Next, we propose an $H(\mathrm{div})$-conforming mixed finite element approximation of the problem and adapt the abstract spectral approximation theory for noncompact operators developed in [36, 37] to prove that the spectrum is correctly approximated and to obtain error estimates.

The discrete analysis relies partly on the techniques used in the Raviart-Thomas mixed approximation of the **grad**-div eigenvalue problem. This spectral problem emerged in the study of coupled fluid-solid systems [12] (see also [69] for a similar setting in elasticity). The **grad**-div spectral problem is posed in H(div) and it is closely related to the Maxwell eigenvalue problem, which is formulated in terms of the **curl-curl** operator in H(**curl**). Although the two spectral problems have been initially studied in isolation from each other, a common framework becomes now clear thanks to the language of differential forms and the approach based on the finite element exterior calculus provided by [6] (see also [20, Part 4]).

The negative impact that material parameters have on the regularity of the solution of the boundary value problem complicates the analysis in this common framework (see [52, Remark 13]). Here, we follow the lines of the methodology presented originally in [12] and use the information about interface singularities of solutions of the Neumann boundary value problem for $\mathrm{div}(\kappa\, \mathbf{grad})$, with κ piecewise constant.

The paper is organized as follows: in Section 3.2, we introduce the spectral problem and the corresponding variational formulation, which leads to a quadratic eigenvalue problem. We introduce an auxiliary unknown to transform the quadratic eigenvalue problem into a linear one. Moreover, we introduce the corresponding solution operator for the spectral problem. In Section 3.3, we provide a thorough spectral characterization of the solution operator, based on the theory developed in [55]. We also consider the limit problem (i.e., the case when the viscosity vanishes) and the relation between the solutions of the dissipative and non-dissipative problems. In Section 3.4, we introduce a finite element discretization using Raviart-Thomas elements for both fluids and imposing the continuity of the corresponding normal components on the interface. We analyze the discrete spectral problem analogously as in the continuous case and introduce the corresponding discrete solution operator. We use the abstract theory from [36] to prove the convergence. We also prove error estimates for our problem by adapting the arguments from [13]. Finally, in Section 5.5, we report some numerical tests which allow us to asses the performance of the proposed method.

Throughout the paper, Ω is a generic Lipschitz bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$, with outer unit normal vector $\boldsymbol{n}$. We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth function compactly supported in Ω. For $r \geq 0$, $\|\cdot\|_{r,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^r(\Omega)$ or $\mathrm{H}^r(\Omega)^d$ with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define the Hilbert space $\mathrm{H}(\mathrm{div};\Omega) := \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^d : \ \mathrm{div}\,\boldsymbol{v} \in \mathrm{L}^2(\Omega)\}$, whose norm is given by $\|\boldsymbol{v}\|^2_{\mathrm{div},\Omega} := \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Omega}$, and its subspace $\mathrm{H}_0(\mathrm{div};\Omega) := \{\boldsymbol{v} \in \mathrm{H}(\mathrm{div};\Omega) : \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

Finally, C represents a generic constant independent of the discretization parameters, which may take different values at different places.

3.2 The model problem

We take as our model problem the case of two immiscible fluids within a rigid cavity. Let Ω_i with $i = 1, 2$ be the polygonal (in the 2D case) or polyhedral (in the 3D case) Lipschitz domains occupied by each of the fluids. Let ρ_i be the corresponding densities, ν_i the fluid viscosities, and c_i the acoustic speeds, which we consider all constant, ρ_i and c_i strictly positive and ν_i non negative. We denote by $\boldsymbol{n}_i$ the outward unit normal vectors corresponding to each subdomain. We define $\Omega := (\overline{\Omega}_1 \cup \overline{\Omega}_2)^\circ$, $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$, and $\Gamma_i := \partial\Omega_i \cap \partial\Omega$, $i = 1, 2$. We assume that each domain Ω_i as well as Ω are simply connected (see Figure 3.1).

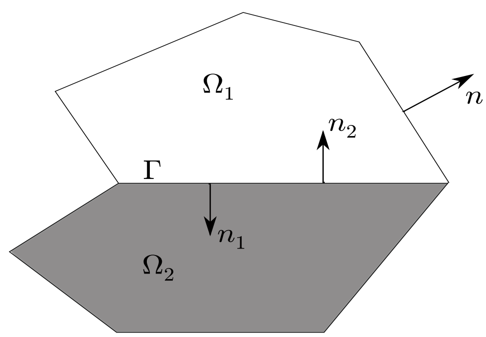

Figure 3.1: 2D sketch of the polygonal domains for the fluids.

We consider small displacements of a compressible viscous fluid at rest neglecting convective terms. The equation of motion derived from the Navier-Stokes equation reads

$$\rho_i \ddot{\boldsymbol{U}}_i = 2\nu_i \Delta \dot{\boldsymbol{U}}_i - \nabla P_i \quad \text{in } \Omega_i$$

where $\boldsymbol{U}_i$ denotes the fluid displacement and P_i the pressure fluctuation in the domain Ω_i, $i = 1, 2$. The dot represents derivation with respect to time. (See [18] and [81] for a more detailed derivation). Moreover, since the fluid is compressible, we consider the isentropic relation

$$P_i + \rho_i c_i^2 \operatorname{div} \boldsymbol{U}_i = 0 \qquad \text{in } \Omega_i. \tag{3.2.1}$$

Since we are considering irrotational fluids, we assume that $\mathbf{curl}\, \boldsymbol{U}_i = \mathbf{0}$. Hence, considering the identity $\Delta \dot{\boldsymbol{U}}_i = \nabla(\operatorname{div} \dot{\boldsymbol{U}}_i) - \mathbf{curl}(\mathbf{curl}\, \dot{\boldsymbol{U}}_i)$, we conclude that $\Delta \dot{\boldsymbol{U}}_i = \nabla(\operatorname{div} \dot{\boldsymbol{U}}_i)$. Then, the equations of our model problem are the following:

$$\rho_1 \ddot{\boldsymbol{U}}_1 - 2\nu_1 \nabla(\operatorname{div} \dot{\boldsymbol{U}}_1) + \nabla P_1 = \mathbf{0} \quad \text{in } \Omega_1 \times (0,T), \tag{3.2.2}$$

$$P_1 + \rho_1 c_1^2 \operatorname{div} \boldsymbol{U}_1 = 0 \quad \text{in } \Omega_1 \times [0,T], \tag{3.2.3}$$

$$\rho_2 \ddot{\boldsymbol{U}}_2 - 2\nu_2 \nabla(\operatorname{div} \dot{\boldsymbol{U}}_2) + \nabla P_2 = \mathbf{0} \quad \text{in } \Omega_2 \times (0,T), \tag{3.2.4}$$

$$P_2 + \rho_2 c_2^2 \operatorname{div} \boldsymbol{U}_2 = 0 \quad \text{in } \Omega_2, \times [0,T], \tag{3.2.5}$$

$$\boldsymbol{U}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{U}_2 \cdot \boldsymbol{n}_2 = 0 \quad \text{on } \Gamma \times [0,T], \tag{3.2.6}$$

$$(2\nu_1 \operatorname{div} \dot{\boldsymbol{U}}_1 + P_1) - (2\nu_2 \operatorname{div} \dot{\boldsymbol{U}}_2 + P_2) = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.2.7}$$

$$\boldsymbol{U}_1 \cdot \boldsymbol{n}_1 = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.2.8}$$

$$\boldsymbol{U}_2 \cdot \boldsymbol{n}_2 = 0 \quad \text{on } \Gamma_2 \times (0,T). \tag{3.2.9}$$

Let us remark that a similar argument leads exactly to the same equations in 2D.

Multiplying equations (3.2.2) and (3.2.4) by a test function $\boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega)$, integrating by parts, and using the boundary conditions and the transmission conditions on Γ, we obtain

$$\int_\Omega \rho \ddot{\boldsymbol{U}} \cdot \boldsymbol{v} + 2\int_\Omega \nu \operatorname{div} \dot{\boldsymbol{U}} \operatorname{div} \boldsymbol{v} - \int_\Omega P \operatorname{div} \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div},\Omega), \tag{3.2.10}$$

where

$$\boldsymbol{U} := \begin{cases} \boldsymbol{U}_1 & \text{in } \Omega_1, \\ \boldsymbol{U}_2 & \text{in } \Omega_2, \end{cases} \quad P := \begin{cases} P_1 & \text{in } \Omega_1, \\ P_2 & \text{in } \Omega_2, \end{cases} \quad \nu := \begin{cases} \nu_1 & \text{in } \Omega_1, \\ \nu_2 & \text{in } \Omega_2, \end{cases}$$

$$\rho := \begin{cases} \rho_1 & \text{in } \Omega_1, \\ \rho_2 & \text{in } \Omega_2, \end{cases} \quad \text{and } c := \begin{cases} c_1 & \text{in } \Omega_1, \\ c_2 & \text{in } \Omega_2. \end{cases}$$

Using (3.2.3) and (3.2.5) we eliminate P in (3.2.10) and write

$$\int_\Omega \rho \ddot{\boldsymbol{U}} \cdot \boldsymbol{v} + 2\int_\Omega \nu \operatorname{div} \dot{\boldsymbol{U}} \operatorname{div} \boldsymbol{v} + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{U} \operatorname{div} \boldsymbol{v} = 0 \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div},\Omega). \tag{3.2.11}$$

The *vibration modes* of this problem are complex solutions of the form $\boldsymbol{U}(\boldsymbol{x},t) = e^{\lambda t}\boldsymbol{u}(\boldsymbol{x})$ with $\lambda \in \mathbb{C}$. Looking for this kind of solutions leads to the following quadratic eigenvalue problem:

Problem 3.2.1 *Find* $\lambda \in \mathbb{C}$ *and* $\mathbf{0} \neq \boldsymbol{u} \in \mathrm{H}_0(\operatorname{div};\Omega)$ *such that*

$$\lambda^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + 2\lambda \int_\Omega \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega).$$

Let us remark that in absence of viscosity (i.e., $\nu = 0$) we are left with the free vibration problem of two inviscid fluids in contact (whose numerical approximation has not been analyzed either). The eigenvalues λ^2 of such a problem are negative real numbers (as will be proved below), so that λ are purely imaginary, namely, $\lambda = \pm i\omega$ with ω being the so called *natural vibration frequencies*, which correspond to periodic in time solutions $\boldsymbol{U}(\boldsymbol{x},t) = e^{-i\omega t}\boldsymbol{u}(\boldsymbol{x})$ of the time domain problem. This is the reason why, for $\nu = 0$, Problem 3.2.1 is usually written as follows: Find $\omega > 0$ and $\mathbf{0} \neq \boldsymbol{u} \in \mathrm{H}_0(\operatorname{div};\Omega)$ such that

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = \omega^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega). \tag{3.2.12}$$

In the applications, ν is typically very small. As we will show below, in such a case there are eigenvalues λ of Problem 3.2.1 that lie close to $\pm i\omega$ with ω being a natural vibration frequency (i.e., a solution of (3.2.12)). Actually, we will prove below that those λ converge to $\pm i\omega$ as $\|\nu\|_{\infty,\Omega}$ goes to zero. On solving Problem 3.2.1, the aim is to compute the eigenvalues λ close to the smallest natural vibration frequencies $\omega > 0$, which are the most relevant in the applications.

Remark 3.2.1 *In the case of a homogeneous viscous fluid, ρ, c and ν are constant in the whole Ω. Then, Problem 3.2.1 can be written as*

$$\lambda^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \frac{2\lambda\nu + \rho c^2}{\rho c^2} \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \qquad \forall \boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}, \Omega).$$

Hence, in such a case, $(\lambda, \boldsymbol{u})$ is an eigenpair of Problem 3.2.1 if and only if $-\frac{\lambda^2 \rho c^2}{2\lambda\nu + \rho c^2} = \omega^2$ with $(\omega, \boldsymbol{u})$ being a solution to problem (3.2.12). Therefore, for a homogeneous viscous fluid, λ can be algebraically computed from the solution of (3.2.12) as follows:

$$\lambda = \frac{-\nu\omega^2 \pm \sqrt{\nu^2\omega^4 - \rho^2 c^4 \omega^2}}{\rho c^2}.$$

We denote $\mathcal{H} := \mathrm{L}^2(\Omega)^d$ endowed with the weighted inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{H}} := \int_\Omega \rho \boldsymbol{v} \cdot \overline{\boldsymbol{w}}$$

and $\mathcal{V} := \mathrm{H}_0(\operatorname{div}; \Omega)$ with the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{V}} := \int_\Omega \rho \boldsymbol{v} \cdot \overline{\boldsymbol{w}} + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{v} \operatorname{div} \overline{\boldsymbol{w}}.$$

Notice that the inner products in $\mathcal{H}$ and $\mathcal{V}$ induce norms $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{V}}$ on each of these spaces equivalent to the classical $\mathrm{L}^2(\Omega)^d$ and $\mathrm{H}(\operatorname{div}; \Omega)$ norms, respectively. Therefore, when it might be convenient, we will use these classical norms.

Clearly $\lambda = 0$ is an eigenvalue of Problem 3.2.1 with associated eigenspace

$$\mathcal{K} = \mathrm{H}_0(\operatorname{div}^0, \Omega) := \{\boldsymbol{v} \in \mathrm{H}_0(\operatorname{div}; \Omega) : \ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}.$$

We define:

$$\mathcal{G} := \mathcal{K}^{\perp_{\mathcal{V}}} = \{\boldsymbol{v} \in \mathcal{V} : (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{V}} = 0 \quad \forall \boldsymbol{w} \in \mathcal{K}\}.$$

Since $\mathcal{K}$ is a closed subspace of $\mathcal{V}$, clearly $\mathcal{V} = \mathcal{G} \oplus \mathcal{K}$. Notice that $\mathcal{G}$ and $\mathcal{K}$ are also orthogonal in the $\mathcal{H}$ inner product. Hence,

$$\mathcal{G} = \{\boldsymbol{v} \in \mathcal{V} : (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{H}} = 0 \ \ \forall \boldsymbol{w} \in \mathcal{K}\}.$$

The following result brings a characterization of the space $\mathcal{G}$.

Lemma 3.2.1 *There holds*

$$\mathcal{G} = \frac{1}{\rho} \nabla(\mathrm{H}^1(\Omega)) \cap \mathcal{V}.$$

Proof. We will prove this result by checking the double inclusion. Let $\boldsymbol{v} \in \mathcal{G}$. Then, for all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)^d$, since $\mathbf{curl}\,\boldsymbol{\psi} \in \mathcal{K}$, we have

$$0 = \int_\Omega \rho\,\mathbf{curl}\,\boldsymbol{\psi} \cdot \overline{\boldsymbol{v}} = \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\,(\rho \overline{\boldsymbol{v}})\,.$$

Thus, $\mathbf{curl}\,(\rho \boldsymbol{v}) = \mathbf{0}$ in Ω. Since Ω is simply connected, this implies that there exists $\varphi \in \mathrm{H}^1(\Omega)$ such that $\rho \boldsymbol{v} = \nabla\varphi$. Hence, $\boldsymbol{v} \in \frac{1}{\rho}\nabla(\mathrm{H}^1(\Omega)) \cap \mathcal{V}$. Conversely, let $\boldsymbol{v} \in \frac{1}{\rho}\nabla(\mathrm{H}^1(\Omega)) \cap \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{K}$. Let $\varphi \in \mathrm{H}^1(\Omega)$ be such that $\boldsymbol{v} = \frac{1}{\rho}\nabla\varphi$. Then,

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{H}} = \int_\Omega \rho \left(\frac{1}{\rho}\nabla\varphi\right) \cdot \overline{\boldsymbol{w}} = -\int_\Omega \varphi \operatorname{div} \overline{\boldsymbol{w}} + \int_{\partial\Omega} \varphi(\overline{\boldsymbol{w}} \cdot \boldsymbol{n}) = 0.$$

Therefore, $\boldsymbol{v} \in \mathcal{G}$. The proof is complete. □

In what follows we prove additional regularity for the functions in $\mathcal{G}$ on each subdomain. From now on, s will denote a positive number such that the following lemma holds true.

Lemma 3.2.2 *There exists $s > 0$ (with s depending on ρ, Ω_1 and Ω_2) such that, for all $\boldsymbol{v} \in \mathcal{G}$, $\boldsymbol{v} \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$ and*

$$\|\boldsymbol{v}\|_{s,\Omega_1} + \|\boldsymbol{v}\|_{s,\Omega_2} \leq C\|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}, \tag{3.2.13}$$

where C is a positive constant independent of $\boldsymbol{v}$.

Proof. According to Lemma 5.2.14, there exists $\varphi \in \mathrm{H}^1(\Omega)$ such that $\boldsymbol{v} = \frac{1}{\rho}\nabla\varphi$. Consequently, $\varphi \in \mathrm{H}^1(\Omega)/\mathbb{C}$ is the unique solution of the following well-posed Neumann problem:

$$\begin{aligned} \operatorname{div}\left(\frac{1}{\rho}\nabla\varphi\right) &= \operatorname{div} \boldsymbol{v} \quad \text{in } \Omega, \\ \frac{1}{\rho}\frac{\partial\varphi}{\partial\boldsymbol{n}} &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}$$

Hence, in the 3D case, the result follows from [78, Lemma 2.20], while in the 2D case, it follows by applying [78, Lemma 4.3]. (See [80] for more details.) □

Remark 3.2.2 *The above lemma establishes the existence of a regularity exponent $s > 0$ that will play a role in the error estimates of the numerical method proposed in this paper. In spite of the fact that we refer to [78] in the proof of this lemma, the value of s that arises from this reference is far from being optimal, since it is valid for global regularity in $\mathrm{H}^s(\Omega)^d$ and for any arbitrary geometrical setting of the subdomains Ω_1 and Ω_2. In most of the applications, the subdomains at which ρ is constant are similar to those shown in Figure 3.1. In such a case, a detailed analysis of how this exponent depends on the geometry of the domain and on the coefficient ρ can be found in [59] for the 2D case and in [58] for 3D problems (see also [19, 77]). Let us remark that, although the analyses of these references is for problems with Dirichlet boundary conditions, similar results hold true for Neumann boundary conditions as in our case (see [59, Remark III.5.2]). In particular, for instance, Lemma 3.2.2 holds true for $s = 1$ in the example reported as Test 1 in Section 5.5 (see Figure 3.2).*

From the physical point of view, the time domain problem (3.2.11) is dissipative in the sense that its solution should decay as t increases. The latter happens if and only if the so called *decay rate*, $\mathrm{Re}(\lambda)$, is negative. The following result shows that this is the case in our formulation.

Lemma 3.2.3 *Let $(\lambda, \boldsymbol{u}) \in \mathbb{C} \times \mathcal{V}$ be a solution of Problem 3.2.1. If $\lambda \neq 0$, then $Re(\lambda) < 0$.*

Proof. Testing Problem 3.2.1 with $\boldsymbol{v} = \boldsymbol{u}$, we have that $A\lambda^2 + B\lambda + C = 0$, with

$$A := \int_\Omega \rho |\boldsymbol{u}|^2, \quad B := 2\int_\Omega \nu |\operatorname{div} \boldsymbol{u}|^2, \quad \text{and} \quad C := \int_\Omega \rho c^2 |\operatorname{div} \boldsymbol{u}|^2.$$

We observe that $A > 0$, $B \geq 0$, and $C \geq 0$. Moreover, since $\lambda = 0$ if and only if $\boldsymbol{u} \in \mathcal{K} = \mathbb{H}_0(\operatorname{div}^0, \Omega)$, for $\lambda \neq 0$ we have that $B, C > 0$, too. Therefore, since $\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$, it is immediate to check that $\mathrm{Re}(\lambda) < 0$. □

Remark 3.2.3 *Any eigenpair $(\lambda, \boldsymbol{u})$ of Problem 3.2.1 satisfies*

$$\lambda^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \int_\Omega (2\lambda\nu + \rho c^2) \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = 0 \qquad \forall \boldsymbol{v} \in \mathcal{V}.$$

Since the coefficients are constant in each subdomain, if $2\lambda\nu + \rho c^2 \neq 0$ in Ω_i, by testing with $\boldsymbol{v} \in \mathcal{D}(\Omega_i)^d$ we obtain that $\operatorname{div} \boldsymbol{u}|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, $i = 1, 2$. On the other hand, if $2\lambda\nu + \rho c^2 = 0$ in Ω_i ($i = 1$ or 2), then, for $\lambda \neq 0$, by testing again with $\boldsymbol{v} \in \mathcal{D}(\Omega_i)^d$, we obtain that $\boldsymbol{u} = \boldsymbol{0}$ in Ω_i. Thus, in any case, $\operatorname{div} \boldsymbol{u}|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, $i = 1, 2$.

For the theoretical analysis it is convenient to transform Problem 3.2.1 into a linear eigenvalue problem. With this aim we introduce the new variable $\widehat{\boldsymbol{u}} := \lambda \boldsymbol{u}$, as usual in quadratic problems, and the space $\widetilde{\mathcal{V}} := \mathcal{V} \times \mathcal{H}$ endowed with the corresponding product norm, which carry us to the following:

Problem 3.2.2 *Find $\lambda \in \mathbb{C}$ and $\boldsymbol{0} \neq (\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \widetilde{\mathcal{V}}$ such that*

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = \lambda \left(-2 \int_\Omega \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} \right) \qquad \forall \boldsymbol{v} \in \mathcal{V}, \tag{3.2.14}$$

$$\int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}} = \lambda \int_\Omega \rho \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}} \qquad \forall \widehat{\boldsymbol{v}} \in \mathcal{H}. \tag{3.2.15}$$

We observe that $\lambda = 0$ is an eigenvalue of Problem 3.2.2 and its associated eigenspace is $\widetilde{\mathcal{K}} := \mathcal{K} \times \{0\}$. Let $\widetilde{\mathcal{G}}$ be the orthogonal complement of $\widetilde{\mathcal{K}}$ in $\mathcal{V} \times \mathcal{H}$. Notice that $\widetilde{\mathcal{G}} = \mathcal{G} \times \mathcal{H}$.

We introduce the sesquilinear continuous form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}},$$

and the sesquilinear continuous forms $\widetilde{a}, \widetilde{b} : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ defined as follows:

$$\widetilde{a}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) := \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}},$$

$$\widetilde{b}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) := -2 \int_\Omega \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} + \int_\Omega \rho \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}}.$$

In what follows we prove that $a(\cdot, \cdot)$ and $\widetilde{a}(\cdot, \cdot)$ are elliptic in $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, respectively.

Lemma 3.2.4 *The sesquilinear form $a : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is $\mathcal{G}$-elliptic and, consequently, $\widetilde{a} : \widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}} \to \mathbb{C}$ is $\widetilde{\mathcal{G}}$-elliptic.*

Proof. For $\boldsymbol{v} \in \mathcal{G}$ we have

$$a(\boldsymbol{v}, \boldsymbol{v}) = \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{v} \operatorname{div} \overline{\boldsymbol{v}} \geq \min_\Omega \{\rho c^2\} \| \operatorname{div} \boldsymbol{v} \|_{0,\Omega}^2.$$

Then, the $\mathcal{G}$-ellipticity of $a(\cdot, \cdot)$ follows from Lemma 3.2.2. From this, the ellipticity of $\widetilde{a}(\cdot, \cdot)$ in $\widetilde{\mathcal{G}} = \mathcal{G} \times \mathcal{H}$ is immediate. □

Let $\boldsymbol{T} : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ be the bounded linear operator defined by $\boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g}) := (\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \widetilde{\mathcal{G}}$, where $(\boldsymbol{u}, \widehat{\boldsymbol{u}})$ is the unique solution of the following problem:

$$\widetilde{a}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) = \widetilde{b}((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) \qquad \forall (\boldsymbol{v}, \widehat{\boldsymbol{v}}) \in \widetilde{\mathcal{G}}.$$

It is easy to check that

$$\widehat{\boldsymbol{u}} = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.2.16}$$

and

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} = -2 \int_\Omega \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}. \tag{3.2.17}$$

As a consequence of the above equalities, we have that $\mu = 0$ is an eigenvalue of $\boldsymbol{T}$ with associated eigenspace $\{\boldsymbol{0}\} \times \mathcal{G}^{\perp_\mathcal{H}}$, which is nontrivial since $\mathcal{G}^{\perp_\mathcal{H}} \supset \mathcal{K}$. The following lemma shows that the nonzero eigenvalues of $\boldsymbol{T}$ are exactly the reciprocals of the nonzero eigenvalues of Problem 3.2.2 with the same corresponding eigenfunctions.

Lemma 3.2.5 *There holds that $(\mu, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ is an eigenpair of $\boldsymbol{T}$ (i.e. $\boldsymbol{T}(\boldsymbol{u}, \widehat{\boldsymbol{u}}) = \mu(\boldsymbol{u}, \widehat{\boldsymbol{u}})$) with $\mu \neq 0$ if and only if $(\lambda, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ is a solution of Problem 3.2.2 with $\lambda = 1/\mu \neq 0$.*

Proof. Let $(\mu, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ be an eigenpair of $\boldsymbol{T}$ with $\mu \neq 0$. Hence

$$\widetilde{a}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) = \frac{1}{\mu} \widetilde{b}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) \qquad \forall (\boldsymbol{v}, \widehat{\boldsymbol{v}}) \in \widetilde{\mathcal{G}}. \tag{3.2.18}$$

Then, according to (3.2.16) we have that $\widehat{\boldsymbol{u}} = \frac{1}{\mu} \boldsymbol{u} \in \mathcal{G}$. Hence, for $(\boldsymbol{v}, \widehat{\boldsymbol{v}}) \in \widetilde{\mathcal{K}} = \mathcal{K} \times \{\boldsymbol{0}\}$, clearly $\widetilde{b}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) = 0$ and $\widetilde{a}((\boldsymbol{u}, \widehat{\boldsymbol{u}}), (\boldsymbol{v}, \widehat{\boldsymbol{v}})) = 0$. So, (3.2.18) holds for all $(\boldsymbol{v}, \widehat{\boldsymbol{v}}) \in \widetilde{\mathcal{V}} = \widetilde{\mathcal{G}} \oplus \widetilde{\mathcal{K}}$; namely, $(\lambda, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ with $\lambda = 1/\mu$ is a solution to Problem 3.2.2.

Conversely, let $(\lambda, (\boldsymbol{u}, \widehat{\boldsymbol{u}}))$ be a solution of Problem 3.2.2 with $\lambda \neq 0$. Taking $\boldsymbol{v} \in \mathcal{K}$ in (5.2.15), we have that $\int_\Omega \rho \widehat{\boldsymbol{u}} \cdot \boldsymbol{v} = 0$, which implies that $\widehat{\boldsymbol{u}} \in \mathcal{G}$. On the other hand, we observe that (5.2.16) implies that $\lambda \boldsymbol{u} = \widehat{\boldsymbol{u}} \in \mathcal{G}$. Hence it is easy to check that $\boldsymbol{T}(\boldsymbol{u}, \widehat{\boldsymbol{u}}) = \mu(\boldsymbol{u}, \widehat{\boldsymbol{u}})$ with $\mu = 1/\lambda$. □

3.3 Spectral Characterization

The goal of this section is to characterize the spectrum of the solution operator $\boldsymbol{T}$. Since the inclusion $\mathrm{H}_0(\mathrm{div};\Omega) \hookrightarrow \mathrm{L}^2(\Omega)^d$ is not compact, it is easy to check from (3.2.16) that $\boldsymbol{T}$ is not compact either. However, we will show that the essential spectrum, has to lie in a small region of the complex plane, well separated from the isolated eigenvalues which, according to Lemma 3.2.5, correspond to the solutions of Problem 3.2.2. With this aim, we will resort to the theory described in [55] to decompose appropriately $\boldsymbol{T}$. Let $\boldsymbol{T}_1, \boldsymbol{T}_2 : \mathcal{G} \to \mathcal{G}$ be the operators given by

$$\boldsymbol{T}_1\boldsymbol{f} = \boldsymbol{u}_1 \in \mathcal{G} : \quad a(\boldsymbol{u}_1, \boldsymbol{v}) = 2\int_\Omega \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}, \tag{3.3.1}$$

$$\boldsymbol{T}_2\boldsymbol{g} = \boldsymbol{u}_2 \in \mathcal{G} : \quad a(\boldsymbol{u}_2, \boldsymbol{v}) = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}. \tag{3.3.2}$$

It is easy to check that these operators are self-adjoint with respect to $a(\cdot,\cdot)$. Moreover $\boldsymbol{T}_1$ is non-negative and $\boldsymbol{T}_2$ is positive with respect to $a(\cdot,\cdot)$ (namely, $a(\boldsymbol{T}_1\boldsymbol{v}, \boldsymbol{v}) \geq 0\ \forall \boldsymbol{v} \in \mathcal{G}$ and $a(\boldsymbol{T}_2\boldsymbol{v}, \boldsymbol{v}) > 0\ \forall \boldsymbol{v} \in \mathcal{G},\ \boldsymbol{v} \neq \boldsymbol{0}$). Moreover, we have the following result.

Lemma 3.3.1 *The operator $\boldsymbol{T}_2 : \mathcal{G} \to \mathcal{G}$ is compact.*

Proof.

Since $a(\cdot,\cdot)$ is $\mathcal{G}$-elliptic (cf. Lemma 3.2.4), applying Lax-Milgram's Lemma, we know that problem (3.3.2) is well posed and has a unique solution $\boldsymbol{u}_2 \in \mathcal{G}$. Moreover, according to Lemma 3.2.2, we know that there exists $s > 0$ such that $\boldsymbol{u}_2 \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$. On the other hand, notice that (3.3.2) also holds for $\boldsymbol{v} \in \mathcal{K}$, since in such a case $a(\boldsymbol{u}_2, \boldsymbol{v}) = 0 = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}}$ for $\boldsymbol{g} \in \mathcal{G}$. Hence, since $\mathcal{V} = \mathcal{G} \oplus \mathcal{K}$, we have that

$$a(\boldsymbol{u}_2, \boldsymbol{v}) = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\boldsymbol{v}} \quad \forall \boldsymbol{v} \in \mathcal{V}.$$

Then, by testing this equation with $\boldsymbol{v} \in \mathcal{D}(\Omega)^d \subset \mathcal{V}$, we have that $-\nabla(\rho c^2 \operatorname{div} \boldsymbol{u}_2) = \rho \boldsymbol{g}$ in Ω, so that $\rho c^2 \operatorname{div} \boldsymbol{u}_2 \in \mathrm{H}^1(\Omega)$. Therefore, since ρ and c are positive constants in each subdomain Ω_1 and Ω_2, we have that $\operatorname{div} \boldsymbol{u}_2|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, $i = 1, 2$. Since the inclusions $\{v \in \mathrm{L}^2(\Omega) : v|_{\Omega_i} \in \mathrm{H}^1(\Omega_i),\ i = 1, 2.\} \subset \mathrm{L}^2(\Omega)$ and $\mathrm{H}^s(\Omega_1 \cup \Omega_2)^d \subset \mathrm{L}^2(\Omega)^d$, are compact, we derive that $\boldsymbol{T}_2$ is compact too. □

The operator $\boldsymbol{T}$ can be written in terms of the operators $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ given above as follows:

$$\boldsymbol{T} = \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2 \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}.$$

Moreover, by defining as in [55] the operators

$$\boldsymbol{S} := \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_2^{1/2} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{H} := \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & \boldsymbol{0} \end{pmatrix},$$

we have that $\boldsymbol{ST} = \boldsymbol{HS}$. We note that the eigenvalues of $\boldsymbol{T}$ and $\boldsymbol{H}$ and their algebraic multiplicities coincide. Moreover the corresponding Jordan chains have the same length. In fact, let $\{\boldsymbol{x}_k\}_{k=1}^r$ be a Jordan chain associated with the eigenvalue μ of $\boldsymbol{T}$. Then, using the identities above, we observe that

$$\boldsymbol{HSx}_k = \boldsymbol{STx}_k = \boldsymbol{S}(\mu\boldsymbol{x}_k + \boldsymbol{x}_{k-1}) = \mu\boldsymbol{Sx}_k + \boldsymbol{Sx}_{k-1}, \quad k = 1, \dots, r.$$

This shows that $\{\boldsymbol{Sx}_k\}_{k=1}^r$ is a Jordan chain of $\boldsymbol{H}$ of the same length. Actually, the whole spectra of $\boldsymbol{T}$ and $\boldsymbol{H}$ coincide as is shown in the following result, which has been proved in Lemma 3.2 of [13].

Lemma 3.3.2 *There holds*

$$\mathrm{sp}(\boldsymbol{T}) = \mathrm{sp}(\boldsymbol{H}).$$

Moreover, $\mathrm{Sp_e}(\boldsymbol{T}) = \mathrm{Sp_e}(\boldsymbol{H})$, *too.*

The operator $\boldsymbol{H}$ can be written as the sum of a self-adjoint operator $\boldsymbol{B}$ and a compact operator $\boldsymbol{C}$:

$$\boldsymbol{H} = \boldsymbol{B} + \boldsymbol{C} \quad \text{with} \quad \boldsymbol{B} := \begin{pmatrix} -\boldsymbol{T}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} := \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & \boldsymbol{0} \end{pmatrix}.$$

Then, applying the classical Weyl's Theorem (see [83]), we have that $\mathrm{Sp_e}(\boldsymbol{H}) = \mathrm{Sp_e}(\boldsymbol{B})$ and the rest of the spectrum $\mathrm{Sp_d}(\boldsymbol{H}) := \mathrm{sp}(\boldsymbol{H}) \setminus \mathrm{Sp_e}(\boldsymbol{H})$ consists of isolated eigenvalues with finite algebraic multiplicity. Moreover, $\mathrm{Sp_e}(\boldsymbol{B}) = \mathrm{Sp_e}(-\boldsymbol{T}_1) \cup \{0\}$.

Our next goal is to show that the essential spectrum of $\boldsymbol{T}_1$ must lie in a small region of the complex plane. Actually, we will localize the whole spectrum of $\boldsymbol{T}_1$. With this aim, we analyze separately for which values $\mu \in \mathbb{C}$, the operator $(\mu\boldsymbol{I} - \boldsymbol{T}_1)$ is not necessarily one-to-one and for which values it is not necessarily onto.

- If $(\mu\boldsymbol{I} - \boldsymbol{T}_1)$ is not one-to-one, then there exists $\boldsymbol{f} \in \mathcal{G}$, $\boldsymbol{f} \neq \boldsymbol{0}$, such that $\boldsymbol{T}_1\boldsymbol{f} = \mu\boldsymbol{f}$, namely,

$$\mu \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} = 2 \int_\Omega \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

  Then, testing with $\boldsymbol{v} = \boldsymbol{f}$ and using that in each subdomain the coefficients ρ and c are positive, we deduce that

$$\mu = \frac{2\int_\Omega \nu \left|\operatorname{div} \boldsymbol{f}\right|^2}{\int_\Omega \rho c^2 \left|\operatorname{div} \boldsymbol{f}\right|^2}$$

  (we recall that for $\boldsymbol{0} \neq \boldsymbol{f} \in \mathcal{G}$, $\int_\Omega |\operatorname{div} \boldsymbol{f}|^2 > 0$ because of Lemma 3.2.2). Hence,

$$\mu \in \left[\frac{2\min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}, \frac{2\max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}}\right].$$

- On the other hand, $(\mu\boldsymbol{I} - \boldsymbol{T}_1)$ is onto if and only if for any $\boldsymbol{g} \in \mathcal{G}$ there exists $\boldsymbol{f} \in \mathcal{G}$ such that $\boldsymbol{T}_1\boldsymbol{f} = \mu\boldsymbol{f} - \boldsymbol{g}$, which from (3.3.1) reads

$$\int_\Omega \rho c^2 \operatorname{div}\boldsymbol{g} \operatorname{div}\overline{\boldsymbol{v}} = \int_\Omega (-2\nu + \mu\rho c^2) \operatorname{div}\boldsymbol{f} \operatorname{div}\overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

By writing $\mu = \alpha + \beta i$ with $\alpha, \beta \in \mathbb{R}$, the equation above reads:

$$\int_\Omega (-2\nu + \alpha\rho c^2 + \rho c^2 \beta i) \operatorname{div}\boldsymbol{f} \operatorname{div}\overline{\boldsymbol{v}} = \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{g} \operatorname{div}\overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

We observe that for all $\beta \neq 0$ the problem above has a solution and hence the operator $(\mu\boldsymbol{I} - \boldsymbol{T}_1)$ is onto. On the other hand, if $\beta = 0$, then μ has to be real. In such a case, the operator $\boldsymbol{T}_1$ will still be onto when $(-2\nu + \mu\rho c^2)$ has the same sign in the whole domain Ω. This happens at least in two cases:

(i) when $\mu > \dfrac{2\max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}}$, in which case $-2\nu + \mu\rho c^2 > 0$,

(ii) when $\mu < \dfrac{2\min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}$, in which case $-2\nu + \mu\rho c^2 < 0$.

Therefore, if $(\mu\boldsymbol{I} - \boldsymbol{T}_1)$ is not onto, then $\mu \in \left[\frac{2\min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}, \frac{2\max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}}\right]$, too.

Now we are in position to write the following spectral characterization of the solution operator $\boldsymbol{T}$.

Theorem 3.3.1 *The spectrum of* $\boldsymbol{T}$ *consists of*

$$\mathrm{Sp_e}(\boldsymbol{T}) = \mathrm{sp}(-\boldsymbol{T}_1) \cup \{0\}$$

with

$$\mathrm{sp}(\boldsymbol{T}_1) \subset \left[\frac{2\min_\Omega\{\nu\}}{\max_\Omega\{\rho c^2\}}, \frac{2\max_\Omega\{\nu\}}{\min_\Omega\{\rho c^2\}}\right]$$

and $\mathrm{Sp_d}(\boldsymbol{T}) := \mathrm{sp}(\boldsymbol{T}) \setminus \mathrm{Sp_e}(\boldsymbol{T})$, *which is a set of isolated eigenvalues of finite algebraic multiplicity.*

Proof. As a consequence of the classical Weyl's Theorem (see [83]) and Lemma 5.3.2,

$$\mathrm{Sp_e}(\boldsymbol{T}) = \mathrm{Sp_e}(\boldsymbol{H}) = \mathrm{Sp_e}(\boldsymbol{B}) = \mathrm{Sp_e}(-\boldsymbol{T}_1) \cup \{0\},$$

whereas the inclusion follows from the above analysis. □

In what follows, we will show that for ν small enough some of the eigenvalues of $\boldsymbol{T}$ are well separated from its essential spectrum. With this end, given $\boldsymbol{f} \in \mathcal{G}$, by testing (3.3.1) with $\boldsymbol{v} = \boldsymbol{u}_1 \in \mathcal{G}$ and using the the definition of $a(\cdot,\cdot)$, we have that

$$\min_\Omega\{\rho c^2\} \|\boldsymbol{u}_1\|_{\mathrm{div},\Omega}^2 \leq a(\boldsymbol{u}_1, \boldsymbol{u}_1) \leq 2\|\nu\|_{\infty,\Omega} \|\operatorname{div}\boldsymbol{f}\|_{0,\Omega} \|\boldsymbol{u}_1\|_{\mathrm{div},\Omega}.$$

Therefore $\|\boldsymbol{T}_1\|_{\mathcal{L}(\mathcal{G}\times\mathcal{G})} \to 0$ as $\|\nu\|_{\infty,\Omega}$ goes to zero. Consequently, $\boldsymbol{H}$ converges in norm to the operator

$$\boldsymbol{H}_0 := \begin{pmatrix} \mathbf{0} & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & \mathbf{0} \end{pmatrix}$$

as $\|\nu\|_{\infty,\Omega}$ goes to zero. Thus, from the classical spectral approximation theory (see [53]), the isolated eigenvalues of $\boldsymbol{H}$ converge to those of $\boldsymbol{H}_0$.

Since the isolated eigenvalues of $\boldsymbol{H}$ and $\boldsymbol{T}$ coincide (cf. Lemma 5.3.2), in order to localize those of $\boldsymbol{T}$, we begin by characterizing those of $\boldsymbol{H}_0$. Let μ be an isolated eigenvalue of $\boldsymbol{H}_0$ and $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{G} \times \mathcal{G}$ the corresponding eigenfunction. It is easy to check that

$$\boldsymbol{H}_0 \begin{pmatrix} \boldsymbol{u} \\ \widehat{\boldsymbol{u}} \end{pmatrix} = \mu \begin{pmatrix} \boldsymbol{u} \\ \widehat{\boldsymbol{u}} \end{pmatrix} \quad \Longleftrightarrow \quad \boldsymbol{T}_2 \boldsymbol{u} = -\mu^2 \boldsymbol{u} \quad \text{and} \quad \boldsymbol{T}_2^{1/2} \boldsymbol{u} = \mu \widehat{\boldsymbol{u}}. \tag{3.3.3}$$

Since $\boldsymbol{T}_2$ is compact, self-adjoint, and positive, its spectrum consists of a sequence of positive eigenvalues that converge to zero and 0 itself. Notice that the spectrum of $\boldsymbol{T}_2$ is related with the solution of the eigenvalue problem (3.2.12). In fact, this problem has 0 as an eigenvalue with corresponding eigenspace $\mathcal{K}$. The rest of the eigenvalues ω^2 are strictly positive and the corresponding eigenfunctions $\boldsymbol{u} \in \mathcal{K}^{\perp_\nu} =: \mathcal{G}$, so that they are also solutions of the following problem: Find $\omega > 0$ and $\boldsymbol{u} \in \mathcal{G}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = \omega^2 \int_\Omega \rho \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

Clearly $(\omega^2, \boldsymbol{u})$ is an eigenpair of the above problem with $\omega > 0$ if and only if $\boldsymbol{T}_2 \boldsymbol{u} = \frac{1}{\omega^2} \boldsymbol{u}$. Thus, by virtue of (3.3.3), we have that the eigenvalues of $\boldsymbol{H}_0$ are given by $\pm i/\omega$ and hence they are purely imaginary.

Now we are in a position to establish the following result.

Theorem 3.3.2 *For each isolated eigenvalue $\pm i/\omega$ of $\boldsymbol{H}_0$ of algebraic multiplicity m, let $r > 0$ be such that the disc $D_r := \{z \in \mathbb{C} : \ |z \mp i/\omega| < r\}$ intersects $\mathrm{sp}(\boldsymbol{H}_0)$ only in $\pm i/\omega$. Then, there exists $\delta > 0$ such that if $\|\nu\|_{\infty,\Omega} < \delta$, there exist m eigenvalues of $\boldsymbol{T}$, $\mu_1, \ldots, \mu_m$, (repeated according to their respective algebraic multiplicities) lying in the disc D_r. Moreover, $\mu_1, \ldots, \mu_m \to \frac{i}{\omega}$ as $\|\nu\|_{\infty,\Omega}$ goes to zero.*

As claimed above, the eigenvalues of $\boldsymbol{T}$ that are relevant in the applications, are those which are close to $\pm i/\omega$ for the smallest positive vibration frequencies ω of (3.2.12). According to the above theorem, these eigenvalues are well separated from the real axis and, hence, from the essential spectrum of $\boldsymbol{T}$ (cf. Theorem 3.3.1).

3.4 Spectral Approximation

In this section, we propose and analyze a finite element method to approximate the solutions of Problem 3.2.1. With this end, we introduce appropriate discrete spaces. Let $\{\mathcal{T}_h(\Omega)\}_{h>0}$ be a family of regular partitions of Ω such that $\mathcal{T}_h(\Omega_i) := \{T \in \mathcal{T}_h : T \subset \overline{\Omega}_i\}$ are partitions of Ω_i, $i = 1, 2$. We introduce the lowest-order Raviart-Thomas finite element space:

$$\mathcal{V}_h := \{\boldsymbol{v} \in \mathcal{V} : \boldsymbol{v}|_T(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x},\ \boldsymbol{a} \in \mathbb{R}^d,\ b \in \mathbb{R},\ \boldsymbol{x} \in T\}.$$

The discretization of Problem 3.2.1 reads as follows:

Problem 3.4.1 *Find* $\lambda_h \in \mathbb{C}$ *and* $\mathbf{0} \neq \boldsymbol{u}_h \in \mathcal{V}_h$ *such that*

$$\lambda_h^2 \int_\Omega \rho \boldsymbol{u}_h \cdot \overline{\boldsymbol{v}}_h + 2\lambda_h \int_\Omega \nu \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h + \int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h = 0 \qquad \forall \boldsymbol{v}_h \in \mathcal{V}_h.$$

We proceed as we did in the continuous case and introduce a new discrete variable $\widehat{\boldsymbol{u}}_h := \lambda_h \boldsymbol{u}_h$ to rewrite the problem above in the following equivalent form:

Problem 3.4.2 *Find* $\lambda_h \in \mathbb{C}$ *and* $\mathbf{0} \neq (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \mathcal{V}_h \times \mathcal{V}_h$ *such that*

$$\int_\Omega \rho c^2 \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h = \lambda_h \left(-2 \int_\Omega \nu \operatorname{div} \boldsymbol{u}_h \operatorname{div} \overline{\boldsymbol{v}}_h - \int_\Omega \rho \widehat{\boldsymbol{u}}_h \cdot \overline{\boldsymbol{v}}_h\right) \qquad \forall \boldsymbol{v}_h \in \mathcal{V}_h,$$

$$\int_\Omega \rho \widehat{\boldsymbol{u}}_h \cdot \overline{\widehat{\boldsymbol{v}}}_h = \lambda_h \int_\Omega \rho \boldsymbol{u}_h \cdot \overline{\widehat{\boldsymbol{v}}}_h \qquad \forall \widehat{\boldsymbol{v}}_h \in \mathcal{V}_h.$$

We observe that $\lambda_h = 0$ is an eigenvalue of this problem and its associated eigenspace is $\widetilde{\mathcal{K}}_h := \mathcal{K}_h \times \{0\}$ with $\mathcal{K}_h := \mathcal{K} \cap \mathcal{V}_h$ being the eigenspace of $\lambda_h = 0$ in Problem 3.4.1. At the beginning of Section 5.5, we will show that Problem 3.4.2 is well posed, in the sense that it is equivalent to a generalized matrix eigenvalue problem with a symmetric positive definite right-hand side matrix.

We introduce the well known Raviart-Thomas interpolation operator, $\Pi_h : \mathcal{V} \cap \mathrm{H}^r(\Omega_1 \cup \Omega_2)^d \to \mathcal{V}_h$, $r \in (0, 1]$ (see [72]), for which there hold the approximation result

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{0,\Omega} \leq C h^r (\|\boldsymbol{v}\|_{r,\Omega_1} + \|\boldsymbol{v}\|_{r,\Omega_2} + \|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}) \tag{3.4.1}$$

and the commuting diagram property

$$\operatorname{div}(\Pi_h \boldsymbol{v}) = \mathbb{P}_h(\operatorname{div} \boldsymbol{v}), \tag{3.4.2}$$

where

$$\mathbb{P}_h : \mathrm{L}^2(\Omega) \to \mathcal{U}_h := \{v_h \in \mathrm{L}^2(\Omega) :\ v_h|_T \in \mathbb{P}_0(T)\ \ \forall T \in \mathcal{T}_h\}$$

is the standard L^2-orthogonal projector. Then, for any $r \in (0, 1]$ we have that

$$\|q - \mathbb{P}_h q\|_{0,\Omega} \leq C h^r \|q\|_{r,\Omega} \qquad \forall q \in \mathrm{H}^r(\Omega). \tag{3.4.3}$$

Let $\mathcal{G}_h$ be the orthogonal complement of $\mathcal{K}_h$ in $\mathcal{V}_h$, and $\widetilde{\mathcal{G}}_h := \mathcal{G}_h \times \mathcal{G}_h \subset \widetilde{\mathcal{V}} = \mathcal{V} \times \mathcal{H}$ endowed with the corresponding product norm. Note that $\mathcal{G}_h \not\subseteq \mathcal{G}$ and hence $\widetilde{\mathcal{G}}_h \not\subseteq \widetilde{\mathcal{G}}$.

The following result provides estimates for the terms in the Helmholtz decomposition of functions in $\mathcal{G}_h$. Let us recall that, here and thereafter, $s > 0$ denotes the optimal regularity exponent such that Lemma 3.2.2 holds true.

Lemma 3.4.1 *For any $\boldsymbol{v}_h \in \mathcal{G}_h$, there exists $\xi \in \mathrm{H}^1(\Omega)$ and $\chi \in \mathcal{K}$ such that*

$$\boldsymbol{v}_h = \frac{1}{\rho}\nabla\xi + \chi$$

with $\frac{1}{\rho}\nabla\xi \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$, and the following estimates holds

$$\left\|\frac{1}{\rho}\nabla\xi\right\|_{s,\Omega_1} + \left\|\frac{1}{\rho}\nabla\xi\right\|_{s,\Omega_2} \leq C\|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega} \qquad \text{and} \qquad \|\chi\|_{0,\Omega} \leq Ch^s\|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega}.$$

Proof. The proof follows by repeating the arguments of the proof of Lemma 4.1 from [13], taking care of the presence of the discontinuous coefficient ρ. □

Remark 3.4.1 *We notice that Lemma 3.4.1 provides (1.1) with $P\boldsymbol{v}_h = \frac{1}{\rho}\nabla\xi$. It is worthwhile to mention that, with the regularity results for ξ at hand (see Lemma 3.2.2), Lemma 3.4.1 may also be deduced by considering in Lemma 5.10 of [6] the case of differential 2-forms.*

The following result is a direct consequence of Lemma 3.4.1.

Lemma 3.4.2 *The sesquilinear form $a : \mathcal{G}_h \times \mathcal{G}_h \to \mathbb{C}$ is $\mathcal{G}_h$-elliptic, with ellipticity constant not depending on h. Consequently, $\widetilde{a} : \widetilde{\mathcal{G}}_h \times \widetilde{\mathcal{G}}_h \to \mathbb{C}$ is $\widetilde{\mathcal{G}}_h$-elliptic uniformly in h.*

Now, we are in position to introduce the discrete version of the operator $\boldsymbol{T}$. Let $\boldsymbol{T}_h : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}$ be defined by $\boldsymbol{T}_h(\boldsymbol{f},\boldsymbol{g}) := (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ with $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) \in \widetilde{\mathcal{G}}_h$ being the solution of

$$\widetilde{a}((\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h), (\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h)) = \widetilde{b}((\boldsymbol{f},\boldsymbol{g}), (\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h)) \qquad \forall (\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in \widetilde{\mathcal{G}}_h.$$

It is easy to check that $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) = \boldsymbol{T}_h(\boldsymbol{f},\boldsymbol{g})$ if and only if

$$\widehat{\boldsymbol{u}}_h = \mathbb{P}_{\mathcal{G}_h}\boldsymbol{f}, \tag{3.4.4}$$

where $\mathbb{P}_{\mathcal{G}_h}$ is the $\mathcal{H}$-orthogonal projection onto $\mathcal{G}_h$, and $\boldsymbol{u}_h \in \mathcal{G}_h$ solves

$$\int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_h \operatorname{div}\boldsymbol{v}_h = -2\int_\Omega \nu \operatorname{div}\boldsymbol{f} \operatorname{div}\overline{\boldsymbol{v}}_h - \int_\Omega \rho\boldsymbol{g}\cdot\overline{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \mathcal{G}_h. \tag{3.4.5}$$

Since $\boldsymbol{T}_h(\widetilde{\mathcal{V}}) \subset \widetilde{\mathcal{G}}_h$, there holds $\operatorname{sp}(\boldsymbol{T}_h) = \operatorname{sp}(\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}) \cup \{0\}$ (cf. [12, Lemma 4.1]). Thus, we will restrict our attention to $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$.

As claimed above, at the beginning of Section 5.5, Problem 3.4.2 will be shown to be equivalent to a well posed generalized matrix eigenvalue problem. This problem has $\lambda_h = 0$ as an eigenvalue with corresponding eigenspace $\widetilde{\mathcal{K}}_h$. The rest of the eigenvalues are related with the spectrum of $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$ according to the following lemma.

Lemma 3.4.3 *There holds that $(\mu_h, (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h))$ is an eigenpair of $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$ with $\mu_h \neq 0$ if and only if $(\lambda_h, (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h))$ is a solution of Problem 3.4.2 with $\lambda_h = 1/\mu_h$.*

Proof. The proof follows essentially as that of Lemma 3.2.5, by using the fact that $\mathcal{V}_h \times \mathcal{V}_h = \widetilde{\mathcal{G}}_h \oplus (\mathcal{K}_h \times \mathcal{K}_h)$. □

Our next goal is to show that any isolated eigenvalue of $\boldsymbol{T}$ with algebraic multiplicity m is approximated by exactly m eigenvalues of $\boldsymbol{T}_h$ (repeated according to their respective algebraic multiplicities) and that spurious eigenvalues do not arise. With this end, we will adapt to our problem the theory from [13], which in turn use arguments introduced in [36, 37] to deal with non compact operators. From now on, let $\mu \in \mathrm{Sp_d}(\boldsymbol{T})$, $\mu \neq 0$, be a fixed isolated eigenvalue of finite algebraic multiplicity m. Let $\mathcal{E}$ be the invariant subspace of $\boldsymbol{T}$ corresponding to μ. Our analysis will be based on proving the following two properties:

$$\text{P1.}\quad \|\boldsymbol{T}-\boldsymbol{T}_h\|_h := \sup_{\boldsymbol{0}\neq(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h} \frac{\|(\boldsymbol{T}-\boldsymbol{T}_h)(\boldsymbol{f}_h,\boldsymbol{g}_h)\|_{\widetilde{\mathcal{V}}}}{\|(\boldsymbol{f}_h,\boldsymbol{g}_h)\|_{\widetilde{\mathcal{V}}}} \to 0 \quad \text{as } h\to 0;$$

$$\text{P2.}\quad \forall(\boldsymbol{v},\widehat{\boldsymbol{v}})\in\mathcal{E},\quad \inf_{(\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)\in\widetilde{\mathcal{G}}_h} \|(\boldsymbol{v},\widehat{\boldsymbol{v}})-(\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)\|_{\widetilde{\mathcal{V}}} \to 0 \quad \text{as } h\to 0.$$

Let $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$ and $(\boldsymbol{u},\widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}_h,\boldsymbol{g}_h)$. From (3.2.17), we can write $\boldsymbol{u}=\boldsymbol{u}_1+\boldsymbol{u}_2$ with $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathcal{G}$ satisfying

$$\boldsymbol{u}_1\in\mathcal{G}:\quad \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_1 \operatorname{div}\overline{\boldsymbol{v}} = -2\int_\Omega \nu \operatorname{div}\boldsymbol{f}_h \operatorname{div}\overline{\boldsymbol{v}} \qquad \forall\boldsymbol{v}\in\mathcal{G}, \tag{3.4.6}$$

and

$$\boldsymbol{u}_2\in\mathcal{G}:\quad \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_2 \operatorname{div}\overline{\boldsymbol{v}} = -\int_\Omega \rho\boldsymbol{g}_h\cdot\overline{\boldsymbol{v}} \qquad \forall\boldsymbol{v}\in\mathcal{G}. \tag{3.4.7}$$

The following result states some properties of the solutions of the problems above.

Lemma 3.4.4 *For $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$, let $(\boldsymbol{u},\widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}_h,\boldsymbol{g}_h)$ and consider the decomposition $\boldsymbol{u}=\boldsymbol{u}_1+\boldsymbol{u}_2$ as above. Hence, $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$, $\operatorname{div}\boldsymbol{u}_1\in\mathcal{U}_h$, $\operatorname{div}\boldsymbol{u}_2\in\mathrm{H}^{1+s}(\Omega_1\cup\Omega_2)$, and the following estimates hold true*

$$\|\boldsymbol{u}_1\|_{s,\Omega_1}+\|\boldsymbol{u}_1\|_{s,\Omega_2}\leq C\|\boldsymbol{f}_h\|_{\operatorname{div},\Omega}, \tag{3.4.8}$$

$$\|\boldsymbol{u}_2\|_{s,\Omega_1}+\|\boldsymbol{u}_2\|_{s,\Omega_2}+\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_1}+\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_2}\leq C\|\boldsymbol{g}_h\|_{\operatorname{div},\Omega}. \tag{3.4.9}$$

Proof. Since $\boldsymbol{u}_1\in\mathcal{G}$, due to Lemma 3.2.2 we have that $\boldsymbol{u}_1\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$ and $\|\boldsymbol{u}_1\|_{s,\Omega_1}+\|\boldsymbol{u}_1\|_{s,\Omega_2}\leq C\|\operatorname{div}\boldsymbol{f}_h\|_{0,\Omega}$. Moreover, note that (3.4.6) also holds for $\boldsymbol{v}\in\mathcal{K}$ and hence for all $\boldsymbol{v}\in\mathcal{V}$. Then, we write

$$\int_\Omega (\rho c^2\operatorname{div}\boldsymbol{u}_1+2\nu\operatorname{div}\boldsymbol{f}_h)\operatorname{div}\overline{\boldsymbol{v}} = 0 \qquad \forall\boldsymbol{v}\in\mathcal{V}.$$

Thus, taking test functions in $\mathcal{D}(\Omega)^d\subset\mathcal{V}$ we have $\nabla(\rho c^2\operatorname{div}\boldsymbol{u}_1+2\nu\operatorname{div}\boldsymbol{f}_h)=0$. Since ρ,c,ν and $\operatorname{div}\boldsymbol{f}_h$ are piecewise constant, we have that $\operatorname{div}\boldsymbol{u}_1$ is piecewise constant as well; namely, $\operatorname{div}\boldsymbol{u}_1\in\mathcal{U}_h$.

On the other hand, since $\boldsymbol{u}_2\in\mathcal{G}$, by applying Lemma 3.2.2 again we have that $\boldsymbol{u}_2\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$ and $\|\boldsymbol{u}_2\|_{s,\Omega_1}+\|\boldsymbol{u}_2\|_{s,\Omega_2}\leq C\|\boldsymbol{g}_h\|_{0,\Omega}$. To prove additional regularity for $\operatorname{div}\boldsymbol{u}_2$, we use Lemma 3.4.1 to write $\boldsymbol{g}_h=\frac{1}{\rho}\nabla\xi+\chi$ with $\chi\in\mathcal{K}$, $\frac{1}{\rho}\nabla\xi\in\mathrm{H}^s(\Omega_1\cup\Omega_2)^d$ and $\|\frac{1}{\rho}\nabla\xi\|_{s,\Omega_1}+$

$\|\frac{1}{\rho}\nabla\xi\|_{s,\Omega_2} \leq C\|\operatorname{div}\boldsymbol{g}_h\|_{0,\Omega}$. Moreover, since ρ is constant in each subdomain Ω_i, also $\nabla\xi|_{\Omega_i} \in \mathrm{H}^s(\Omega_i)^d$, $i=1,2$. Then, from (3.4.7) we have that

$$\int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_2 \operatorname{div}\overline{\boldsymbol{v}} = -\int_\Omega \nabla\xi\cdot\overline{\boldsymbol{v}} \qquad \forall\boldsymbol{v}\in\mathcal{G}.$$

Since the above equation trivially holds for $\boldsymbol{v}\in\mathcal{K}$ too, it holds for all $\boldsymbol{v}\in\mathcal{V}$. Then, by testing it with $\boldsymbol{v}\in\mathcal{D}(\Omega)^d$ we have that $\nabla(\rho c^2\operatorname{div}\boldsymbol{u}_2) = -\nabla\xi\in\Omega$. Therefore, by restricting to Ω_i, $i=1,2$, we have that $\nabla(\rho c^2\operatorname{div}\boldsymbol{u}_2|_{\Omega_i}) = -\nabla\xi|_{\Omega_i}\in\mathrm{H}^s(\Omega_i)^d$. Since ρ and c are piecewise constant, we conclude that $\operatorname{div}\boldsymbol{u}_2|_{\Omega_i}\in\mathrm{H}^{1+s}(\Omega_1\cup\Omega_2)$, and

$$\|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_1} + \|\operatorname{div}\boldsymbol{u}_2\|_{1+s,\Omega_2} \leq C\|\nabla\xi\|_{0,\Omega} \leq C\|\operatorname{div}\boldsymbol{g}_h\|_{0,\Omega}.$$

Hence, we conclude the proof. □

We consider a similar decomposition in the discrete case. For $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$, let $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h) := \boldsymbol{T}_h(\boldsymbol{f}_h,\boldsymbol{g}_h)$. We write $\boldsymbol{u}_h = \boldsymbol{u}_{1h}+\boldsymbol{u}_{2h}$ with $\boldsymbol{u}_{1h}$ and $\boldsymbol{u}_{2h}$ satisfying

$$\boldsymbol{u}_{1h}\in\mathcal{G}_h: \quad \int_\Omega \rho c^2\operatorname{div}\boldsymbol{u}_{1h}\operatorname{div}\overline{\boldsymbol{v}}_h = -2\int_\Omega \nu\operatorname{div}\boldsymbol{f}_h\operatorname{div}\overline{\boldsymbol{v}}_h \qquad \forall\boldsymbol{v}_h\in\mathcal{G}_h, \tag{3.4.10}$$

and

$$\boldsymbol{u}_{2h}\in\mathcal{G}_h: \quad \int_\Omega \rho c^2\operatorname{div}\boldsymbol{u}_{2h}\operatorname{div}\overline{\boldsymbol{v}}_h = -\int_\Omega \rho\boldsymbol{g}_h\cdot\overline{\boldsymbol{v}}_h \qquad \forall\boldsymbol{v}_h\in\mathcal{G}_h. \tag{3.4.11}$$

These are the finite element discretization of problems (3.4.6) and (3.4.7), respectively, and the following error estimates hold true.

Lemma 3.4.5 *Let $(\boldsymbol{f}_h,\boldsymbol{g}_h)\in\widetilde{\mathcal{G}}_h$. Let $\boldsymbol{u}_1,\boldsymbol{u}_2$ be the solutions of problems* (3.4.6) *and* (3.4.7)*, respectively, and $\boldsymbol{u}_{1h},\boldsymbol{u}_{2h}$ those of problems* (3.4.10) *and* (3.4.11)*, respectively. Then, the following estimates hold true:*

$$\|\boldsymbol{u}_1-\boldsymbol{u}_{1h}\|_{\mathrm{div},\Omega} \leq Ch^s\|\boldsymbol{f}_h\|_{\mathcal{V}}, \tag{3.4.12}$$

$$\|\boldsymbol{u}_2-\boldsymbol{u}_{2h}\|_{\mathrm{div},\Omega} \leq Ch^s\|\boldsymbol{g}_h\|_{\mathcal{H}}. \tag{3.4.13}$$

Proof. Since $\mathcal{G}_h\nsubseteq\mathcal{G}$, we will resort to the second Strang Lemma, which for problems (3.4.6) and (3.4.10) reads as follows:

$$\|\boldsymbol{u}_1-\boldsymbol{u}_{1h}\|_{\mathrm{div},\Omega} \leq C\left[\inf_{\boldsymbol{v}_h\in\mathcal{G}_h}\|\boldsymbol{u}_1-\boldsymbol{v}_h\|_{\mathrm{div},\Omega} + \sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\mathcal{G}_h}\frac{a(\boldsymbol{u}_1-\boldsymbol{u}_{1h},\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\mathrm{div},\Omega}}\right]. \tag{3.4.14}$$

Because of Lemma 3.4.4, $\Pi_h\boldsymbol{u}_1$ is well defined. Since $\Pi_h\boldsymbol{u}_1\in\mathcal{V}_h=\mathcal{G}_h\oplus\mathcal{K}_h$, there exists $\widetilde{\boldsymbol{u}}_{1h}\in\mathcal{G}_h$ and $\breve{\boldsymbol{u}}_h\in\mathcal{K}_h$ such that $\Pi_h\boldsymbol{u}_1 = \widetilde{\boldsymbol{u}}_{1h}+\breve{\boldsymbol{u}}_h$. Then, since $\boldsymbol{u}_1-\widetilde{\boldsymbol{u}}_{1h}$ is orthogonal to $\breve{\boldsymbol{u}}_h$, we observe that

$$\begin{aligned}\|\boldsymbol{u}_1-\widetilde{\boldsymbol{u}}_{1h}\|_{\mathcal{V}}^2 &\leq \|\boldsymbol{u}_1-\widetilde{\boldsymbol{u}}_{1h}\|_{\mathcal{V}}^2+\|\breve{\boldsymbol{u}}_h\|_{\mathcal{V}}^2\\ &= \|(\widetilde{\boldsymbol{u}}_{1h}-\boldsymbol{u}_1)+\breve{\boldsymbol{u}}_h\|_{\mathcal{V}}^2 = \|\boldsymbol{u}_1-\Pi_h\boldsymbol{u}_1\|_{\mathcal{V}}^2\\ &\leq C\left(\|\boldsymbol{u}_1-\Pi_h\boldsymbol{u}_1\|_{0,\Omega}^2+\|\operatorname{div}\boldsymbol{u}_1-\operatorname{div}(\Pi_h\boldsymbol{u}_1)\|_{0,\Omega}^2\right).\end{aligned}$$

The first term on the right hand side above is bounded as follows:

$$\|\boldsymbol{u}_1 - \Pi_h \boldsymbol{u}_1\|_{0,\Omega} \le Ch^s(\|\boldsymbol{u}_1\|_{s,\Omega_1} + \|\boldsymbol{u}_1\|_{s,\Omega_2} + \|\operatorname{div}\boldsymbol{u}_1\|_{0,\Omega}) \le Ch^s\|\boldsymbol{f}_h\|_{\mathcal{V}},$$

where we have used (3.4.1), (3.4.8), and the fact that $\|\operatorname{div}\boldsymbol{u}_1\|_{0,\Omega} \le C\|\operatorname{div}\boldsymbol{f}_h\|_{0,\Omega}$, which in turn follows from (3.4.6) by taking $\boldsymbol{v} = \boldsymbol{f}_h$. On the other hand, the second term vanishes because of (3.4.2) since $\operatorname{div}\boldsymbol{u}_1 \in \mathcal{U}_h$ (cf. Lemma 3.4.4). Hence, $\|\boldsymbol{u}_1 - \widetilde{\boldsymbol{u}}_{1h}\|_{\operatorname{div},\Omega} \le Ch^s\|\boldsymbol{f}_h\|_{\mathcal{V}}$, which allows us to control the approximation term in (3.4.14).

For the consistency term, it is enough to recall that (3.4.6) holds for all $\boldsymbol{v} \in \mathcal{V}$. Then, by using (3.4.10), it is easy to check that $a(\boldsymbol{u}_1 - \boldsymbol{u}_{1h}, \boldsymbol{v}_h) = 0$ for all $\boldsymbol{v}_h \in \mathcal{G}_h \subset \mathcal{V}$. From this, the Strang estimate for $\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega}$ reads as follows:

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega} \le C \inf_{\boldsymbol{v}_h \in \mathcal{G}_h} \|\boldsymbol{u}_1 - \boldsymbol{v}_h\|_{\operatorname{div},\Omega} \le Ch^s\|\boldsymbol{f}_h\|_{\mathcal{V}}.$$

Thus (3.4.12) holds true.

To prove (3.4.13), we resort again to the second Strang Lemma:

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_{2h}\|_{\operatorname{div},\Omega} \le C\left[\inf_{\boldsymbol{v}_h \in \mathcal{G}_h} \|\boldsymbol{u}_2 - \boldsymbol{v}_h\|_{\operatorname{div},\Omega} + \sup_{\boldsymbol{0} \ne \boldsymbol{v}_h \in \mathcal{G}_h} \frac{a(\boldsymbol{u}_2 - \boldsymbol{u}_{2h}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}}\right]. \tag{3.4.15}$$

Since $\boldsymbol{u}_2 \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$ (cf. Lemma 3.4.4), we have that $\Pi_h \boldsymbol{u}_2$ is well defined. We proceed as above and write $\Pi_h \boldsymbol{u}_2 = \widetilde{\boldsymbol{u}}_{2h} + \check{\boldsymbol{u}}_h$ with $\widetilde{\boldsymbol{u}}_{2h} \in \mathcal{G}_h$ and $\check{\boldsymbol{u}}_h \in \mathcal{K}_h$ to obtain

$$\|\boldsymbol{u}_2 - \widetilde{\boldsymbol{u}}_{2h}\|_{\operatorname{div},\Omega} \le C\left[\|\boldsymbol{u}_2 - \Pi_h \boldsymbol{u}_2\|_{0,\Omega} + \|\operatorname{div}\boldsymbol{u}_2 - \operatorname{div}(\Pi_h \boldsymbol{u}_2)\|_{0,\Omega}\right]. \tag{3.4.16}$$

For the first term on the right hand side above, (3.4.1) and Lemma 3.4.4 yield

$$\|\boldsymbol{u}_2 - \Pi_h \boldsymbol{u}_2\|_{0,\Omega} \le Ch^s(\|\boldsymbol{u}_2\|_{s,\Omega_1} + \|\boldsymbol{u}_2\|_{s,\Omega_2} + \|\operatorname{div}\boldsymbol{u}_2\|_{0,\Omega}) \le Ch^s\|\boldsymbol{g}_h\|_{\mathcal{H}}.$$

For the second term, we have from (5.4.25) and from Lemma 3.4.4 again

$$\begin{aligned}\|\operatorname{div}\boldsymbol{u}_2 - \operatorname{div}\Pi_h \boldsymbol{u}_2\|_{0,\Omega}^2 &= \|\operatorname{div}\boldsymbol{u}_2 - \mathbb{P}_h(\operatorname{div}\boldsymbol{u}_2)\|_{0,\Omega}^2, \\ &\le Ch(\|\operatorname{div}\boldsymbol{u}_2\|_{1,\Omega_1} + \|\operatorname{div}\boldsymbol{u}_2\|_{1,\Omega_2}) \le Ch\|\boldsymbol{g}_h\|_{\mathcal{H}}.\end{aligned}$$

Hence, $\|\boldsymbol{u}_2 - \widetilde{\boldsymbol{u}}_{2h}\|_{\operatorname{div},\Omega} \le Ch^s\|\boldsymbol{g}_h\|_{\mathcal{H}}$, which allows us to bound the approximation term in (3.4.15).

For the consistency term, given $\boldsymbol{v}_h \in \mathcal{G}_h$ we apply Lemma 3.4.1 to write $\boldsymbol{v}_h = \frac{1}{\rho}\nabla\xi + \chi$ with $\frac{1}{\rho}\nabla\xi \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$, $\chi \in \mathcal{K}$, and $\|\chi\|_{0,\Omega} \le Ch^s\|\operatorname{div}\boldsymbol{v}_h\|_{0,\Omega}$. Then, from (3.4.7) we have

$$a(\boldsymbol{u}_2, \boldsymbol{v}_h) = \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_2 \operatorname{div}\overline{\boldsymbol{v}}_h = \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_2 \operatorname{div}\left(\frac{1}{\rho}\nabla\overline{\xi}\right) = \int_\Omega \boldsymbol{g}_h \cdot \nabla\overline{\xi}.$$

On the other hand, from (3.4.11),

$$a(\boldsymbol{u}_{2h}, \boldsymbol{v}_h) = \int_\Omega \rho c^2 \operatorname{div}\boldsymbol{u}_{2h} \operatorname{div}\overline{\boldsymbol{v}}_h = \int_\Omega \rho\boldsymbol{g}_h \cdot \overline{\boldsymbol{v}}_h = \int_\Omega \boldsymbol{g}_h \cdot \nabla\overline{\xi} + \int_\Omega \rho\boldsymbol{g}_h \cdot \overline{\chi}.$$

Therefore,

$$a(\boldsymbol{u}_2 - \boldsymbol{u}_{2h}, \boldsymbol{v}_h) = -\int_\Omega \rho \boldsymbol{g}_h \cdot \overline{\chi} \leq Ch^s \|\boldsymbol{g}_h\|_{0,\Omega} \|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}$$

and, hence,

$$\sup_{\mathbf{0}\neq \boldsymbol{v}_h \in \mathcal{G}_h} \frac{a(\boldsymbol{u}_2 - \boldsymbol{u}_{2h}, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}} \leq Ch^s \|\boldsymbol{g}_h\|_{0,\Omega},$$

which allows us to complete the proof. □

Now, we are in a position to establish the following result.

Lemma 3.4.6 *Property P1 holds true. Moreover,*

$$\|\boldsymbol{T} - \boldsymbol{T}_h\|_h \leq Ch^s.$$

Proof. For $(\boldsymbol{f}_h, \boldsymbol{g}_h) \in \widetilde{\mathcal{G}}_h$, let $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}_h, \boldsymbol{g}_h)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) := \boldsymbol{T}_h(\boldsymbol{f}_h, \boldsymbol{g}_h)$. From (3.2.16) and (3.4.4) we have that $\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h = \boldsymbol{f}_h - \mathbb{P}_{\mathcal{G}_h} \boldsymbol{f}_h = 0$. Hence, by writing $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ and $\boldsymbol{u}_h = \boldsymbol{u}_{1h} + \boldsymbol{u}_{2h}$ as in Lemma 3.4.5, we have from this lemma

$$\|\boldsymbol{T} - \boldsymbol{T}_h\|_h \leq \sup_{\mathbf{0}\neq(\boldsymbol{g}_h, \boldsymbol{f}_h)\in\widetilde{\mathcal{G}}_h} \frac{\|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\operatorname{div},\Omega} + \|\boldsymbol{u}_2 - \boldsymbol{u}_{2h}\|_{\operatorname{div},\Omega}}{\|(\boldsymbol{f}_h, \boldsymbol{g}_h)\|_{\widetilde{\mathcal{V}}}} \leq Ch^s.$$

Thus, we conclude the proof. □

Our next goal is to prove property P2. With this aim, first we will prove the following additional regularity result,.

Lemma 3.4.7 *Let* $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{E}$. *Then,* $\boldsymbol{u}, \widehat{\boldsymbol{u}} \in \mathcal{G} \subset \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$, $\operatorname{div} \boldsymbol{u}, \operatorname{div} \widehat{\boldsymbol{u}} \in \mathrm{H}^{1+s}(\Omega_1 \cup \Omega_2)$, *and*

$$\|\boldsymbol{u}\|_{s,\Omega_1} + \|\boldsymbol{u}\|_{s,\Omega_2} + \|\operatorname{div} \boldsymbol{u}\|_{1+s,\Omega_1} + \|\operatorname{div} \boldsymbol{u}\|_{1+s,\Omega_2} \leq C \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}, \tag{3.4.17}$$

$$\|\widehat{\boldsymbol{u}}\|_{s,\Omega_1} + \|\widehat{\boldsymbol{u}}\|_{s,\Omega_2} + \|\operatorname{div} \widehat{\boldsymbol{u}}\|_{1+s,\Omega_1} + \|\operatorname{div} \widehat{\boldsymbol{u}}\|_{1+s,\Omega_2} \leq C \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}. \tag{3.4.18}$$

Proof. We prove the above inequalities for all the generalized eigenfunctions of $\boldsymbol{T}$. Let $\{(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\}_{k=1}^p$ be a Jordan chain of the operator $\boldsymbol{T}$ associated with μ. Then, $\boldsymbol{T}(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k) = \mu(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k) + (\boldsymbol{u}_{k-1}, \widehat{\boldsymbol{u}}_{k-1})$, $k = 1, \ldots, p$, with $(\boldsymbol{u}_0, \widehat{\boldsymbol{u}}_0) = \mathbf{0}$. We will use an induction argument on k. Assume that $\boldsymbol{u}_{k-1}$ and $\widehat{\boldsymbol{u}}_{k-1}$ belong to $\mathcal{G}$ and satisfy (3.4.17) and (3.4.18), respectively (which obviously hold for $k = 1$). First note that, because of the boundedness of $\boldsymbol{T}$, we have

$$\|(\boldsymbol{u}_{k-1}, \widehat{\boldsymbol{u}}_{k-1})\|_{\widetilde{\mathcal{V}}} \leq C \|(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\|_{\widetilde{\mathcal{V}}}. \tag{3.4.19}$$

On the other hand, by using (3.2.16) and (3.2.17) we have that

$$\mu \widehat{\boldsymbol{u}}_k + \widehat{\boldsymbol{u}}_{k-1} = \boldsymbol{u}_k \quad \text{in } \Omega \tag{3.4.20}$$

and that $\mu \boldsymbol{u}_k + \boldsymbol{u}_{k-1} \in \mathcal{G}$ satisfies

$$\int_\Omega \rho c^2 \operatorname{div}(\mu \boldsymbol{u}_k + \boldsymbol{u}_{k-1}) \operatorname{div} \overline{\boldsymbol{v}} = -2 \int_\Omega \nu \operatorname{div} \boldsymbol{u}_k \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \widehat{\boldsymbol{u}}_k \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{G}.$$

Hence, $\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k \in \mathcal{G}$.

We observe that the equation above also holds for any $\boldsymbol{v} \in \mathcal{K}$. Then,

$$\int_\Omega \rho c^2 \operatorname{div}(\mu \boldsymbol{u}_k + \boldsymbol{u}_{k-1}) \operatorname{div} \overline{\boldsymbol{v}} = -2 \int_\Omega \nu \operatorname{div} \boldsymbol{u}_k \operatorname{div} \overline{\boldsymbol{v}} - \int_\Omega \rho \widehat{\boldsymbol{u}}_k \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \mathcal{V}.$$

Thus, considering test functions in $\mathcal{D}(\Omega)^d \subset \mathcal{V}$ we obtain

$$\nabla((\mu\rho c^2 + 2\nu) \operatorname{div} \boldsymbol{u}_k) = \rho \widehat{\boldsymbol{u}}_k - \nabla(\rho c^2 \operatorname{div} \boldsymbol{u}_{k-1}). \tag{3.4.21}$$

Let us assume that $\mu\rho c^2 + 2\nu \neq 0$ in both Ω_1 and Ω_2 (we discuss the other case at the end of the proof). Hence, since ρ, c, and ν are constant in each Ω_i, $\rho_i \widehat{\boldsymbol{u}}_k - \nabla(\rho_i c_i^2 \operatorname{div} \boldsymbol{u}_{k-1}) \in \mathrm{L}^2(\Omega_i)^d$, $\operatorname{div} \boldsymbol{u}_k|_{\Omega_i} \in \mathrm{H}^1(\Omega_i)$, and

$$\|\operatorname{div} \boldsymbol{u}_k\|_{1,\Omega_i} \leq C \left(\|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1,\Omega_i} + \|(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\|_{\widetilde{\mathcal{V}}} \right), \qquad i = 1, 2.$$

Now, since $\boldsymbol{u}_k \in \mathcal{G}$, due to Lemma 3.2.2 we have that $\boldsymbol{u}_k \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$. Then, from (3.2.13) and the previous estimate we have

$$\|\boldsymbol{u}_k\|_{s,\Omega_1} + \|\boldsymbol{u}_k\|_{s,\Omega_2} \leq C \left(\|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1,\Omega_1} + \|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1,\Omega_2} + \|(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k)\|_{\widetilde{\mathcal{V}}} \right). \tag{3.4.22}$$

On the other hand, from (3.4.20) we obtain

$$\|\widehat{\boldsymbol{u}}_k\|_{s,\Omega_i} \leq \frac{1}{\mu} \left(\|\widehat{\boldsymbol{u}}_{k-1}\|_{s,\Omega_i} + \|\boldsymbol{u}_k\|_{s,\Omega_i} \right), \qquad i = 1, 2, \tag{3.4.23}$$

and, from (3.4.21),

$$\|\operatorname{div} \boldsymbol{u}_k\|_{1+s,\Omega_i} \leq C(\|\widehat{\boldsymbol{u}}_k\|_{s,\Omega_i} + \|\operatorname{div} \boldsymbol{u}_{k-1}\|_{1+s,\Omega_i}), \qquad i = 1, 2. \tag{3.4.24}$$

Finally, from (3.4.20) again,

$$\|\operatorname{div} \widehat{\boldsymbol{u}}_k\|_{1+s,\Omega_i} \leq \frac{1}{\mu}(\|\operatorname{div} \boldsymbol{u}_k\|_{1+s,\Omega_i} + \|\operatorname{div} \widehat{\boldsymbol{u}}_{k-1}\|_{1+s,\Omega_i}), \qquad i = 1, 2. \tag{3.4.25}$$

Hence, from inequalities (3.4.22)–(3.4.25), the inductive assumption, and (3.4.19), we derive (3.4.17) and (3.4.18) provided $\mu\rho c^2 + 2\nu \neq 0$ in both Ω_1 and Ω_2.

In case that $\mu\rho c^2 + 2\nu$ vanishes in Ω_i, $i = 1$ or 2, arguing as in Remark 3.2.3 we obtain that $\boldsymbol{u}_1|_{\Omega_i} = \widehat{\boldsymbol{u}}_1|_{\Omega_i} = \boldsymbol{0}$ and, once again, an induction argument allow us to conclude that $\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k = \boldsymbol{0}$ in Ω_i, $k = 1, \ldots, p$. The proof is complete. □

Now, we are in position to establish property P2.

Lemma 3.4.8 *Property P2 holds true. Moreover, for any* $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{E}$*, there exists* $\widetilde{\boldsymbol{u}}_h$, $\widetilde{\widehat{\boldsymbol{u}}}_h \in \mathcal{G}_h$ *such that*

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\mathrm{div},\Omega} + \|\widehat{\boldsymbol{u}} - \widetilde{\widehat{\boldsymbol{u}}}_h\|_{\mathrm{div},\Omega} \leq C h^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}.$$

Proof. Let $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in \mathcal{E}$. According to Lemma 3.4.7, we have that $\boldsymbol{u}, \widehat{\boldsymbol{u}} \in \mathrm{H}^s(\Omega_1 \cup \Omega_2)^d$ and $\operatorname{div} \boldsymbol{u}, \operatorname{div} \widehat{\boldsymbol{u}} \in \mathrm{H}^{1+s}(\Omega_1 \cup \Omega_2)$. Let $\Pi_h \boldsymbol{u} \in \mathcal{V}_h$ be the Raviart-Thomas interpolant of $\boldsymbol{u}$. Since $\mathcal{V}_h = \mathcal{G}_h \oplus \mathcal{K}_h$, we decompose $\Pi_h \boldsymbol{u} = \widetilde{\boldsymbol{u}}_h + \breve{\boldsymbol{u}}_h$ with $\widetilde{\boldsymbol{u}}_h \in \mathcal{G}_h$ and $\breve{\boldsymbol{u}}_h \in \mathcal{K}_h$. The same arguments from the proof of Lemma 3.4.5 that lead to (3.4.13) apply in this case and combined with Lemma 3.4.7 allow us to prove that $\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\operatorname{div},\Omega} \leq C h^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}$. A similar procedure can be used to define $\widetilde{\widehat{\boldsymbol{u}}}_h$ and to prove that $\|\widehat{\boldsymbol{u}} - \widetilde{\widehat{\boldsymbol{u}}}_h\|_{\operatorname{div},\Omega} \leq C h^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}}$. □

We also have the following auxiliary result when the source terms are in $\mathcal{E}$.

Lemma 3.4.9 *For $(\boldsymbol{f}, \boldsymbol{g}) \in \mathcal{E}$, let $(\boldsymbol{u}, \widehat{\boldsymbol{u}}) := \boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g})$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) := \boldsymbol{T}_h(\boldsymbol{f}, \boldsymbol{g})$. Then,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\operatorname{div},\Omega} + \|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{0,\Omega} \leq C h^s \|(\boldsymbol{f}, \boldsymbol{g})\|_{\widetilde{\mathcal{V}}}.$$

Proof. Since $\mathcal{G}_h \not\subseteq \mathcal{G}$, we resort once more to the second Strang Lemma, which applied now to (3.2.17) and (3.4.5) leads to

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\operatorname{div},\Omega} \leq C \left[\inf_{\boldsymbol{v}_h \in \mathcal{G}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\operatorname{div},\Omega} + \sup_{\boldsymbol{0} \neq \boldsymbol{v}_h \in \mathcal{G}_h} \frac{a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\operatorname{div},\Omega}} \right].$$

From Lemma 3.4.8 we know that there exists $\widetilde{\boldsymbol{u}}_h \in \mathcal{G}_h$ such that

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\operatorname{div},\Omega} \leq C h^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}} \leq C h^s \|(\boldsymbol{f}, \boldsymbol{g})\|_{\widetilde{\mathcal{V}}}.$$

Moreover, the consistency term above vanishes. In fact, consider $\boldsymbol{v}_h \in \mathcal{G}_h$ and the decomposition $\boldsymbol{v}_h = \frac{1}{\rho} \nabla \xi + \chi$ as in Lemma 3.4.1. Using the same arguments as in the proof of Lemma 3.4.5, we prove that

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \rho \boldsymbol{g} \cdot \overline{\chi} = 0,$$

where the last equality holds because $\boldsymbol{g} \in \mathcal{G}$ and $\chi \in \mathcal{K}$.

On the other hand, we know from (3.2.16) and (3.4.4) that $\widehat{\boldsymbol{u}} = \boldsymbol{f}$ and $\widehat{\boldsymbol{u}}_h = \mathbb{P}_{\mathcal{G}_h} \boldsymbol{f}$, respectively. Then, since $\mathbb{P}_{\mathcal{G}_h}$ is the $\mathcal{H}$-orthogonal projection onto $\mathcal{G}_h$, we have that $\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{\mathcal{H}} \leq \|\widehat{\boldsymbol{u}} - \widetilde{\widehat{\boldsymbol{u}}}_h\|_{\mathcal{H}}$, with $\widetilde{\widehat{\boldsymbol{u}}}_h \in \mathcal{G}_h$ as in Lemma 3.4.8. Hence, we obtain

$$\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{0,\Omega} \leq C h^s \|(\boldsymbol{u}, \widehat{\boldsymbol{u}})\|_{\widetilde{\mathcal{V}}} \leq C h^s \|(\boldsymbol{f}, \boldsymbol{g})\|_{\widetilde{\mathcal{V}}}.$$

The proof is complete. □

The above lemmas are the ingredients to prove spectral convergence and to obtain error estimates. Our first result is the following theorem which has been proved in [36] as a consequence of property P1 (cf. Lemma 5.4.8) and which shows that the proposed method is free of spurious modes.

Theorem 3.4.1 *Let $K \subset \mathbb{C}$ be a compact set such that $K \cap \operatorname{sp}(\boldsymbol{T}) = \emptyset$. Then, there exists $h_0 > 0$ such that, for all $h \leq h_0$, $K \cap \operatorname{sp}(\boldsymbol{T}_h) = \emptyset$.*

Let $D \subset \mathbb{C}$ be a closed disk centered at μ, such that $D \cap \operatorname{sp}(\boldsymbol{T}) = \{\mu\}$. Let $\mu_{1h}, \ldots, \mu_{m(h)h}$ be the eigenvalues of $\boldsymbol{T}_h$ contained in D (repeated according to their algebraic multiplicities).

Under assumptions P1 and P2, it is proved in [36] that $m(h) = m$ for h small enough and that $\lim_{h\to 0} \mu_{kh} = \mu$ for $k = 1, \ldots, m$.

On the other hand the arguments used in Section 5 of [13] can be readily adapted to our problem, to obtain error estimates. We recall the definition of the gap between two closed subspaces $\mathcal{W}$ and $\mathcal{Y}$ of $\widetilde{\mathcal{V}}$:

$$\widehat{\delta}(\mathcal{W}, \mathcal{Y}) := \max\{\delta(\mathcal{W}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{W})\},$$

with

$$\delta(\mathcal{W}, \mathcal{Y}) := \sup_{\substack{(\boldsymbol{\phi},\boldsymbol{\psi})\in\mathcal{W} \\ \|(\boldsymbol{\phi},\boldsymbol{\psi})\|_{\widetilde{\mathcal{V}}}=1}} \left[\inf_{(\widehat{\boldsymbol{\phi}},\widehat{\boldsymbol{\psi}})\in\mathcal{Y}} \|(\boldsymbol{\phi},\boldsymbol{\psi}) - (\widehat{\boldsymbol{\phi}},\widehat{\boldsymbol{\psi}})\|_{\widetilde{\mathcal{V}}} \right].$$

Let $\mathcal{E}_h$ be the invariant subspace of $\boldsymbol{T}_h$ relative to the eigenvalues $\mu_{1h}, \ldots, \mu_{mh}$ converging to μ. From Lemmas 5.4.8–3.4.9, we derive the following results for which we do not include proofs to avoid repeating step by step those of [13, Section 5].

Theorem 3.4.2 *There exist constants $h_0 > 0$ and $C > 0$ such that, for all $h \leq h_0$,*

$$\widehat{\delta}\,(\mathcal{E}_h, \mathcal{E}) \leq Ch^s.$$

Theorem 3.4.3 *There exist constants $h_0 > 0$ and $C > 0$ such that, for all $h \leq h_0$,*

$$\left| \mu - \frac{1}{m} \sum_{k=1}^{m} \mu_{kh} \right| \leq Ch^{2s},$$

$$\left| \frac{1}{\mu} - \frac{1}{m} \sum_{k=1}^{m} \frac{1}{\mu_{kh}} \right| \leq Ch^{2s},$$

$$\max_{k=1,\ldots,m} |\mu - \mu_{kh}| \leq Ch^{2s/p},$$

where p is the ascent of the eigenvalue μ of $\boldsymbol{T}$.

3.5 Numerical Results

We implemented the proposed method in a MATLAB code. We report in this section the results of some numerical tests, in order to assess its performance. With this end, first we introduce a convenient matrix form of the discrete problem which allows us to use standard eigensolvers. As a by-product, this matrix form also allows us to prove that Problems 3.4.1 and 3.4.2 are well posed.

Let $\{\phi_j\}_{j=1}^N$ be a nodal basis of $\mathcal{V}_h$. We define the matrices $\mathbf{K}_1 := (\mathbf{K}_{ij}^{(1)})$, $\mathbf{K}_2 := (\mathbf{K}_{ij}^{(2)})$ and $\mathbf{M} := (\mathbf{M}_{ij})$ as follows:

$$\mathbf{K}_{ij}^{(1)} := 2\int_\Omega \nu \operatorname{div}\phi_i \operatorname{div}\phi_j, \quad \mathbf{K}_{ij}^{(2)} := \int_\Omega \rho c^2 \operatorname{div}\phi_i \operatorname{div}\phi_j, \quad \text{and} \quad \mathbf{M}_{ij} := \int_\Omega \rho\phi_i \cdot \phi_j.$$

The matrix form of Problem 3.4.1 reads

$$(\lambda_h^2 \mathbf{M} + \lambda_h \mathbf{K}_1 + \mathbf{K}_2)\vec{\boldsymbol{u}}_h = \mathbf{0}, \tag{3.5.1}$$

where we denote by $\vec{\boldsymbol{u}}_h$ the vector of components of $\boldsymbol{u}_h$ in the nodal basis of $\mathcal{V}_h$.

Analogously, the matrix form of Problem 3.4.2 reads

$$\begin{pmatrix} \mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\widehat{\boldsymbol{u}}}_h \end{pmatrix} = \lambda_h \begin{pmatrix} -\mathbf{K}_1 & -\mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\widehat{\boldsymbol{u}}}_h \end{pmatrix},$$

with $\vec{\widehat{\boldsymbol{u}}}_h$ being the vector of components of $\widehat{\boldsymbol{u}}_h$. However, this problems is not suitable to be solved with standard eigensolvers, since neither the right-hand side nor the left-hand side matrix are Hermitian and positive definite.

Alternatively, for $\lambda_h \neq 0$, let $\mu_h := \frac{1}{\lambda_h}$. Then, problem (3.5.1) is equivalent to

$$(\mathbf{M} + 2\mu_h \mathbf{K}_1 + \mu_h^2 \mathbf{K}_2)\vec{\boldsymbol{u}}_h = \mathbf{0}.$$

Introducing $\vec{\boldsymbol{w}}_h := \mu_h \vec{\boldsymbol{u}}_h$, the problem above is equivalent to

$$\begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\boldsymbol{w}}_h \end{pmatrix} = \mu_h \begin{pmatrix} -\mathbf{K}_1 & -\mathbf{K}_2 \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\boldsymbol{w}}_h \end{pmatrix},$$

which in turn is equivalent to

$$\begin{pmatrix} -\mathbf{K}_1 & -\mathbf{K}_2 \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\boldsymbol{w}}_h \end{pmatrix} = \lambda_h \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\boldsymbol{w}}_h \end{pmatrix}.$$

Thus, the last problem is equivalent to Problem 3.4.1 except for $\lambda_h = 0$ and the matrix in its right-hand side is Hermitian and positive definite. Hence, it is well posed and can be safely solved by standard eigensolvers.

Test 1. We applied it to a 2D rectangular rigid cavity filled with two fluids with different physical parameters as shown in Figure 3.2. The domain occupied by the fluids are $\Omega_1 := (0, A) \times (0, H)$ and $\Omega_2 := (0, A) \times (H, B)$. For such a simple geometry, it is possible to calculate an analytical solution which will be used to validate our method.

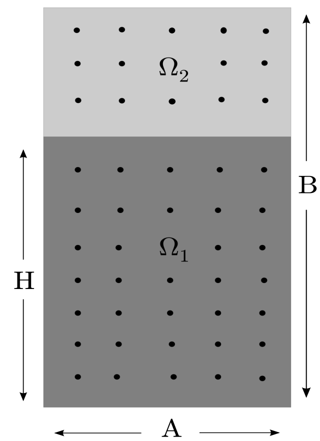

Figure 3.2: Test 1. Two fluids in a rectangular rigid cavity.

Let $\boldsymbol{u} \in \mathrm{H}_0(\mathrm{div}, \Omega)$ be a solution of Problem 3.2.1. Testing it with $\boldsymbol{v} \in \mathcal{D}(\Omega)^2$ we have $\nabla((2\lambda\nu + \rho c^2)\,\mathrm{div}\,\boldsymbol{u}) = -\lambda^2\rho\boldsymbol{u} \in \mathrm{L}^2(\Omega)^2$. Then, $\widehat{p} := -(2\nu\lambda + \rho c^2)\,\mathrm{div}\,\boldsymbol{u} \in \mathrm{H}^1(\Omega)$. Hence, $\widehat{p}_1|_\Gamma = \widehat{p}_2|_\Gamma$. Moreover, $\boldsymbol{u} = -\dfrac{1}{\lambda^2\rho}\nabla\widehat{p}$, which implies that $\dfrac{1}{\rho_1}\dfrac{\partial\widehat{p}_1}{\partial\nu} = \dfrac{1}{\rho_2}\dfrac{\partial\widehat{p}_2}{\partial\nu}$ on Γ. Then, we write problem (3.2.2)–(3.2.9), in terms of $\widehat{p}_i$ as follows:

$$
\begin{aligned}
\Delta\widehat{p}_i &= \frac{\lambda^2\rho_i}{\rho_i c_i^2 + 2\nu_i\lambda}\widehat{p}_i && \text{in } \Omega_i,\ \ i = 1,2,\\
\frac{\partial\widehat{p}_i}{\partial\boldsymbol{n}_i} &= 0 && \text{on } \Gamma_i,\ \ i = 1,2,\\
\widehat{p}_1 &= \widehat{p}_2 && \text{on } \Gamma,\\
\frac{1}{\rho_1}\frac{\partial\widehat{p}_1}{\partial\boldsymbol{n}} &= \frac{1}{\rho_2}\frac{\partial\widehat{p}_2}{\partial\boldsymbol{n}} && \text{on } \Gamma.
\end{aligned}
$$

We proceed by separation of variables. Assuming that $\widehat{p}_i(x,y) = X_i(x)Y_i(y)$, we are left with the following problem:

$$\frac{X_i''(x)}{X_i(x)} + \frac{Y_i''(y)}{Y_i(y)} = \frac{\lambda^2\rho_i}{\rho_i c_i^2 + 2\nu_i\lambda} \quad \text{in } \Omega_i, \tag{3.5.2}$$

$$X_i'(0) = X_i'(A) = 0, \qquad i = 1,2, \tag{3.5.3}$$

$$Y_1'(0) = Y_2'(B) = 0, \tag{3.5.4}$$

$$\frac{1}{\rho_1}X_1(x)Y_1'(H) = \frac{1}{\rho_2}X_2(x)Y_2'(H),\ \ 0 < x < A, \tag{3.5.5}$$

$$X_1(x)Y_1(H) = X_2(x)Y_2(H), \quad 0 < x < A. \tag{3.5.6}$$

From (3.5.2) we have that $X_i(x)''/X_i(x)$ and $Y_i(y)''/Y_i(y)$ are constant. Moreover, from (3.5.5) and (3.5.6), it is easy to check that $Y_i(H)$ and $Y_i'(H)$ cannot vanish simultaneously and $X_1(x) = X_2(x)$ (actually, it is derived that $X_1(x) = CX_2(x)$, but the constant C can be chosen equal to one without loss of generality).

From the fact that $X_i(x)''/X_i(x)$ is constant and (3.5.3), we have that

$$X_1(x) = X_2(x) = \cos\left(\frac{m\pi x}{A}\right), \qquad m = 0, 1, 2, \ldots.$$

On the other hand, from the fact that $Y_i(y)''/Y_i(y)$ is also constant and (3.5.4) we derive

$$Y_1(y) = C_1 \cosh(r_m^{(1)}(\lambda)y) \qquad \text{and} \qquad Y_2(y) = C_2 \cosh(r_m^{(2)}(\lambda)(y - B)), \tag{3.5.7}$$

where C_1 and C_2 are constants and

$$r_m^{(i)} := \sqrt{\frac{\lambda^2 \rho_i}{\rho_i c_i^2 + 2\nu_i \lambda} + \frac{m^2\pi^2}{A^2}}, \qquad m = 0, 1, 2, \ldots, \qquad i = 1, 2.$$

Since $Y_i(H)$ and $Y_i'(H)$ cannot vanish simultaneously, (3.5.5) and (3.5.6) lead to

$$\frac{1}{\rho_1} Y_1'(H) = \frac{1}{\rho_2} Y_2'(H) \qquad \text{and} \qquad Y_1(H) = Y_2(H),$$

respectively. Thus, substituting (3.5.7) into these equation yields the following linear system for the coefficients C_1 and C_2:

$$\begin{aligned} C_1 \cosh(r_m^{(1)}(\lambda)H) &= C_2 \cosh(r_m^{(2)}(\lambda)(H - B)), \\ \frac{C_1 r_m^{(1)}(\lambda)}{\rho_1} \sinh(r_m^{(1)}(\lambda)H) &= \frac{C_2 r_m^{(2)}(\lambda)}{\rho_2} \sinh(r_m^{(2)}(\lambda)(H - B)). \end{aligned}$$

For this system to have non trivial solutions, its determinant must vanish, which yields the following non linear equation in λ for $m = 0, 1, 2, \ldots$ whose roots are the eigenvalues of Problem 3.2.1:

$$\begin{aligned} f_m(\lambda) := \frac{r_m^{(1)}(\lambda)}{\rho_1} \sinh(r_m^{(1)}(\lambda)H) \cosh(r_m^{(2)}(\lambda)(H - B)) \\ - \frac{r_m^{(2)}(\lambda)}{\rho_2} \sinh(r_m^{(2)}(\lambda)(H - B)) \cosh(r_m^{(1)}(\lambda)H) = 0. \end{aligned}$$

We have computed some roots of the above equation and used these roots as exact eigenvalues to compare those obtained with the method proposed in this paper. For the geometrical parameters, we have taken $A = 1\,\mathrm{m}$, $B = 2\,\mathrm{m}$ and $H = 1.25\,\mathrm{m}$.

We have used physical parameters of water and air for the density and acoustic speed of the fluids in Ω_1 and Ω_2, respectively: $c_1 = 1430\,\mathrm{m/s}$, $\rho_1 = 1000\ \mathrm{kg/m^3}$, $c_2 = 340\,\mathrm{m/s}$ and $\rho_2 = 1\,\mathrm{kg/m^3}$. We have used uniform meshes as those shown in Figure 4.1. The refinement parameter N refers to the number of elements per width of the rectangle.

Figure 3.3: Test 1. Meshes for $N = 4$ (left) and $N = 8$ (right).

In presence of dissipation ($\nu \neq 0$), the eigenvalues λ are complex numbers $\lambda = \eta + i\omega$, with $\eta < 0$ being the decay rate and ω the vibration frequency. In absence of dissipation ($\nu = 0$), the eigenvalues λ are purely imaginary ($\eta = 0$). The same holds for the computed eigenvalues λ_h.

In our first test, we neglected the viscosity damping effects by taking $\nu_1 = \nu_2 = 0$. In this case, the eigenvalues λ are actually purely imaginary Accurate values of the zeros of $f_m(\lambda)$ have been obtained with the MATLAB command `fminsearch` applied to $|f_m(\lambda)|$.

Table 3.1 shows the four smallest eigenvalues computed with the proposed method on successively refined meshes. Accurate values obtained with the MATLAB command `fminsearch` applied to $|f_m(\lambda)|$ are also reported on the last line of the table as exact eigenvalues.

m	1	0	1	0
$N = 8$	$1066.07\,i$	$1418.42\,i$	$1784.37\,i$	$1796.61\,i$
$N = 16$	$1067.78\,i$	$1422.52\,i$	$1781.49\,i$	$1797.09\,i$
$N = 32$	$1068.21\,i$	$1423.54\,i$	$1780.73\,i$	$1797.21\,i$
$N = 64$	$1068.33\,i$	$1423.79\,i$	$1780.55\,i$	$1797.23\,i$
Order	2.00	2.00	1.99	2.00
Exact	$1068.36\,i$	$1423.87\,i$	$1780.49\,i$	$1797.24\,i$

Table 3.1: Test 1. Computed and exact eigenvalues for dissipative fluids in a rigid cavity.

As predicted by the theory, these eigenvalues are purely imaginary. The high accuracy of the computed eigenvalues can be observed from Table 3.1 even for the coarsest mesh. We have used a least squares fitting to estimate the convergence rate for each eigenvalue, which are also reported in Table 3.1. A clear order $\mathcal{O}(h^2)$ can be seen in all cases.

Secondly, test we have used the same physical parameters as above for both fluids, but considering now non vanishing viscosities. In order to make the dissipation effects more visible, we have used unrealistically large viscosity values: $\nu_1 = 9\,\mathrm{N/m}s^2$ and $\nu_2 = 1\,\mathrm{N/m}s^2$. We have repeated the scheme used above. We report in Table 5.2 the computed and 'exact' eigenvalues and the estimated convergence rates, which are in accordance with the theory once again. Notice that now all λ have negative real parts (decay rate) as predicted by the theory.

m	1	0	1	0
$N = 8$	$-9.83127 + 1066.03\,i$	$-17.38526 + 1418.31\,i$	$-27.54238 + 1784.16\,i$	$-0.04746 + 1796.61\,i$
$N = 16$	$-9.86298 + 1067.74\,i$	$-17.48513 + 1422.41\,i$	$-27.45337 + 1781.27\,i$	$-0.04875 + 1797.08\,i$
$N = 32$	$-9.87090 + 1068.17\,i$	$-17.50995 + 1423.43\,i$	$-27.43029 + 1780.53\,i$	$-0.04908 + 1797.20\,i$
$N = 64$	$-9.87288 + 1068.38\,i$	$-17.51614 + 1423.78\,i$	$-27.42447 + 1780.34\,i$	$-0.04916 + 1797.23\,i$
Order	2.00	2.00	1.99	2.00
Exact	$-9.87354 + 1068.32\,i$	$-17.51817 + 1423.76\,i$	$-27.42236 + 1780.27\,i$	$-0.04919 + 1797.24\,i$

Table 3.2: Test 1. Computed and exact eigenvalues for dissipative fluids in a rigid cavity.

It can be seen from Table 5.2 that even in the coarsest mesh the vibration frequencies are computed with at least three correct significant digits. In turn, the decay rates are computed with at least two correct significant digits, in spite of the fact that they are much smaller than the vibration frequencies. Moreover, a least square fitting of the computed decay rates shows that they also converge quadratically. As a consequence, the decay rates computed with the finest mesh attain three or four significant correct digits.

Finally, Figure 5.5 show the real and imaginary parts of the computed pressure as defined in (3.2.1) for the smallest eigenvalue.

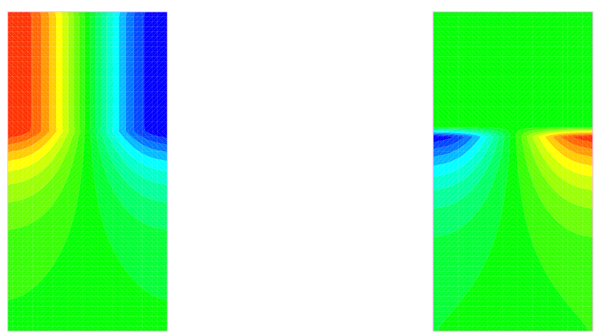

Figure 3.4: Test 1. Real (left) and imaginary (right) parts of the computed pressure for the first eigenvalue.

Test 2. As a second test, we have applied our code to solve a problem with a curved interface. In spite of the fact that such a problem does not lie in our theoretical framework, we will report experimental results which will allow us to asses the performance of the method in this case. The whole domain Ω is the same as in the previous experiment, but with a curved interface Γ. We report in Table 4.3 the computed eigenvalues. In this case, there is no analytical solution available. Therefore, we have obtained more accurate approximations of the exact eigenvalues by means of a least square fitting. These more accurate values are reported on the last line of Table 4.3 as 'Exact' We also report in this table the estimated order of convergence, which once more is clearly quadratic.

ω_h^i	ω_h^1	ω_h^2	ω_h^3	ω_h^4
$N = 8$	$-10.40320 + 1096.60\,i$	$-10.45430 + 1099.28\,i$	$-10.46725 + 1099.96\,i$	$-10.47051 + 1100.13\,i$
$N = 16$	$-13.49718 + 1249.28\,i$	$-13.42914 + 1246.13\,i$	$-13.41172 + 1245.32\,i$	$-13.40737 + 1245.12\,i$
$N = 32$	$-26.06756 + 1735.74\,i$	$-25.88822 + 1729.76\,i$	$-25.84076 + 1728.18\,i$	$-25.82875 + 1727.78\,i$
$N = 64$	$-0.05457 + 1969.38\,i$	$-0.05782 + 1974.72\,i$	$-0.05890 + 1976.45\,i$	$-0.05918 + 1976.93\,i$
Order	1.98	1.97	1.93	1.67
'Exact'	$-10.47163 + 1100.19\,i$	$-13.40581 + 1245.04\,i$	$-25.82423 + 1727.63\,i$	$-0.05934 + 1977.19\,i$

Table 3.3: Test 2. Computed and 'Exact' eigenvalues for dissipative fluids in a rigid cavity with a curved interface.

Finally, Figure 5.6 show the real and imaginary parts of the computed pressure as defined in (3.2.1) for the smallest eigenvalue.The curved interface can be clearly appreciated in the imaginary part of the pressure.

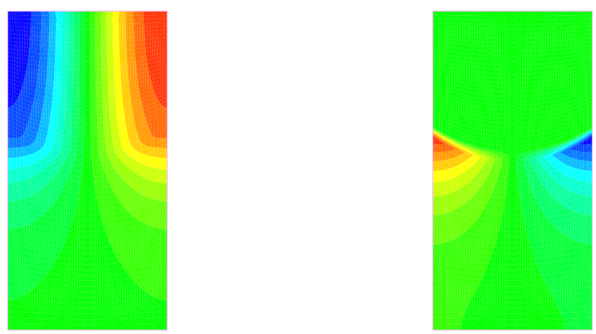

Figure 3.5: Test 2. Real (left) and imaginary (right) parts of the pressure for the first eigenvalue.

Chapter 4

Mixed discontinuous Galerkin approximation of the elasticity eigenproblem

4.1 Introduction

In this paper, we present a discontinuous Galerkin (DG) approximation of the linearized vibrations of an elastic structure. We are interested in the dual-mixed formulation of this problem because it delivers a direct finite element approximation of the Cauchy stress tensor (which is variable of interest in many applications). Moreover, the mixed formulation permits to deal safely with nearly incompressible materials.

Recently, a mixed finite element approximation of the eigenvalue elasticity problem with reduced symmetry has been analyzed in [69]. It is based on a formulation that only maintains the stress tensor as primary unknown, besides the rotation whose role is the weak imposition of the symmetry restriction. It is shown that the lowest order Arnorld-Falk-Whinter element provides a correct spectral approximation for this formulation. It also gives quasi optimal asymptotic error estimates for eigenvalues and eigenfunctions.

The ability of DG methods to handle efficiently hp-adaptive strategies make them well-suited for the numerical simulation of physical systems related to elastodynamics. Our aim here is to introduce an interior penalty discontinuous Galerkin version for the H(div)-conforming finite element space employed in [69]. The k^{th}-order of this method amounts to approximate the Cauchy stress tensor and the rotation by discontinuous finite element spaces of degree k and $k-1$ respectively. We point out that a H(curl)-based interior penalty discontinuous Galerkin method has also been introduced in [29] for the Maxwell eigensystem. The DG approximation we are considering here may be regarded as its counterpart in the H(div)-setting. As in [1, 29], our analysis requires conforming meshes, but the DG scheme still allows different (polynomial) orders of approximation in different elements. A further advantage of this DG method is that it allows to implement high-order elements in a mixed formulation by using standard shape functions.

It is well known that the solution operator corresponding to mixed formulations is generally not compact. In our case, this operator admits a non physical vibration mode (with non physical meaning) whose eigenspace is infinite dimensional. It is then essential to use a scheme that is safe from the pollution that may appear in the form of fictitious eigenvalues interspersed between the physically relevant ones. It turns out that [20], for mixed eigenvalue problems, the conditions guarantying the convergence of the source problem doesn't ensure (as for compact operators [8]) a correct spectral approximation.

It has been shown in [29] that DG methods can also benefit from the general theory developed in [36, 37] to deal with the spectral numerical analysis of non-compact operators. We follow here the same strategy, combined with techniques from [69, 68], to prove that our numerical scheme is spurious free. We also establish asymptotic error estimates for the eigenvalues and eigenfunctions. We treat with special care the analysis of the limit problem obtained when the Lamé coefficient tends to infinity.

Our contribution is the application of the DG method to the spectral elasticity problem, considering the dual-mixed formulation studied in [69] where the associated solution operator is non-compact. This can be consider as a first step to apply discontinuous methods to solve the elasticity eigenproblem. We introduce the discontinuous polynomial spaces in the skeleton, in order to approximate the Cauchy tensor with polynomials of degree k and the rotation with polynomials of degree $k-1$ $(k \geq 1)$, and a discrete bilinear form which includes an interior penalty parameter that is called a *stabilization parameter*. In order to obtain spectral correctness, approximation results and error estimates, we introduce properly mesh-dependent norms and we adapt the theory from [36, 37] for this norms, proving that the constants are independent of the size of the mesh.

The outline of the paper is the following: in Section 4.2 we introduce the elasticity eigenvalue problem, recalling the dual-mixed formulation studied in [68]. We introduce the solution operator, and its spectral characterization. In Section 4.3 we introduce the DG spaces and interior penalty mixed discrete formulation. Section 4.4 deals with the well posedness of the discrete formulation, in order to introduce the corresponding discrete solution operator. In section 4.5 we study the spectral correctness of the method, proving the non pollution of the spectrum and error estimates for the eigenfunctions and eigenvalues, with the corresponding order of convergence. In Section 5.5 we report some numerical test to observe the asses of the method. Finally, we incorporate an appendix where we study the limit spectral elasticity problem, this means that is possible to consider spectral convergence of the formulation for λ large enough. The eigenvalues and eigenfunction of our model problems are proved to converge to corresponding ones when the Lamé constant λ goes to infinity. Additional regularity for the limit eigenfunctions is proved.

We end this section with some of the notations that we will use below. Given any Hilbert space V, let V^n and $V^{n\times n}$ denote, respectively, the space of vectors and tensors of order n $(n = 2, 3)$ with entries in V. In particular, $\boldsymbol{I}$ is the identity matrix of $\mathbb{R}^{n\times n}$ and $\mathbf{0}$ denotes a generic null vector or tensor. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{n\times n}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{n} \tau_{ii}$, the deviatoric tensor $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{n}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I}$, and the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^{n} \tau_{ij}\sigma_{ij}$.

Let Ω be a polyhedral Lipschitz bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$. For $s \geq 0$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathrm{H}^s(\Omega)^n$ or $\mathrm{H}^s(\Omega)^{n\times n}$, with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define for $s \geq 0$ the Hilbert space $\mathrm{H}^s(\mathbf{div}, \Omega) := \{\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^{n\times n} : \ \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^n\}$, whose norm is given by $\|\boldsymbol{\tau}\|^2_{\mathrm{H}^s(\mathbf{div},\Omega)} := \|\boldsymbol{\tau}\|^2_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{s,\Omega}$ and denote $\mathrm{H}(\mathbf{div}, \Omega) := \mathrm{H}^0(\mathbf{div}; \Omega)$.

Henceforth, we denote by C generic constants independent of the discretization parameter, which may take different values at different places.

4.2 The model problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be an open bounded Lipschitz polygon/polyhedron representing a solid domain. We denote by $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$ and assume that $\partial\Omega = \Gamma_R \cup \Gamma_F$, with $\mathrm{int}(\Gamma_R) \cap \mathrm{int}(\Gamma_F) = \emptyset$. The solid is supposed to be isotropic and linearly elastic with mass density ρ and Lamé constants μ and λ. We assume that the structure is fixed at $\Gamma_R \neq \emptyset$ and free of stress on Γ_F. We can combine the constitutive law

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega, \tag{4.2.1}$$

and the equilibrium equation

$$\omega^2\boldsymbol{u} = \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma} \qquad \text{in } \Omega, \tag{4.2.2}$$

to eliminate either the displacement field $\boldsymbol{u}$ or the Cauchy stress tensor $\boldsymbol{\sigma}$ from the global spectral formulation of the elasticity problem. Here, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\Big\{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\Big\}$ is the linearized strain tensor, and $\mathcal{C} : \ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the Hooke operator, which is given in terms of the Lamé coefficients λ and μ by

$$\mathcal{C}\boldsymbol{\tau} := \lambda\,(\mathrm{tr}\,\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{R}^{n\times n}.$$

Opting for the elimination of the displacement $\boldsymbol{u}$ and maintaining the stress tensor $\boldsymbol{\sigma}$ as a main variable leads to the following dual mixed formulation of the elasticity eigenproblem: Find $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{n\times n}$ symmetric, $\boldsymbol{r} : \Omega \to \mathbb{R}^{n\times n}$ skew symmetric and $\omega \in \mathbb{R}$ such that,

$$\begin{array}{rcll} -\nabla\left(\rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}\right) & = & \omega^2\left(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}\right) & \text{in } \Omega, \\ \mathbf{div}\,\boldsymbol{\sigma} & = & \mathbf{0} & \text{on } \Gamma_R, \\ \boldsymbol{\sigma}\boldsymbol{n} & = & \mathbf{0} & \text{on } \Gamma_F \end{array} \tag{4.2.3}$$

We notice that the additional variable $\boldsymbol{r} := \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ is the rotation. It acts as a Lagrange multiplier for the symmetry restriction. We also point out that the displacement can be recovered and also post-processed at the discrete level by using identity (4.2.2).

Taking into account that the Neumann boundary condition becomes essential in the mixed formulation, we consider the closed subspace $\boldsymbol{\mathcal{W}}$ of $\mathrm{H}(\mathbf{div}, \Omega)$ given by

$$\boldsymbol{\mathcal{W}} := \{\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div}; \Omega), \quad \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_F\}\,.$$

The rotation $\boldsymbol{r}$ will be sought in the space

$$\boldsymbol{\mathcal{Q}} := \{\boldsymbol{s} \in \mathrm{L}^2(\Omega)^{n\times n} : \ \boldsymbol{s}^{\mathtt{t}} = -\boldsymbol{s}\}.$$

We introduce the symmetric bilinear forms

$$B\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{r} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{s} : \boldsymbol{\sigma}$$

and

$$A\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big) := \int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + B\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)$$

and denote the Hilbertian product norm on $\mathrm{H}(\mathbf{div};\Omega)\times\mathrm{L}^2(\Omega)^{n\times n}$ by

$$\|(\boldsymbol{\tau},\boldsymbol{s})\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div};\Omega)} + \|\boldsymbol{s}\|^2_{0,\Omega}.$$

The variational formulation of the eigenvalue problem (4.2.3) is given as follows in terms of $\kappa := 1+\omega^2$ (see [69] for more details): Find $\kappa\in\mathbb{R}$ and $\mathbf{0}\neq(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ such that

$$A\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big) = \kappa\, B\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big) \quad \forall(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}. \tag{4.2.4}$$

We notice that the bilinear form

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{C},\mathbf{div}} := \int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau}$$

also defines an inner product on $\boldsymbol{\mathcal{W}}$. Moreover, the following well-known result establishes that the norm induced by $(\cdot,\cdot)_{\mathcal{C},\mathbf{div}}$ is equivalent to $\|\cdot\|_{\mathrm{H}(\mathbf{div};\Omega)}$ uniformly in the Lamé coefficient λ.

Proposition 4.2.1 *There exist constants $c_2\geq c_1>0$ independent of λ such that*

$$c_1\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)} \leq \|\boldsymbol{\tau}\|_{\mathcal{C},\mathbf{div}} \leq c_2\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)} \quad \forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}},$$

where $\|\boldsymbol{\tau}\|_{\mathcal{C},\mathbf{div}} := \sqrt{(\boldsymbol{\tau},\boldsymbol{\tau})_{\mathcal{C},\mathbf{div}}}$.

Proof. The bound from above follows immediately from the fact that

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} = \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \frac{1}{n(n\lambda+2\mu)}\int_\Omega (\operatorname{tr}\boldsymbol{\sigma})(\operatorname{tr}\boldsymbol{\tau}) \tag{4.2.5}$$

is bounded by a constant independent of λ. The left inequality may be found, for example, in [69, Lemma 2.1]. □

As a consequence of Proposition 4.2.1, there exists a constant $M>0$ independent of λ such that

$$\left|A\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)\right| \leq M\,\|(\boldsymbol{\sigma},\boldsymbol{r})\|\|(\boldsymbol{\tau},\boldsymbol{s})\| \quad \forall(\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}. \tag{4.2.6}$$

Proposition 4.2.2 *There exists a constant $\alpha>0$, depending on ρ, μ and Ω (but not on λ), such that*

$$\sup_{(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}} \frac{A\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)}{\|(\boldsymbol{\tau},\boldsymbol{s})\|} \geq \alpha\|(\boldsymbol{\sigma},\boldsymbol{r})\| \quad \forall(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}. \tag{4.2.7}$$

Proof. It follows from Proposition 4.2.1 that

$$A\Big((\boldsymbol{\tau},\mathbf{0}),(\boldsymbol{\tau},\mathbf{0})\Big)=(\boldsymbol{\tau},\boldsymbol{\tau})_{\mathcal{C},\mathrm{div}}\geq C_1^2\|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div};\Omega)}\qquad\forall\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}},\tag{4.2.8}$$

with $C_1>0$ independent of λ. On the other hand, there exists a constant $\beta>0$ depending only on Ω (see, for instance, [22]) such that

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}}\frac{\int_\Omega \boldsymbol{s}:\boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathrm{div};\Omega)}}\geq\beta\|\boldsymbol{s}\|_{0,\Omega},\qquad\forall\boldsymbol{s}\in\boldsymbol{\mathcal{Q}}.\tag{4.2.9}$$

Consequently, the Babuška-Brezzi theory shows that, for any bounded linear form $L\in\mathcal{L}(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$, the problem: find $(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ such that

$$A\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)=L\big((\boldsymbol{\tau},\boldsymbol{s})\big)\qquad\forall(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$$

is well-posed, which proves (4.2.7). □

We deduce from Proposition 4.2.2 and from the symmetry of $A(\cdot,\cdot)$ that the operator $\boldsymbol{T}:[\mathrm{L}^2(\Omega)^{n\times n}]^2\to\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ defined for any $(\boldsymbol{f},\boldsymbol{g})\in[\mathrm{L}^2(\Omega)^{n\times n}]^2$, by

$$A\Big(\boldsymbol{T}(\boldsymbol{f},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)=B\Big((\boldsymbol{f},\boldsymbol{g}),(\boldsymbol{\tau},\boldsymbol{s})\Big)\quad\forall(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}\tag{4.2.10}$$

is well-defined and symmetric with respect to $A(\cdot,\cdot)$. Moreover, there exists a constant $C>0$ independent of λ such that

$$\|\boldsymbol{T}(\boldsymbol{f},\boldsymbol{g})\|\leq C\|(\boldsymbol{f},\boldsymbol{g})\|_{0,\Omega},\quad\forall(\boldsymbol{f},\boldsymbol{g})\in[\mathrm{L}^2(\Omega)^{n\times n}]^2.\tag{4.2.11}$$

It is clear that $(\kappa,(\boldsymbol{\sigma},\boldsymbol{r}))$ is a solution of Problem (4.2.4) if and only if $\big(\eta=\frac{1}{\kappa},(\boldsymbol{\sigma},\boldsymbol{r})\big)$ is an eigenpair for $\boldsymbol{T}$. Let

$$\boldsymbol{\mathcal{K}}:=\{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}:\ \mathbf{div}\,\boldsymbol{\tau}=\mathbf{0}\ \text{ in }\Omega\}.$$

From the definition of $\boldsymbol{T}$, it is clear that $\boldsymbol{T}|_{\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}}:\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}\longrightarrow\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$ reduces to the identity. Thus, $\eta=1$ is an eigenvalue of $\boldsymbol{T}$ with eigenspace $\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$. We introduce the orthogonal subspace to $\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$ in $\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ with respect to the bilinear form B,

$$[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_B}:=\Big\{(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}:\ B\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)=0\quad\forall(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}\Big\}.\tag{4.2.12}$$

Lemma 4.2.1 *The subspace $[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_B}$ is invariant for $\boldsymbol{T}$, i.e.,*

$$\boldsymbol{T}([\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_B})\subset[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_B}.\tag{4.2.13}$$

Moreover, we have the direct and stable decomposition

$$\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}=[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]\oplus[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_B}.\tag{4.2.14}$$

Proof. See Lemma 3.3 and Lemma 3.4 of [69]. □

We deduce from Lemma 4.2.1 that there exists a unique projection $\boldsymbol{P}:\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}\to\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}$ with range $[\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}]^{\perp_B}$ and kernel $\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}$ associated to the splitting (4.2.14).

Let us consider the elasticity problem posed in Ω with a volume load in $\mathrm{L}^2(\Omega)^n$ and with homogeneous Dirichlet and Neumann boundary conditions on Γ_R and Γ_F (respectively). According to [35, 48], there exists $\widehat{s} > 0$ that depends on Ω, λ and μ such that the displacement field that solves this problem belongs to $\mathrm{H}^{1+s}(\Omega)^n$ for all $s \in (0, \widehat{s})$. The following result shows that $\boldsymbol{P}$ and $\boldsymbol{T} \circ \boldsymbol{P}$ are regularizing operators.

Lemma 4.2.2 *For all $s \in (0, \widehat{s})$, $\boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \subset \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n}$ and $\boldsymbol{T} \circ \boldsymbol{P}(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \subset \mathrm{H}^s(\mathrm{div}, \Omega) \times \mathrm{H}^s(\Omega)^{n\times n}$. Moreover, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{P}(\boldsymbol{\tau}, \boldsymbol{s})\|_{\mathrm{H}^s(\Omega)^{n\times n}\times \mathrm{H}^s(\Omega)^{n\times n}} \leq C\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \tag{4.2.15}$$

and

$$\|\boldsymbol{T} \circ \boldsymbol{P}(\boldsymbol{\tau}, \boldsymbol{s})\|_{\mathrm{H}^s(\Omega)\times \mathrm{H}^s(\Omega)^{n\times n}} \leq C\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}. \tag{4.2.16}$$

Proof. Estimate (4.2.16) is a consequence of (4.2.15) as can be seen in the proof of Proposition 3.5 from [69]. In turn, estimate (4.2.15) was proved in Lemma 3.2 of the same reference. This proof follows from the fact that $(\widetilde{\boldsymbol{\tau}}, \widetilde{\boldsymbol{s}}) = \boldsymbol{P}(\boldsymbol{\tau}, \boldsymbol{s})$ if and only if there exists $\widetilde{\boldsymbol{u}} \in \mathbb{H}^1(\Omega)^n$ such that

$$\begin{aligned}
-\,\mathbf{div}\,\widetilde{\boldsymbol{\tau}} &= -\,\mathbf{div}\,\boldsymbol{\tau} && \text{in}\,\Omega,\\
\widetilde{\boldsymbol{\tau}} &= \varepsilon(\widehat{\boldsymbol{u}}) && \text{in}\,\Omega,\\
\widetilde{\boldsymbol{\tau}}\nu &= 0 && \text{in}\,\Gamma_N,\\
\widehat{\boldsymbol{u}} &= 0 && \text{in}\,\Gamma_D.
\end{aligned}$$

Therefore, (4.2.15) follows from the classical a priori estimate for the linear elasticity equations (see for instance [48]). □

In principle, the exponent $\widehat{s}$ and the constant C in (4.2.15) depend od the Lamé coefficient λ. Since a similar estimate holds true for $\lambda = +\infty$ (see from appendix) it is reasonable to expect that there exists $\widehat{s}$ and C such that (4.2.15) holds true for all $\lambda > 0$. However, to the bests of the authors knowledge, this has not been proved. From now and on we make this assumption:

Assumption 4.2.1 *There exists $\widehat{s}$ and $\widehat{C}$ independent of λ such that the following estimate holds true*

$$\|\boldsymbol{P}(\boldsymbol{\tau}, \boldsymbol{s})\|_{\mathrm{H}^s(\Omega)^{n\times n}\times \mathrm{H}^s(\Omega)^{n\times n}} \leq \widehat{C}\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \tag{4.2.17}$$

The next result stablish a spectral characterization for operator $\boldsymbol{T}$.

Proposition 4.2.3 *The spectrum $\mathrm{sp}(\boldsymbol{T})$ of $\boldsymbol{T}$ decomposes as follows*

$$\mathrm{sp}(\boldsymbol{T}) = \{0, 1\} \cup \{\eta_k\}_{k\in\mathbb{N}}$$

where $\{\eta_k\}_k \subset (0, 1)$ is a real sequence of finite-multiplicity eigenvalues of $\boldsymbol{T}$ which converges to 0. The ascent of each of these eigenvalues is 1 and the corresponding eigenfunctions lie in $\boldsymbol{P}(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})$. Moreover, $\eta = 1$ is an infinite-multiplicity eigenvalue of $\boldsymbol{T}$ with associated eigenspace $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ and $\eta = 0$ is not an eigenvalue.

Proof. See [69, Theorem 3.7]. □

The following result provides a bound of the resolvent $\big(z\boldsymbol{I}-\boldsymbol{T}\big)^{-1}$.

Proposition 4.2.4 *If $z \notin \operatorname{sp}(\boldsymbol{T})$, there exists a constant $C>0$ independent of λ and z such that*

$$\|\big(z\boldsymbol{I}-\boldsymbol{T}\big)(\boldsymbol{\sigma},\boldsymbol{r})\| \geq C \operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)\,\|(\boldsymbol{\sigma},\boldsymbol{r})\| \quad \forall(\boldsymbol{\sigma},\boldsymbol{r})\in\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}}. \tag{4.2.18}$$

where $\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)$ represents the distance between z and the spectrum of $\boldsymbol{T}$ in the complex plane, which in principle depends on λ.

4.3 A discontinuous Galerkin discretization

We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega}$ into triangles/tetrahedra K of diameter h_K. The parameter $h:=\max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. Hereafter, given an integer $m\geq 0$ and a domain $D\subset\mathbb{R}^n$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D.

We say that a closed subset $F\subset\overline{\Omega}$ is an interior edge/face if F has a positive $(n-1)$-dimensional measure and if there are distinct elements K and K' such that $F=\bar{K}\cap\bar{K}'$. A closed subset $F\subset\overline{\Omega}$ is a boundary edge/face if there exists $K\in\mathcal{T}_h$ such that F is an edge/face of K and $F=\bar{K}\cap\partial\Omega$. We consider the set $\mathcal{F}_h^0$ of interior edges/faces and the set $\mathcal{F}_h^\partial$ of boundary edges/faces. We assume that the boundary mesh $\mathcal{F}_h^\partial$ is compatible with the partition $\partial\Omega=\Gamma_R\cup\Gamma_F$, i.e.,

$$\cup_{F\in\mathcal{F}_h^R}F=\Gamma_R \qquad\text{and}\qquad \cup_{F\in\mathcal{F}_h^F}F=\Gamma_F$$

where $\mathcal{F}_h^R:=\{F\in\mathcal{F}_h^\partial;\quad F\subset\Gamma_R\}$ and $\mathcal{F}_h^F:=\{F\in\mathcal{F}_h^\partial;\quad F\subset\Gamma_F\}$. We denote

$$\mathcal{F}_h:=\mathcal{F}_h^0\cup\mathcal{F}_h^\partial \qquad\text{and}\qquad \mathcal{F}_h^*:=\mathcal{F}_h^0\cup\mathcal{F}_h^F,$$

and for any element $K\in\mathcal{T}_h$, we introduce the set

$$\mathcal{F}(K):=\{F\in\mathcal{F}_h;\quad F\subset\partial K\}$$

of edges/faces composing the boundary of K.

The space of piecewise polynomial functions of degree at most m relatively to $\mathcal{T}_h$ is denoted by

$$\mathcal{P}_m(\mathcal{T}_h):=\{v\in\mathrm{L}^2(\Omega);\quad v|_K\in\mathcal{P}_m(K),\quad\forall K\in\mathcal{T}_h\}.$$

For any $k\geq 1$, we consider the finite element spaces

$$\boldsymbol{\mathcal{W}}_h:=\mathcal{P}_k(\mathcal{T}_h)^{n\times n} \qquad \boldsymbol{\mathcal{W}}_h^c:=\boldsymbol{\mathcal{W}}_h\cap\boldsymbol{\mathcal{W}} \qquad\text{and}\qquad \boldsymbol{\mathcal{Q}}_h:=\mathcal{P}_{k-1}(\mathcal{T}_h)^{n\times n}\cap\boldsymbol{\mathcal{Q}}.$$

Let us now recall some well-known properties of the Brezzi-Douglas-Marini (BDM) mixed finite element [27]. For $t>1/2$, the tensorial version of the BDM-interpolation operator $\Pi_h:\mathrm{H}^t(\Omega)^{n\times n}\to\boldsymbol{\mathcal{W}}_h^c$, satisfies the following classical error estimate, see [21, Proposition 2.5.4],

$$\|\boldsymbol{\tau}-\Pi_h\boldsymbol{\tau}\|_{0,\Omega}\leq Ch^{\min(t,k+1)}\|\boldsymbol{\tau}\|_{t,\Omega} \qquad \forall\boldsymbol{\tau}\in\mathrm{H}^t(\Omega)^{n\times n},\quad t>1/2. \tag{4.3.19}$$

For less regular tensorial fields we also have the following error estimate

$$\|\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}\|_{0,\Omega} \leq C h^t (\|\boldsymbol{\tau}\|_{t,\Omega} + \|\boldsymbol{\tau}\|_{\mathbb{H}(\mathrm{div};\Omega)}) \quad \forall \boldsymbol{\tau} \in \mathbb{H}^t(\Omega)^{n\times n} \cap \mathbb{H}(\mathrm{div};\Omega), \quad t \in (0, 1/2]. \tag{4.3.20}$$

Moreover, thanks to the commutativity property, if $\mathbf{div}\, \boldsymbol{\tau} \in \mathrm{H}^t(\Omega)^n$, then

$$\|\mathbf{div}(\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau})\|_{0,\Omega} = \|\mathbf{div}\, \boldsymbol{\tau} - \mathcal{R}_h \,\mathbf{div}\, \boldsymbol{\tau}\|_{0,\Omega} \leq C h^{\min(t,k)} \|\mathrm{div}\, \boldsymbol{v}\|_{0,\Omega}, \tag{4.3.21}$$

where $\mathcal{R}_h$ is the $\mathrm{L}^2(\Omega)^n$-orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{T}_h)^n$. Finally, we denote by $\mathcal{S}_h : \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $\mathrm{L}^2(\Omega)^{n\times n}$-norm. It is well-known that, for any $t > 0$, we have

$$\|\boldsymbol{r} - \mathcal{S}_h \boldsymbol{r}\|_{0,\Omega} \leq C h^{\min(t,k)} \|\boldsymbol{r}\|_{t,\Omega} \qquad \forall \boldsymbol{r} \in \mathrm{H}^t(\Omega)^{n\times n} \cap \boldsymbol{\mathcal{Q}}. \tag{4.3.22}$$

For the analysis we need to decompose adequately the space $\boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$. We consider,

$$\boldsymbol{\mathcal{K}}_h = \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h^c; \quad \mathbf{div}\, \boldsymbol{\tau} = 0\} \subset \boldsymbol{\mathcal{K}}.$$

Lemma 4.3.1 *There exists a projection $\boldsymbol{P}_h : \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h \to \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$ with kernel $\boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h$ such that for all $s \in (0, \widehat{s})$, there exists a constant C independent of h and λ such that*

$$\|(\boldsymbol{P} - \boldsymbol{P}_h)(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \leq C\, h^s \|\mathbf{div}\, \boldsymbol{\sigma}_h\|_{0,\Omega} \quad \forall (\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h.$$

Proof. See [69, Lemma 4.2] □

For any $t \geq 0$, we consider the broken Sobolev space

$$\mathrm{H}^t(\mathcal{T}_h) := \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^n; \quad \boldsymbol{v}|_K \in \mathrm{H}^t(K)^n \quad \forall K \in \mathcal{T}_h\}.$$

For each $\boldsymbol{v} := \{\boldsymbol{v}_K\} \in \mathrm{H}^t(\mathcal{T}_h)^n$ and $\boldsymbol{\tau} := \{\boldsymbol{\tau}_K\} \in \mathrm{H}^t(\mathcal{T}_h)^{n\times n}$ the components $\boldsymbol{v}_K$ and $\boldsymbol{\tau}_K$ represent the restrictions $\boldsymbol{v}|_K$ and $\boldsymbol{\tau}|_K$. When no confusion arises, the restrictions of these functions will be written without any subscript. We will also need the space given on the skeletons of the triangulations $\mathcal{T}_h$ by

$$\mathrm{L}^2(\mathcal{F}_h) := \prod_{F\in\mathcal{F}_h} \mathrm{L}^2(F).$$

Similarly, the components χ_F of $\chi := \{\chi_F\} \in \mathrm{L}^2(\mathcal{F}_h)$ coincide with the restrictions $\chi|_F$ and we denote

$$\int_{\mathcal{F}_h} \chi := \sum_{F\in\mathcal{F}_h} \int_F \chi_F \quad \text{and} \quad \|\chi\|^2_{0,\mathcal{F}_h} := \int_{\mathcal{F}_h} \chi^2, \qquad \forall \chi \in \mathrm{L}^2(\mathcal{F}_h).$$

Similarly, $\|\chi\|^2_{0,\mathcal{F}_h^*} := \sum_{F\in\mathcal{F}_h^*} \int_F \chi_F^2$ for all $\chi \in \mathrm{L}^2(\mathcal{F}_h^*) := \prod_{F\in\mathcal{F}_h^*} \mathrm{L}^2(F)$.

From now on, $h_{\mathcal{F}} \in \mathrm{L}^2(\mathcal{F}_h)$ is the piecewise constant function defined by $h_{\mathcal{F}}|_F := h_F$ for all $F \in \mathcal{F}_h$ with h_F denoting the diameter of edge/face F.

Given a vector valued function $\boldsymbol{v} \in \mathrm{H}^t(\mathcal{T}_h)^n$, with $t > 1/2$, we define averages $\{\boldsymbol{v}\} \in \mathrm{L}^2(\mathcal{F}_h)^n$ and jumps $[\![\boldsymbol{v}]\!] \in \mathrm{L}^2(\mathcal{F}_h)$ by

$$\{\boldsymbol{v}\}_F := (\boldsymbol{v}_K + \boldsymbol{v}_{K'})/2 \quad \text{and} \quad [\![\boldsymbol{v}]\!]_F := \boldsymbol{v}_K \cdot \boldsymbol{n}_K + \boldsymbol{v}_{K'} \cdot \boldsymbol{n}_{K'} \quad \forall F \in \mathcal{F}(K) \cap \mathcal{F}(K'),$$

where $\boldsymbol{n}_K$ is the outward unit normal vector to ∂K. On the boundary of Ω we use the following conventions for averages and jumps:

$$\{\boldsymbol{v}\}_F := \boldsymbol{v}_K \quad \text{and} \quad [\![\boldsymbol{v}]\!]_F := \boldsymbol{v}_K \cdot \boldsymbol{n} \quad \forall F \in \mathcal{F}(K) \cap \partial\Omega.$$

Similarly, for matrix valued functions $\boldsymbol{\tau} \in \mathrm{H}^t(\mathcal{T}_h)^{n\times n}$, we define $\{\boldsymbol{\tau}\} \in \mathrm{L}^2(\mathcal{F}_h)^{n\times n}$ and $[\![\boldsymbol{\tau}]\!] \in \mathrm{L}^2(\mathcal{F}_h)^n$ by

$$\{\boldsymbol{\tau}\}_F := (\boldsymbol{\tau}_K + \boldsymbol{\tau}_{K'})/2 \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_F := \boldsymbol{\tau}_K \boldsymbol{n}_K + \boldsymbol{\tau}_{K'} \boldsymbol{n}_{K'} \quad \forall F \in \mathcal{F}(K) \cap \mathcal{F}(K')$$

and on the boundary of Ω we set

$$\{\boldsymbol{\tau}\}_F := \boldsymbol{\tau}_K \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_F := \boldsymbol{\tau}_K \boldsymbol{n} \quad \forall F \in \mathcal{F}(K) \cap \partial\Omega.$$

Given $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}_h$ we define $\mathbf{div}_h\, \boldsymbol{\tau} \in \mathrm{L}^2(\Omega)^n$ by $\mathbf{div}_h\, \boldsymbol{\tau}|_K = \mathbf{div}(\boldsymbol{\tau}|_K)$ for all $K \in \mathcal{T}_h$ and endow $\boldsymbol{\mathcal{W}}(h) := \boldsymbol{\mathcal{W}} + \boldsymbol{\mathcal{W}}_h$ with the seminorm

$$|\boldsymbol{\tau}|^2_{\boldsymbol{\mathcal{W}}(h)} := \|\mathbf{div}_h\, \boldsymbol{\tau}\|^2_{0,\Omega} + \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{F}_h^*}$$

and the norm

$$\|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{W}}(h)} := |\boldsymbol{\tau}|^2_{\boldsymbol{\mathcal{W}}(h)} + \|\boldsymbol{\tau}\|^2_{0,\Omega}.$$

For the sake of simplicity, we will also use the notation

$$\|(\boldsymbol{\tau}, \boldsymbol{s})\|^2_{DG} := \|\boldsymbol{\tau}\|^2_{\boldsymbol{\mathcal{W}}(h)} + \|\boldsymbol{s}\|^2_{0,\Omega}.$$

The following result will be used in the sequel to ultimately derive a method free of spurious modes. Since according to Proposition 4.2.3 the spectrum of $\boldsymbol{T}$ lies in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \le 1\}$, we restrict our attention to this subset of the complex plane.

Lemma 4.3.2 *There exists a constant $C > 0$ independent of h and λ such that for all $z \in \mathbb{C} \setminus \mathrm{sp}(\boldsymbol{T})$ with $|z| \le 1$, there holds*

$$\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau}, \boldsymbol{s})\|_{DG} \ge C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)|z|\, \|(\boldsymbol{\tau}, \boldsymbol{s})\|_{DG} \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}}(h) \times \boldsymbol{\mathcal{Q}}.$$

Proof. We introduce

$$(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) := \boldsymbol{T}(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$$

and notice that

$$(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) = \boldsymbol{T}(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau}, \boldsymbol{s}).$$

By virtue of Proposition 4.2.3 and the boundedness of $\boldsymbol{T} : [\mathrm{L}^2(\Omega)^{n\times n}]^2 \to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ we have that

$$\begin{aligned} C \operatorname{dist}\big(z, \mathrm{sp}(\boldsymbol{T})\big)\|(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)\| &\le \|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\sigma}^*, \boldsymbol{r}^*)\| \le \|\boldsymbol{T}(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau}, \boldsymbol{s})\| \\ &\le \|\boldsymbol{T}\|\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau}, \boldsymbol{s})\|_0 \le \|\boldsymbol{T}\|\|(z\boldsymbol{I} - \boldsymbol{T})(\boldsymbol{\tau}, \boldsymbol{s})\|_{DG}. \end{aligned}$$

Finally, by the triangle inequality,

$$\begin{aligned}\|(\boldsymbol{\tau},\boldsymbol{s})\|_{DG} \leq |z|^{-1}\|(\boldsymbol{\sigma}^*,\boldsymbol{r}^*)\| + &|z|^{-1}\|(z\boldsymbol{I}-\boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}\\ \leq &|z|^{-1}\left(1+\frac{\|\boldsymbol{T}\|}{C\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)}\right)\|(z\boldsymbol{I}-\boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}\\ \leq &|z|^{-1}\left(\frac{C\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)+\|\boldsymbol{T}\|}{C\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)}\right)\|(z\boldsymbol{I}-\boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}\end{aligned}$$

hence

$$C|z|\left(\frac{\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)}{\|\boldsymbol{T}\|+\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)}\right)\leq\|(z\boldsymbol{I}-\boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}.$$

Since $\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)\leq|z|\leq 1$ and $\|\boldsymbol{T}\|\leq C'$ (with C' independent of λ), from the estimate above we derive

$$\frac{C|z|}{1+C'}\operatorname{dist}\big(z,\operatorname{sp}(\boldsymbol{T})\big)\leq\leq\|(z\boldsymbol{I}-\boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}.$$

and the result follows with $C'':=\dfrac{C}{1+C'}$. □

Remark 4.3.1 *If F is a compact subset of $\mathbb{D}\setminus\operatorname{sp}\{\boldsymbol{T}\}$, we deduce from Lemma 4.3.2 that there exists a constant $C>0$ independent of h and λ such that, for all $z\in F$,*

$$\|\big(z\boldsymbol{I}-\boldsymbol{T}\big)^{-1}\|_{\mathcal{L}(\boldsymbol{\mathcal{W}}(h)\times\boldsymbol{\mathcal{Q}},\boldsymbol{\mathcal{W}}(h)\times\boldsymbol{\mathcal{Q}})}\leq\frac{C}{\operatorname{dist}\big(F,\operatorname{sp}(\boldsymbol{T})\big)|z|}. \tag{4.3.23}$$

Given a parameter $\mathtt{a}_S>0$, we introduce the symmetric bilinear form

$$\begin{aligned}A_h\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big):=\int_\Omega\rho^{-1}\,\mathbf{div}_h\,\boldsymbol{\sigma}\cdot\mathbf{div}_h\,\boldsymbol{\tau}+B\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)+\\ \int_{\mathcal{F}_h^*}\mathtt{a}_S h_{\mathcal{F}}^{-1}[\![\boldsymbol{\sigma}]\!]\cdot[\![\boldsymbol{\tau}]\!]-\int_{\mathcal{F}_h^*}\left(\{\rho^{-1}\,\mathbf{div}_h\,\boldsymbol{\sigma}\}\cdot[\![\boldsymbol{\tau}]\!]+\{\rho^{-1}\,\mathbf{div}_h\,\boldsymbol{\tau}\}\cdot[\![\boldsymbol{\sigma}]\!]\right)\end{aligned}$$

and consider the DG method: find $\kappa_h\in\mathbb{R}$ and $0\neq(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h$ such that

$$A_h\Big((\boldsymbol{\sigma}_h,\boldsymbol{r}_h),(\boldsymbol{\tau},\boldsymbol{s})\Big)=\kappa_h B\Big((\boldsymbol{\sigma}_h,\boldsymbol{r}_h),(\boldsymbol{\tau},\boldsymbol{s})\Big)\qquad\forall(\boldsymbol{\tau},\boldsymbol{s})\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h. \tag{4.3.24}$$

We notice that, as it is usually the case for DG methods, the essential boundary condition is directly incorporated within the scheme.

A straightforward application of the Cauchy-Schwarz inequality shows that, for all $(\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\in\mathrm{H}^t(\mathbf{div};\mathcal{T}_h)\times\boldsymbol{\mathcal{Q}}$ $(t>1/2)$, there exists a constant $M^*>0$ independent of h and λ such that

$$\Big|A_h\Big((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\tau},\boldsymbol{s})\Big)\Big|\leq M^*\|(\boldsymbol{\sigma},\boldsymbol{r})\|_{DG}^*\,\|(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}^*, \tag{4.3.25}$$

where

$$\|(\boldsymbol{\sigma},\boldsymbol{r})\|_{DG}^*:=\Big(\|(\boldsymbol{\sigma},\boldsymbol{r})\|_{DG}^2+\|h_{\mathcal{F}}^{1/2}\{\mathbf{div}\,\boldsymbol{\sigma}\}\|_{0,\mathcal{F}_h^*}^2\Big)^{1/2}.$$

Moreover, we deduce from the discrete trace inequality (See [38])

$$\|h_{\mathcal{F}}^{1/2}\{v\}\|_{0,\mathcal{F}_h} \leq C\|v\|_{0,\Omega} \quad \forall v \in \mathcal{P}_k(\mathcal{T}_h), \tag{4.3.26}$$

that for all $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathrm{H}^t(\mathbf{div}; \mathcal{T}_h) \times \boldsymbol{\mathcal{Q}}$ $(t > 1/2)$, and $(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$,

$$\left| A_h\Big((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) \right| \leq M_{DG}\|(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG}^{*} \, \|(\boldsymbol{\tau}, \boldsymbol{s})\|_{DG}, \tag{4.3.27}$$

with $M_{DG} > 0$ is independent of h and λ.

4.4 The DG-discrete source operator

The following discrete projection operator from the DG-space $\boldsymbol{\mathcal{W}}_h$ onto the $\mathrm{H}(\mathbf{div}; \Omega)$-conforming mixed finite element space $\boldsymbol{\mathcal{W}}^c$ in essential in the forthcoming analysis.

Proposition 4.4.1 *There exists a projection $\mathcal{I}_h : \boldsymbol{\mathcal{W}}_h \to \boldsymbol{\mathcal{W}}_h^c$ such that the norm equivalence*

$$\underline{C}\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{W}}(h)} \leq \left(\|\mathcal{I}_h\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)}^2 + \|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}]\!]\|_{0,\mathcal{F}_h^*}^2\right)^{1/2} \leq \bar{C}\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{W}}(h)} \tag{4.4.28}$$

holds true on $\boldsymbol{\mathcal{W}}_h$ with constants $\underline{C} > 0$ and $\bar{C} > 0$ independent of h. Moreover, we have that

$$\|\mathbf{div}_h(\boldsymbol{\tau} - \mathcal{I}_h\boldsymbol{\tau})\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} h_K^{-2}\|\boldsymbol{\tau} - \mathcal{I}_h\boldsymbol{\tau}\|_{0,K}^2 \leq C_0\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}]\!]\|_{0,\mathcal{F}_h^*}^2, \tag{4.4.29}$$

with $C_0 > 0$ independent of h.

Proof. See [68, Proposition 5.2]. □

Now we prove that the bilinear form A_h satisfies an inf-sup condition, to prove the stability of the DG method.

Proposition 4.4.2 *There exists a positive parameter $\mathtt{a}_S^*$ such that, for all $\mathtt{a}_S \geq \mathtt{a}_S^*$,*

$$\sup_{(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h} \frac{A_h\Big((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big)}{\|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG}} \geq \alpha_{DG}\|(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG}, \quad \forall(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \tag{4.4.30}$$

with $\alpha_{DG} > 0$ independent of h and λ.

Proof. It is shown in [68, Proposition 3.1] that there exists a constant $\alpha_A^c > 0$ independent of h and λ such that

$$\sup_{(\boldsymbol{\tau}_h,\boldsymbol{s}_h)\in\boldsymbol{\mathcal{W}}_h^c\times\boldsymbol{\mathcal{Q}}_h} \frac{A\Big((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big)}{\|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|} \geq \alpha_A^c\|(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \quad \forall(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h. \tag{4.4.31}$$

It follows that there exists an operator $\Theta_h : \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h \to \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$ satisfying

$$A\Big((\boldsymbol{\sigma}_h, \boldsymbol{r}_h), \Theta_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\Big) = \alpha_A^c\|(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|^2 \quad \text{and} \quad \|\Theta_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \leq \|(\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\| \tag{4.4.32}$$

for all $(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) \in \boldsymbol{\mathcal{W}}_h^c \times \boldsymbol{\mathcal{Q}}_h$.

Given $(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$, the decomposition $\boldsymbol{\tau}_h = \boldsymbol{\tau}_h^c + \tilde{\boldsymbol{\tau}}_h$, with $\boldsymbol{\tau}_h^c := \mathcal{I}_h \boldsymbol{\tau}_h$ and $\tilde{\boldsymbol{\tau}}_h := \boldsymbol{\tau}_h - \mathcal{I}_h \boldsymbol{\tau}_h$, and (4.4.32) yield

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) = \alpha_A^c \|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 + \\ A_h\Big((\boldsymbol{\tau}_h^c, \boldsymbol{s}_h), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) + A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\Big) + A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big). \quad (4.4.33)$$

By the Cauchy-Schwarz inequality,

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) = \rho^{-1} \|\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\|_{0,\Omega}^2 + \mathtt{a}_S \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}^2 + \int_\Omega \mathcal{C}^{-1} \tilde{\boldsymbol{\tau}}_h : \tilde{\boldsymbol{\tau}}_h \\ - 2 \int_{\mathcal{F}_h^*} \{\rho^{-1}\, \mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\} \cdot [\![\tilde{\boldsymbol{\tau}}_h]\!] \;\geq\; \mathtt{a}_S \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}^2 \\ - 2\rho^{-1} \|h_{\mathcal{F}}^{1/2} \{\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\}\|_{0,\mathcal{F}_h^*} \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}$$

and we deduce from (4.3.26) and (4.4.29) that

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq (\mathtt{a}_S - C_1) \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}^2.$$

with a constant C_1 independent of h and λ.

We proceed similarly for the terms in the right-hand side of (4.4.33). Indeed, it is straightforward that

$$A_h\Big((\boldsymbol{\tau}_h^c, \boldsymbol{s}_h), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq -\rho^{-1} \|\mathbf{div}\, \boldsymbol{\tau}_h^c\|_{0,\Omega} \|\mathbf{div}_h\, \tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} - C_2 \|\tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} (\|\boldsymbol{\tau}_h^c\|_{0,\Omega} + \|\boldsymbol{s}_h\|_{0,\Omega}) - \\ \rho^{-1} \|h_{\mathcal{F}}^{1/2} \{\mathbf{div}\, \boldsymbol{\tau}_h^c\}\|_{0,\mathcal{F}_h^*} \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}$$

and using again (4.3.26) and (4.4.29) we obtain

$$A_h\Big((\boldsymbol{\tau}_h^c, \boldsymbol{s}_h), (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq -C_3 \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*} \|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\| \geq \\ - \frac{\alpha_A^c}{4} \|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 - C_4 \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}^2$$

with $C_4 > 0$ independent of h and λ. Similar estimates lead to

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\Big) \geq -C_5 \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*} \|\Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\| \geq \\ - C_5 \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*} \|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|,$$

where the last inequality follows from (4.4.32). We conclude that there exists $C_6 > 0$ independent of h and λ such that

$$A_h\Big((\tilde{\boldsymbol{\tau}}_h, \mathbf{0}), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\Big) \geq -\frac{\alpha_D^c}{4} \|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 - C_6 \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h^*}^2.$$

We then have shown that,

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq \frac{\alpha_A^c}{2} \|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 + \big(\mathtt{a}_S - C_7\big) \|h_{\mathcal{F}}^{-1/2} [\![\boldsymbol{\tau}_h]\!]\|_{0,\mathcal{F}_h}^2,$$

with $C_7 := C_1 + C_4 + C_6$. Consequently, if $\mathtt{a}_S > \mathtt{a}_S^* := C_7 + \frac{\alpha_A^c}{2}$,

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq \frac{\alpha_A^c}{2}\Big(\|(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|^2 + \|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}]\!]\|_{0,\mathcal{F}_h^*}^2\Big),$$

and thanks to (4.4.28), we conclude that there exists $\alpha_{DG} > 0$ such that,

$$A_h\Big((\boldsymbol{\tau}_h, \boldsymbol{s}_h), \Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\Big) \geq \alpha_{DG}\|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG}\Big(\|\Theta_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\|_{DG}\Big),$$

which gives (4.4.30). □

In the sequel, we assume that the stabilization parameter is big enough $\mathtt{a}_S > \mathtt{a}_S^*$ so that the inf-sup condition (4.4.30) is guaranteed. The first consequence of this inf-sup condition is that the operator $\boldsymbol{T}_h : \mathrm{L}^2(\Omega)^{n\times n} \times \mathrm{L}^2(\Omega)^{n\times n} \to \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ characterized, for any $(\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2$, by

$$A_h\Big(\boldsymbol{T}_h(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) = B\Big((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}, \boldsymbol{s})\Big) \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \tag{4.4.34}$$

is well-defined, symmetric with respect to $A_h(\cdot, \cdot)$ and there exists a constant $C > 0$ independent of λ and h such that

$$\|\boldsymbol{T}_h(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq C\|(\boldsymbol{f}, \boldsymbol{g})\|_0, \quad \forall (\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2. \tag{4.4.35}$$

We observe that if $(\kappa_h, (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)) \in \mathbb{R} \times \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}$ is a solution of problem (4.4.34) if and only if $(\mu_h, (\boldsymbol{\sigma}_h, \boldsymbol{r}_h))$, with $\mu_h = 1/(1 + \kappa_h)$ is an eigenpair of $\boldsymbol{T}_h$, i.e.

$$\boldsymbol{T}_h(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) = \frac{1}{1 + \kappa_h}(\boldsymbol{\sigma}_h, \boldsymbol{r}_h).$$

Analogously to the continuous case, we prove that the discrete resolvent associated to the discrete operator $\boldsymbol{T}_h$ is bounded.

Theorem 4.4.1 *Assume that* $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}) := \boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g}) \in \mathrm{H}^t(\mathbf{div}, \Omega) \times \mathrm{H}^t(\Omega)^{n\times n}$ *for some* $t > 1/2$. *Then,*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq \left(1 + \frac{M_{DG}}{\alpha_{DG}}\right) \inf_{(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h} \|\boldsymbol{T}(\boldsymbol{f}, \boldsymbol{g}) - (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG}^*. \tag{4.4.36}$$

Moreover, the error estimate

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq C\, h^{\min(t,k)} \Big(\|\tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^t(\mathbf{div}, \Omega)} + \|\tilde{\boldsymbol{r}}\|_{\mathrm{H}^t(\Omega)^{n\times n}}\Big), \tag{4.4.37}$$

holds true with a constant $C > 0$ *independent of* h *and* λ.

Proof. We first notice that the DG approximation (4.4.34) is consistent with regards to its continuous counterpart (4.2.10) in the sense that

$$A_h\Big((\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) = 0 \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h. \tag{4.4.38}$$

Indeed, by definition,

$$A_h\Big((\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) = \int_\Omega \rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}} \cdot \operatorname{\mathbf{div}}_h \boldsymbol{\tau}_h + B\Big((\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) - \int_{\mathcal{F}_h^*} \{\rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\} \cdot [\![\boldsymbol{\tau}_h]\!]. \tag{4.4.39}$$

It is straightforward to deduce from (4.2.10)

$$\nabla\left(\rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\right) = \mathcal{C}^{-1}(\tilde{\boldsymbol{\sigma}} - \boldsymbol{f}) + \tilde{\boldsymbol{r}} - \boldsymbol{g} \quad \text{and} \quad (\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}^{\mathtt{t}})/2 = (\boldsymbol{f} - \boldsymbol{f}^{\mathtt{t}})/2. \tag{4.4.40}$$

Moreover, an integration by parts yields

$$\begin{aligned}\int_\Omega \rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}} \cdot \operatorname{\mathbf{div}}_h \boldsymbol{\tau}_h &= -\sum_{K\in\mathcal{T}_h} \int_K \nabla(\rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}) : \boldsymbol{\tau}_h + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\tau}_h \boldsymbol{n}_K \\ &= -\sum_{K\in\mathcal{T}_h} \int_K \nabla(\rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}) : \boldsymbol{\tau}_h + \int_{\mathcal{F}_h^*} \{\rho^{-1} \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\} \cdot [\![\boldsymbol{\tau}_h]\!]\end{aligned}$$

Substituting back the last identity and (4.4.40) into (4.4.39) we obtain

$$A_h\Big((\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) = B\Big((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}_h, \boldsymbol{s}_h)\Big) \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$$

and (4.4.38) follows.

The Céa estimate (4.4.36) follows now in the usual way by taking advantage of (4.4.38), the inf-sup condition (4.4.30), estimate (4.3.27) and the triangle inequality.

It follows from (4.4.36) that

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{f}, \boldsymbol{g})\|_{DG} \leq \left(1 + \frac{M_{DG}}{\alpha_{DG}}\right) \|(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}) - (\Pi_h \tilde{\boldsymbol{\sigma}}, \mathcal{S}_h \tilde{\boldsymbol{r}})\|_{DG}^*. \tag{4.4.41}$$

Using the interpolation error estimates (4.3.19), (4.3.21) and (4.3.22) we immediately obtain

$$\|(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}) - (\Pi_h \tilde{\boldsymbol{\sigma}}, \mathcal{S}_h \tilde{\boldsymbol{r}})\|_{DG} = \|(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{r}}) - (\Pi_h \tilde{\boldsymbol{\sigma}}, \mathcal{S}_h \tilde{\boldsymbol{r}})\| \leq C_0\, h^{\min(t,k)} \Big(\|\tilde{\boldsymbol{\sigma}}\|_{\mathrm{H}^t(\operatorname{\mathbf{div}},\Omega)} + \|\tilde{\boldsymbol{r}}\|_{\mathrm{H}^t(\Omega)^{n\times n}}\Big). \tag{4.4.42}$$

Moreover, we notice that

$$\|h_{\mathcal{F}}^{1/2} \{\operatorname{\mathbf{div}}(\tilde{\boldsymbol{\sigma}} - \Pi_h \tilde{\boldsymbol{\sigma}})\}\|_{0,\mathcal{F}_h^*} \leq \sum_{K\in\mathcal{T}_h} \sum_{F\in\mathcal{F}(K)} h_F \|\operatorname{\mathbf{div}}(\tilde{\boldsymbol{\sigma}} - \Pi_h \tilde{\boldsymbol{\sigma}})\|_{0,F}^2.$$

Under the regularity hypotheses on $\tilde{\boldsymbol{\sigma}}$, the commuting diagram property satisfied by Π_h, the trace theorem and standard scaling arguments give

$$h_F^{1/2} \|\operatorname{\mathbf{div}}(\tilde{\boldsymbol{\sigma}} - \Pi_h \tilde{\boldsymbol{\sigma}})\|_{0,F} = h_F^{1/2} \|\operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}} - \mathcal{R}_K \operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\|_{0,F} \leq C_2 h_K^{\min(k,t)} \|\operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\|_{t,K}$$

for all $F \in \mathcal{F}(K)$, where the $\mathrm{L}^2(K)$-orthogonal projection $\mathcal{R}_K := \mathcal{R}_h|_K$ onto $\mathcal{P}_{k-1}(K)$ is applied componentwise. It follows that

$$\|h_{\mathcal{F}}^{1/2} \{\operatorname{\mathbf{div}}(\tilde{\boldsymbol{\sigma}} - \Pi_h \tilde{\boldsymbol{\sigma}})\}\|_{0,\mathcal{F}_h^*} \leq C_3 h_K^{\min(k,r)} \left(\sum_{K\in\mathcal{T}_h} \|\operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\|_{t,K}^2\right)^{1/2} \leq C_3 h_K^{\min(k,r)} \|\operatorname{\mathbf{div}} \tilde{\boldsymbol{\sigma}}\|_{r,\Omega}. \tag{4.4.43}$$

Combining (4.4.43) and (4.4.42) with (4.4.41) proves the asymptotic error estimate (4.4.37). □

Lemma 4.4.1 *For all $s \in (0, \widehat{s})$, there exists a constant $C > 0$ independent of h and λ, such that for all $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\sigma}, \boldsymbol{r})\|_{DG} \leq C\, h^s \, \|\mathbf{div}\, \boldsymbol{\sigma}\|_{0,\Omega}.$$

Proof. The result is a consequence of Theorem 4.4.1 by noticing that, by virtue of Lemma 4.2.2, $\boldsymbol{T} \circ \boldsymbol{P}(\boldsymbol{\sigma}, \boldsymbol{r}) \in \mathrm{H}^s(\mathbf{div}; \Omega) \times \mathrm{H}^s(\Omega)$ for some $s \in (1/2, 1]$. □

Lemma 4.4.2 *For all $s \in (0, \widehat{s})$, there exists a constant $C > 0$ independent of h and λ such that*

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG} \leq C\, h^s \, \|(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG} \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h.$$

Proof. For any $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h$ we consider the splitting $\boldsymbol{\tau}_h = \boldsymbol{\tau}_h^c + \tilde{\boldsymbol{\tau}}_h$ with $\boldsymbol{\tau}_h^c := \mathcal{I}_h \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h^c$. We have that

$$\begin{aligned}(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h) &= (\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0}) + (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) \\ &= (\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0}) + (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}_h(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h),\end{aligned}$$

where the last identity is due to the fact that $(\boldsymbol{I} - \boldsymbol{P}_h)(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) \in \boldsymbol{\mathcal{K}}_h \times \boldsymbol{\mathcal{Q}}_h$ and $\boldsymbol{T} - \boldsymbol{T}_h$ vanishes identically on this subspace. It follows that

$$(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h) \;=\; (\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0}) + (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h) + (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h),$$

and the triangle inequality together with (4.2.11) and (4.4.35) yield

$$\begin{aligned}\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{\tau}_h, \boldsymbol{s}_h)\|_{DG} &\leq \|(\boldsymbol{T} - \boldsymbol{T}_h)(\tilde{\boldsymbol{\tau}}_h, \mathbf{0})\|_{DG} + \|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG} \\ &+ \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG} \leq \Big(\|\boldsymbol{T}\|_{\mathcal{L}([\mathrm{L}^2(\Omega)^{n\times n}]^2, \boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})} + \|\boldsymbol{T}_h\|_{\mathcal{L}([\mathrm{L}^2(\Omega)^{n\times n}]^2, \boldsymbol{\mathcal{W}}_h\times\boldsymbol{\mathcal{Q}}_h)} \Big) \\ &\Big(\|\tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_0 \Big) + \|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG}.\end{aligned}$$

Using (4.4.29), Lemma 4.3.1 and Lemma 4.4.1 we have that

$$\|\tilde{\boldsymbol{\tau}}_h\|_{0,\Omega} \leq Ch \|\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{W}}(h)},$$

$$\|(\boldsymbol{P}_h - \boldsymbol{P})(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_0 \leq Ch^s \|\mathbf{div}\, \boldsymbol{\tau}_h^c\|_{0,\Omega} \leq Ch^s \|\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{W}}(h)}$$

and

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}(\boldsymbol{\tau}_h^c, \boldsymbol{s}_h)\|_{DG} \leq Ch^s \|\mathbf{div}\, \boldsymbol{\tau}_h^c\|_{0,\Omega} \leq Ch^s \|\boldsymbol{\tau}_h\|_{\boldsymbol{\mathcal{W}}(h)}$$

respectively, which gives the result. □

4.5 Spectral correctness of the DG method

The convergence analysis follows the same steps introduced in [36, 37], we only need to adapt it to the DG context, cf. also [29].

For the sake of brevity, we will denote in this section $\mathbb{X} := \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, $\mathbb{X}_h := \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h$ and $\mathbb{X}(h) := \boldsymbol{\mathcal{W}}(h) \times \boldsymbol{\mathcal{Q}}$. Moreover, when no confusion can arise, we will use indistinctly $\boldsymbol{x}$, $\boldsymbol{y}$, etc. to denote elements in $\mathbb{X}$ and, analogously, $\boldsymbol{x}_h$, $\boldsymbol{y}_h$, etc. for those in $\mathbb{X}_h$. Finally, we will use $\|\cdot\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}$ to denote the norm of an operator restricted to the discrete subspace $\mathbb{X}_h$; namely, if $\boldsymbol{S} : \mathbb{X}(h) \to \mathbb{X}(h)$, then

$$\|\boldsymbol{S}\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))} := \sup_{\boldsymbol{0}\neq\boldsymbol{x}_h\in\mathbb{X}_h} \frac{\|\boldsymbol{S}\boldsymbol{x}_h\|_{DG}}{\|\boldsymbol{x}_h\|_{DG}}. \tag{4.5.44}$$

Lemma 4.5.1 *If $z \in \mathbb{D} \setminus \operatorname{sp}(\boldsymbol{T})$, there exists $h_0 > 0$ such that if $h \leq h_0$,*

$$\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h\|_{DG} \geq C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z|\, \|\boldsymbol{x}_h\|_{DG} \quad \forall \boldsymbol{x}_h \in \mathbb{X}_h.$$

with $C > 0$ independent of h and λ.

Proof. It follows from

$$(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h = (z\boldsymbol{I} - \boldsymbol{T})\boldsymbol{x}_h + (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{x}_h$$

and Lemma 4.3.2 that

$$\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h\|_{DG} \geq \Big(C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z| - \|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\Big)\|\boldsymbol{x}_h\|_{DG}$$

and the result follows from Lemma 4.4.2. □

Lemma 4.5.2 *If $z \in \mathbb{D} \setminus \operatorname{sp}(\boldsymbol{T})$ and h is sufficiently small,*

$$\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG} \geq C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z|^2\, \|\boldsymbol{x}\|_{DG} \quad \forall \boldsymbol{x} \in \mathbb{X}(h).$$

with $C > 0$ independent of h.

Proof. Given $\boldsymbol{x} \in \mathbb{X}(h)$ we let

$$\boldsymbol{x}_h^* = \boldsymbol{T}_h\boldsymbol{x} \in \mathbb{X}_h.$$

We deduce from the identity

$$(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h^* = \boldsymbol{T}_h(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}$$

and Lemma 4.5.1 that

$$C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z|\|\boldsymbol{x}_h^*\|_{DG} \leq \|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}_h^*\|_{DG} \leq \|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG},$$

and the result follows from the triangle inequality and the boundedness of $\boldsymbol{T}_h$

$$\begin{aligned}\|\boldsymbol{x}\|_{DG} &\leq |z|^{-1}\|\boldsymbol{x}_h^*\|_{DG} + |z|^{-1}\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG} \\ &\leq |z|^{-1}\left(1 + \frac{\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}}{C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z|}\right)\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG}. \\ &\leq |z|^{-1}\left(\frac{C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z| + \|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}}{C \operatorname{dist}\big(z, \operatorname{sp}(\boldsymbol{T})\big)|z|}\right)\|(z\boldsymbol{I} - \boldsymbol{T}_h)\boldsymbol{x}\|_{DG}.\end{aligned}$$

hence

$$C|z|\left(\frac{C\,\mathrm{dist}\,\big(z,\mathrm{sp}(\boldsymbol{T})\big)|z|}{\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}+C\,\mathrm{dist}\,\big(z,\mathrm{sp}(\boldsymbol{T})\big)|z|}\right)\|\boldsymbol{x}\|_{DG}\leq\|(z\boldsymbol{I}-\boldsymbol{T}_h)\boldsymbol{x}\|_{DG}.$$

Since $\mathrm{dist}\,\big(z,\mathrm{sp}(\boldsymbol{T})\big)\leq|z|\leq 1$ and $\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}_h)}\leq C''$ (with C' independent of λ), from the estimate above we derive

$$C|z|^2\,\mathrm{dist}\,\big(z,\mathrm{sp}(\boldsymbol{T})\big)\|\boldsymbol{x}\|_{DG}\leq\|(z\boldsymbol{I}-\boldsymbol{T})(\boldsymbol{\tau},\boldsymbol{s})\|_{DG}.$$

Hence we conclude the proof. □

Remark 4.5.1 *If F is a compact subset of $\mathbb{D}\setminus\mathrm{sp}\{\boldsymbol{T}\}$ and h is small enough, we deduce from Lemma 4.5.2 that $(z\boldsymbol{I}-\boldsymbol{T}_h)$ is invertible. Hence, $F\subset\mathbb{D}\setminus\mathrm{sp}(\boldsymbol{T}_h)$. Consequently, for h small enough, the numerical method does not introduce spurious eigenvalues. Moreover, we have*

$$\|\big(z\boldsymbol{I}-\boldsymbol{T}_h\big)^{-1}\|_{\mathcal{L}(\boldsymbol{\mathcal{W}}(h)\times\boldsymbol{\mathcal{Q}},\boldsymbol{\mathcal{W}}(h)\times\boldsymbol{\mathcal{Q}})}\leq\frac{C}{\mathrm{dist}(F,\mathrm{sp}(\boldsymbol{T}))|z|^2}.\tag{4.5.45}$$

For $\boldsymbol{x}\in\mathbb{X}(h)$ and $\mathbb{E}$ and $\mathbb{F}$ closed subspaces of $\mathbb{X}(h)$, we set $\delta(\boldsymbol{x},\mathbb{E}):=\inf_{\boldsymbol{y}\in\mathbb{E}}\|\boldsymbol{x}-\boldsymbol{y}\|_{DG}$, $\delta(\mathbb{E},\mathbb{F}):=\sup_{\boldsymbol{y}\in\mathbb{E}:\,\|\boldsymbol{y}\|=1}\delta(\boldsymbol{y},\mathbb{F})$, and $\widehat{\delta}(\mathbb{E},\mathbb{F}):=\max\{\delta(\mathbb{E},\mathbb{F}),\delta(\mathbb{F},\mathbb{E})\}$, the latter being the so called *gap* between subspaces $\mathbb{E}$ and $\mathbb{F}$.

Given an isolated eigenvalue $\kappa\neq 1$ of $\boldsymbol{T}$, we define $\mathtt{d}_\kappa:=\frac{1}{2}\,\mathrm{dist}\,\big(\kappa,\mathrm{sp}(\boldsymbol{T})\big)$. Let $D_\kappa:=\{z\in\mathbb{C};\quad|z-\kappa|\leq\mathtt{d}_\kappa\}$ be a closed disk in the complex plane centered at κ with boundary γ not containing the origin and such that $D_\kappa\cap\mathrm{sp}\,\boldsymbol{T}=\{\kappa\}$. We deduce from Remark 4.3.1 that the operator $\boldsymbol{\mathcal{E}}:=\frac{1}{2\pi i}\int_\gamma(z\boldsymbol{I}-\boldsymbol{T})^{-1}\,dz:\mathbb{X}(h)\longrightarrow\mathbb{X}(h)$ is well-defined and bounded uniformly in h. It is well known that $\boldsymbol{\mathcal{E}}$ is a spectral projection in $\mathbb{X}$ onto the (finite dimensional) eigenspace $\boldsymbol{\mathcal{E}}(\mathbb{X})$ corresponding to the eigenvalue κ of $\boldsymbol{T}$. It is important to notice that

$$\boldsymbol{\mathcal{E}}(\mathbb{X}(h))=\boldsymbol{\mathcal{E}}(\mathbb{X}).\tag{4.5.46}$$

Indeed, if $\kappa^*\in D_\gamma$ is an eigenvalue of $\boldsymbol{T}:\mathbb{X}(h)\to\mathbb{X}(h)$ and $\boldsymbol{x}^*\in\mathbb{X}(h)$ is a corresponding eigenfunction, as $\kappa^*\neq 0$ and $\boldsymbol{T}(\mathbb{X}(h))\subset\mathbb{X}$, we actually have that $\boldsymbol{x}^*\in\mathbb{X}$. Then, necessarily, $\kappa^*=\kappa$ and taking into account that $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is the eigenspace associated with η we deduce (4.5.46).

Similarly, we deduce from Remark 4.5.1 that, for h small enough, the operator $\boldsymbol{\mathcal{E}}_h:=\frac{1}{2\pi i}\int_\gamma(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\,dz:\mathbb{X}(h)\longrightarrow\mathbb{X}(h)$ is also well-defined and bounded uniformly in h. Moreover, $\boldsymbol{\mathcal{E}}_h$ is a projector in $\mathbb{X}_h$ onto the eigenspace $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ corresponding to the eigenvalues of $\boldsymbol{T}_h:\mathbb{X}_h\to\mathbb{X}_h$ contained in γ. The same reasoning as above shows that we also have,

$$\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))=\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h).\tag{4.5.47}$$

Our aim now is to compare $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ to $\boldsymbol{\mathcal{E}}(\mathbb{X})$ in terms of the distance $\widehat{\delta}$.

Lemma 4.5.3 *There exists $C>0$ independent of h such that*

$$\|\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\leq\frac{C}{\mathtt{d}_\kappa}\|\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}.\tag{4.5.48}$$

Proof. We deduce from the identity

$$(z\boldsymbol{I}-\boldsymbol{T})^{-1}-(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}=(z\boldsymbol{I}-\boldsymbol{T})^{-1}(\boldsymbol{T}-\boldsymbol{T}_h)(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1} \tag{4.5.49}$$

that, for any $\boldsymbol{x}_h\in\mathbb{X}_h$,

$$\begin{aligned}\|(\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h)\boldsymbol{x}_h\|_{DG}\leq&\frac{1}{2\pi}\int_\gamma\|[(z\boldsymbol{I}-\boldsymbol{T})^{-1}-(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}]\boldsymbol{x}_h\|_{DG}dz\\ =&\frac{1}{2\pi}\int_\gamma\|[(z\boldsymbol{I}-\boldsymbol{T})^{-1}(\boldsymbol{T}-\boldsymbol{T}_h)(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}]\boldsymbol{x}_h\|dz\\ \leq&\|(z\boldsymbol{I}-\boldsymbol{T})^{-1}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))}\|\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\|(z\boldsymbol{I}-\boldsymbol{T}_h)^{-1}\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}_h)}\|\boldsymbol{x}_h\|_{DG}\frac{1}{2\pi}\int_\gamma dz\end{aligned}$$

and the result follows from Lemmas 4.3.2, 4.5.2 and the definition (4.5.44). □

Theorem 4.5.1 *There exists a constant $C>0$ independent of h such that*

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}),\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h))\leq C\Big(\frac{\|\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}}{d_\kappa}+\delta(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h)\Big).$$

Proof. As $\boldsymbol{\mathcal{E}}_h$ is a projector, for h sufficiently small, we have that $\boldsymbol{\mathcal{E}}_h\boldsymbol{x}_h=\boldsymbol{x}_h$ for all $\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$. It follows from (4.5.46) that $\boldsymbol{\mathcal{E}}\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}(\mathbb{X})$, which leads to

$$\delta(\boldsymbol{x}_h,\boldsymbol{\mathcal{E}}(\mathbb{X}))\leq\|\boldsymbol{\mathcal{E}}_h\boldsymbol{x}_h-\boldsymbol{\mathcal{E}}\boldsymbol{x}_h\|_{DG}\leq\|\boldsymbol{\mathcal{E}}_h-\boldsymbol{\mathcal{E}}\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\|\boldsymbol{x}_h\|_{DG}$$

for all $\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$. We deduce from (4.5.48) that

$$\delta(\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h),\boldsymbol{\mathcal{E}}(\mathbb{X}))\leq\frac{C}{\mathsf{d}_\kappa}\|\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}. \tag{4.5.50}$$

On the other hand, as $\boldsymbol{\mathcal{E}}\boldsymbol{x}=\boldsymbol{x}$ for all $\boldsymbol{x}\in\boldsymbol{\mathcal{E}}(\mathbb{X})$, for h small enough and $\boldsymbol{y}_h\in\mathbb{X}_h$,

$$\begin{aligned}\|\boldsymbol{x}-\boldsymbol{\mathcal{E}}_h\boldsymbol{y}_h\|_{DG}\leq\|\boldsymbol{\mathcal{E}}(\boldsymbol{x}-\boldsymbol{y}_h)\|_{DG}+\|(\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h)\boldsymbol{y}_h\|_{DG}\leq&\\ \|\boldsymbol{\mathcal{E}}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))}\|(\boldsymbol{x}-\boldsymbol{y}_h)\|_{DG}+\|(\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h)\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\|\boldsymbol{y}_h\|_{DG}&\\ \leq\big(\|\boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))}+2\|\boldsymbol{\mathcal{E}}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))}\big)\|\boldsymbol{x}-\boldsymbol{y}_h\|_{DG}+\|\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))}\|\boldsymbol{x}\|_{DG}.&\end{aligned}$$

Consequently,

$$\delta(\boldsymbol{x},\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h))\leq C(\delta(\boldsymbol{x},\mathbb{X}_h)+\|\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))})$$

for all $\boldsymbol{x}\in\boldsymbol{\mathcal{E}}(\mathbb{X})$ such that $\|\boldsymbol{x}\|_{DG}=1$ and using that the eigenspace $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is finite dimensional we deduce that

$$\delta(\boldsymbol{\mathcal{E}}(\mathbb{X}),\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h))\leq C(\delta(\boldsymbol{\mathcal{E}}(\mathbb{X}),\mathbb{X}_h)+\|\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}_h\|_{\mathcal{L}(\mathbb{X}_h,\mathbb{X}(h))})$$

and the result follows from the last estimate and (4.5.50). □

Theorem 4.5.2 *Let $\kappa \neq 1$ be an eigenvalue of $\boldsymbol{T}$ of algebraic multiplicity m and let D_γ be a closed disk in the complex plane centered at κ with boundary γ such that $D_\kappa \cap \operatorname{sp} \boldsymbol{T} = \{\kappa\}$. Let $\kappa_{1,h}, \cdots, \kappa_{m(h),h}$ be the eigenvalues of $\boldsymbol{T}_h : \mathbb{X}_h \to \mathbb{X}_h$ lying in D_γ and repeated according to their algebraic multiplicity. Then, we have that $m(h) = m$ for h sufficiently small and*

$$\lim_{h\to 0} \max_{1\le i\le m} |\kappa - \kappa_{i,h}| = 0.$$

Moreover, if $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is the eigenspace corresponding to κ and $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ is the $\boldsymbol{T}_h$-invariant subspace of $\mathbb{X}_h$ spanned by the eigenspaces corresponding to $\{\kappa_{i,h},\ i = 1, \ldots, m\}$ then

$$\lim_{h\to 0} \widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) = 0.$$

Proof. We deduce from Lemma 4.4.2 that

$$\lim_{h\to 0} \|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathbb{X}_h, \mathbb{X}(h))} = 0.$$

Moreover, as $\boldsymbol{\mathcal{E}}(\mathbb{X}) \subset \mathrm{H}^s(\mathbf{div}, \Omega) \times \mathrm{H}^s(\Omega)^{n\times n}$, it follows from (4.4.37) that

$$\lim_{h\to 0} \delta(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) = 0.$$

Hence, by virtue of Theorem 4.5.1, we have that

$$\lim_{h\to 0} \widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) = 0,$$

and, as a consequence, $\boldsymbol{\mathcal{E}}(\mathbb{X})$ and $\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$ have the same dimension provided h is sufficiently small. Finally, being κ an isolated eigenvalue and the radius of the circle γ arbitrary, we deduce that

$$\lim_{h\to 0} \max_{1\le i\le m} |\kappa - \kappa_{i,h}| = 0.$$

□

4.6 Asymptotic error estimates

Along this section we fix a particular eigenvalue $\kappa \neq 1$ of $\boldsymbol{T}$. We wish to obtain error estimates for the eigenfunctions and the eigenvalues in terms of the quantity

$$\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) := \sup_{\boldsymbol{x}\in\boldsymbol{\mathcal{E}}(\mathbb{X}), \|\boldsymbol{x}\|=1} \inf_{\boldsymbol{x}_h\in\mathbb{X}_h} \|\boldsymbol{x} - \boldsymbol{x}_h\|^*_{DG}.$$

Theorem 4.6.1 *For h small enough, there exists a constant C independent of h such that*

$$\widehat{\delta}\big(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)\big) \le \frac{C}{\boldsymbol{d}_\kappa} \delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h).$$

Proof. As $\boldsymbol{\mathcal{E}}(\mathbb{X}(h)) = \boldsymbol{\mathcal{E}}(\mathbb{X})$ and $\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)) = \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)$, it is equivalent to show that

$$\widehat{\delta}\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)), \boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big) \leq \frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h). \tag{4.6.51}$$

To this end, we consider as in the previous section a disk D_κ centered at κ with radius $\mathtt{d}_\kappa$. We first notice that for all $z \in \gamma$

$$(z\boldsymbol{I} - \boldsymbol{T})^{-1} - (z\boldsymbol{I} - \boldsymbol{T}_h)^{-1} = (z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}(\boldsymbol{T} - \boldsymbol{T}_h)(z\boldsymbol{I} - \boldsymbol{T})^{-1},$$

which implies

$$\begin{aligned}
\|(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\| \leq & \frac{1}{2\pi}\int_\gamma \|(z\boldsymbol{I} - \boldsymbol{T})^{-1} - (z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\| dz \\
= & \frac{1}{2\pi}\int_\gamma \|(z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}(\boldsymbol{T} - \boldsymbol{T}_h)(z\boldsymbol{I} - \boldsymbol{T})^{-1}|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\| dz \\
\leq & \|(z\boldsymbol{I} - \boldsymbol{T}_h)^{-1}\|_{\mathcal{L}(\mathbb{X}(h),\mathbb{X}(h))}\|(\boldsymbol{T} - \boldsymbol{T}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}\|(z\boldsymbol{I} - \boldsymbol{T})^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}\frac{1}{2\pi}\int_\gamma dz \\
\leq & \frac{C}{\mathtt{d}_\kappa}\|(\boldsymbol{T} - \boldsymbol{T}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}
\end{aligned} \tag{4.6.52}$$

Now, on the one hand, it is clear that

$$\delta\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)), \boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big) \leq \|(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))}.$$

On the other hand, (4.6.52), the Céa estimate given by Theorem 4.4.1 and the fact that $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is finite dimensional yield

$$\|(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}_h)|_{\boldsymbol{\mathcal{E}}(\mathbb{X})}\|_{\mathcal{L}(\mathbb{X},\mathbb{X}(h))} \leq \frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h), \tag{4.6.53}$$

which proves that

$$\delta\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)), \boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big) \leq \frac{C}{\mathtt{d}_\kappa}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h). \tag{4.6.54}$$

Consequently, as $\boldsymbol{\mathcal{E}}(\mathbb{X}) \subset \mathrm{H}^t(\mathbf{div}, \Omega) \times \mathrm{H}^t(\Omega)^{n\times n}$, we have that

$$\lim_{h\to 0}\delta\Big(\boldsymbol{\mathcal{E}}(\mathbb{X}(h)), \boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))\Big) = 0. \tag{4.6.55}$$

It is shown in [37] that (4.6.55) implies that, for h small enough, $\Lambda_h := \boldsymbol{\mathcal{E}}_h|_{\boldsymbol{\mathcal{E}}(\mathbb{X})} : \boldsymbol{\mathcal{E}}(\mathbb{X}) \to \boldsymbol{\mathcal{E}}_h(\mathbb{X}(h))$ is bijective and Λ_h^{-1} exists and is uniformly bounded with respect to h. Furthermore, it holds that,

$$\sup_{\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)),\|\boldsymbol{x}_h\|_{DG}=1}\|\Lambda_h^{-1}\boldsymbol{x} - \boldsymbol{x}\|_{DG} \leq 2\sup_{\boldsymbol{y}\in\boldsymbol{\mathcal{E}}(\mathbb{X}(h)),\|\boldsymbol{y}\|_{DG}=1}\|\Lambda_h\boldsymbol{y} - \boldsymbol{y}\|_{DG}.$$

Hence,

$$\delta\Big(\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)), \boldsymbol{\mathcal{E}}(\mathbb{X}(h))\Big) \leq \sup_{\boldsymbol{x}_h\in\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)),\|\boldsymbol{x}_h\|_{DG}=1}\|\boldsymbol{x}_h - \Lambda_h^{-1}\boldsymbol{x}\|_{DG} \leq 2\sup_{\boldsymbol{y}\in\boldsymbol{\mathcal{E}}(\mathbb{X}),\|\boldsymbol{y}\|_{DG}=1}\|\boldsymbol{\mathcal{E}}\boldsymbol{y} - \boldsymbol{\mathcal{E}}_h\boldsymbol{y}\|_{DG},$$

and (4.6.53) shows that we also have $\delta(\boldsymbol{\mathcal{E}}_h(\mathbb{X}(h)), \boldsymbol{\mathcal{E}}(\mathbb{X}(h))) \leq \frac{C}{\mathtt{d}}\delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h)$, and the result follows from this last estimate and (4.6.54). □

Theorem 4.6.2 *Assume that $\boldsymbol{\mathcal{E}}(\mathbb{X}) \subset \mathbb{H}^t(\mathbf{div}, \Omega) \times \mathbb{H}^t(\Omega)^{n\times n}$, then there exists $C > 0$ independent of h and λ such that*

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}_h(\mathbb{X}_h), \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq \frac{C}{\mathtt{d}_\kappa} h^{\min\{t,k\}}. \tag{4.6.56}$$

Moreover, there exists $C' > 0$ independent of h such that

$$\max_{1\leq i\leq m} |\kappa - \kappa_{i,h}| \leq \frac{C'}{\mathtt{d}_\kappa}\, h^{2\min\{t,k\}} \tag{4.6.57}$$

Proof. Let $\kappa_{1,h}, \cdots, \kappa_{m,h}$ be the eigenvalues of $\boldsymbol{T}_h : \mathbb{X}_h \to \mathbb{X}_h$ lying in D_γ and repeated according to their algebraic multiplicity. We denote by $\boldsymbol{x}_{i,h}$ the eigenfunction corresponding to $\kappa_{i,h}$ and satisfying $\|\boldsymbol{x}_{i,h}\|_{DG} = 1$. We know from Theorem 4.6.1 that, if h is sufficiently small,

$$\delta(\boldsymbol{x}_{i,h}, \boldsymbol{\mathcal{E}}(\mathbb{X})) \leq \frac{C}{\mathtt{d}_\kappa} \delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h).$$

Then, there exists an eigenfunction $\boldsymbol{x} := (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{E}}(\mathbb{X})$ satisfying

$$\|\boldsymbol{x}_{i,h} - \boldsymbol{x}\|_{DG} \leq \frac{C}{\mathtt{d}_\kappa} \delta^*(\boldsymbol{\mathcal{E}}(\mathbb{X}), \mathbb{X}_h) \to 0 \quad \text{as } h \to 0,$$

which proves that $\|\boldsymbol{x}\|_{DG}$ is bounded from below and above by constant independent of h. Proceeding as in the proof of the consistency property in Theorem 4.4.1 we readily obtain that

$$A_h(\boldsymbol{x}, \boldsymbol{y}_h) = \kappa B(\boldsymbol{x}, \boldsymbol{y}_h) \tag{4.6.58}$$

for all $\boldsymbol{y}_h := (\boldsymbol{\tau}_h, \boldsymbol{s}_h) \in \mathbb{X}_h$. With the aid of (4.6.58), it is easy to show that the identity

$$A_h(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h}) - \kappa B(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h}) = (\kappa_{i,h} - \kappa)\, B(\boldsymbol{x}_{i,h}, \boldsymbol{x}_{i,h})$$

holds true. Now, according to Lemma 3.6 of [69], for any $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}(\mathbb{X})$, $\boldsymbol{x} \neq 0$, it holds that $B(\boldsymbol{x}, \boldsymbol{x}) > 0$. Thus, since $\boldsymbol{\mathcal{E}}(\mathbb{X})$ is finite-dimensional, there exists $c > 0$, independent of h, such that $B(\boldsymbol{x}, \boldsymbol{x}) \geq c\|\boldsymbol{x}\|_{DG}$. This proves that $B(\boldsymbol{x}_{ih}, \boldsymbol{x}_{ih}) \geq \frac{c}{2}$ for h sufficiently small. We obtain from (4.3.25) that

$$\frac{c}{2}|\kappa_{i,h} - \kappa| \leq |A_h(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h})| + |\kappa||B(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h})| \leq C(\|\boldsymbol{x} - \boldsymbol{x}_{i,h}\|_{DG}^*)^2. \tag{4.6.59}$$

Since $\boldsymbol{x} := (\boldsymbol{\sigma}, \boldsymbol{r})$ and $\boldsymbol{x}_{i,h} := (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)$, and by definition of $\|\cdot\|_{DG}^*$ we have

$$\|\boldsymbol{x} - \boldsymbol{x}_{i,h}\|_{DG}^* := \|(\boldsymbol{\sigma}, \boldsymbol{r}) - (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG}^* = \|(\boldsymbol{\sigma}, \boldsymbol{r}) - (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG} + \|h_{\mathcal{F}}^{1/2}\{\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*}.$$

It follows from Theorem 4.5.1, Lemma 4.4.2 and the interpolation error estimates (4.3.19)-(4.3.22) that

$$\|(\boldsymbol{\sigma}, \boldsymbol{r}) - (\boldsymbol{\sigma}_h, \boldsymbol{r}_h)\|_{DG} \leq C_0\widehat{\delta}(\boldsymbol{\mathcal{E}}(\mathbb{X}), \boldsymbol{\mathcal{E}}_h(\mathbb{X}_h)) \leq C_1 h^{\min\{t,k\}}\Big(1 + \|\boldsymbol{\sigma}\|_{\mathrm{H}^t(\mathbf{div},\Omega)} + \|\boldsymbol{r}\|_{\mathrm{H}^t(\Omega)^{n\times n}}\Big) \tag{4.6.60}$$

On the other hand,

$$\|h_{\mathcal{F}}^{1/2}\{\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*} \leq \|h_{\mathcal{F}}^{1/2}\{\operatorname{div}(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\}\|_{\mathcal{F}_h^*} + \|h_{\mathcal{F}}^{1/2}\{\operatorname{div}(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*} \tag{4.6.61}$$

and it follows from (4.4.43) that

$$\|h_{\mathcal{F}}^{1/2}\{\mathrm{div}(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\}\|_{\mathcal{F}_h^*} \leq C_2 h_K^{\min(k,t)}\|\mathbf{div}\,\boldsymbol{\sigma}\|_{t,\Omega} \tag{4.6.62}$$

Finally, using (4.3.26), (4.3.21) and (4.6.60) yield

$$\begin{aligned}\|h_{\mathcal{F}}^{1/2}\{\mathrm{div}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\}\|_{\mathcal{F}_h^*} &\leq C_3\|\,\mathbf{div}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}\\ &\leq C_3\big(\|\,\mathbf{div}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma})\|_{0,\Omega}+\|\,\mathbf{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}\big)\\ &\leq C_3\big(\|\,\mathbf{div}(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma})\|_{0,\Omega}+\|(\boldsymbol{\sigma},\boldsymbol{r})-(\boldsymbol{\sigma}_h,\boldsymbol{r}_h)\|_{DG}\big)\\ &\leq C_4 h^{\min\{k,t\}}\Big(1+\|\boldsymbol{\sigma}\|_{\mathrm{H}^t(\mathbf{div},\Omega)}+\|\boldsymbol{r}\|_{\mathrm{H}^t(\Omega)^{n\times n}}\Big).\end{aligned} \tag{4.6.63}$$

Combining (4.6.59), (4.6.61)-(4.6.63) and (4.6.60) with we obtain (4.6.57). □

Remark 4.6.1 *The constant C' in* (4.6.57) *is not necessarily independent of λ. Indeed, according to the proof of Lemma 3.6 from [69] we know that*

$$B((\boldsymbol{\sigma},\boldsymbol{r}),(\boldsymbol{\sigma},\boldsymbol{r})) = \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma} \geq \min\left\{\frac{n}{n\lambda+2\mu},\frac{1}{2\mu}\right\}\|\boldsymbol{\sigma}\|_{0,\Omega}^2 \geq 0.$$

Therefore, as λ goes to infinity, the constant c in the proof above goes to zero and, consequently, constant C' in Theorem 4.6.2 goes to infinity, too. However, the numerical experiments suggest that (4.6.57) *holds true with a constant that does not deteriorate with λ. To prove it is a subject of future research.*

Remark 4.6.2 *We point out that, thanks to Lemma 4.2.2, we always have that $\boldsymbol{\mathcal{E}}(\mathbb{X}) \subset \{(\boldsymbol{\tau},\boldsymbol{r}) \in (\mathrm{H}^s(\Omega)^{n\times n})^2 : \mathbf{div}\,\boldsymbol{\tau} \in \mathrm{H}^1(\Omega)^n\}$ for all $s \in (0,\widehat{s})$. Consequently, the error estimates given in Theorem 4.6.2 will always hold true for any $t \in (0,\widehat{s})$ even if $\widehat{s} \leq 1/2$. However, it may happen that some eigenspaces satisfy the regularity assumption of the theorem with $t \geq \widehat{s}$.*

4.7 Numerical Results

In this section we report some numerical experiments which allowed us to assess the performance of the method. With this purpose, we implemented the method in a FEniCS code [63]. According to the theoretical results, the proposed method does not introduce spurious modes provided the meshsize h is sufficiently small (cf. Remark 4.5.1). On the other hand, for the theoretical results to hold true, it is assumed that the stabilization parameter $\mathtt{a}_S$ is larger that a threshold $\mathtt{a}_S^*$. Therefore, the absence of spurious modes is actually guaranteed provided h is small enough and $\mathtt{a}_S$ is large enough.

We observed spurious modes when the stabilization parameter is not sufficiently large as is usual in other stabilized methods (see for example [73]).

Since it is not possible to compute explicitly the threshold $\mathtt{a}_S^*$, we made some numerical experiments to determine a safe value of $\mathtt{a}_S$ that allowed us to solve the eigenvalue problem without the presence of spurious modes.

We have used as geometrical domain in all the reported tests the unit square $\Omega := (0,1)^2$, fixed at its bottom. The Lamé coefficients of the material are defined in terms of the Poisson ratio ν and the Young modulus E as follows:

$$\lambda := \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu := \frac{E}{2(1+\nu)}.$$

We have taken for the experiments $\rho = 1$, $E = 1$ and different values of $\nu \in (0, 1/2)$. In fact, we have also applied the proposed method to the limit case $\nu = 1/2$. In such a case, $\lambda = +\infty$ and the definition of the bilinear form B change as described in the appendix. As will be shown in what follows, the method works for $\nu = 1/2$, as well as $\nu < 1/2$.

We have used 'uniform' meshes as those shown in Figure 4.1. Each mesh is identified by the refinement parameter N (the number of element edges on each side of the square domain).

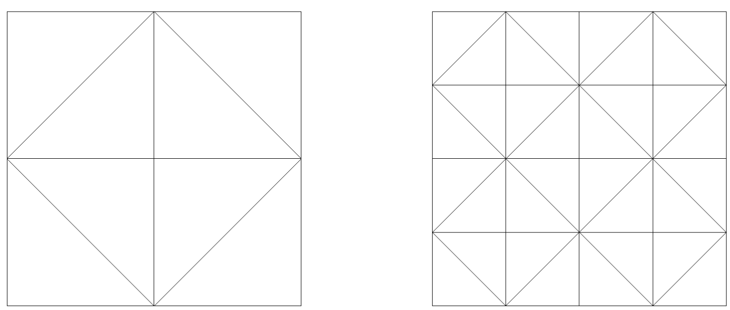

Figure 4.1: Meshes for $N = 2$ (left) and $N = 4$ (right).

In the first tests we are concerned with the determination of a reliable stabilization parameter $\mathtt{a}_S$. We know that the spectral correctness of the method can only be guaranteed if $\mathtt{a}_S$ is sufficiently large (Proposition 4.4.2) and if the meshsize h is sufficiently small (cf. Remark 4.5.1). In a first stage, we fix the refinement level to $N = 8$ and report in Tables 5.1, 5.2 and 4.3 the 10 smallest vibration frequencies computed for different values of $\mathtt{a}_S$. The polynomial degrees are given by $k = 3, 4, 5$ respectively. The boxed numbers are spurious eigenvalues. We observe that they emerge at random positions when we vary $\mathtt{a}_S$ and k and they disappear completely when $\mathtt{a}_S$ is sufficiently large.

$\mathtt{a}_S = 5$	$\mathtt{a}_S = 10$	$\mathtt{a}_S = 20$	$\mathtt{a}_S = 40$	$\mathtt{a}_S = 80$
0.6804474	0.6804497	0.6804460	0.6804472	0.6804472
1.6988814	1.6988904	1.6988615	1.6988797	1.6988800
1.8222056	1.8222073	1.8221859	1.8222050	1.8222052
2.9476938	2.9476927	2.3856290	2.9476928	2.9476933
3.0174161	3.0174530	2.3862301	3.0174095	3.0174114
3.4432120	3.4432156	2.5833172	3.4432158	3.4432168
4.1417685	4.1417626	2.5839852	4.1417697	4.1417750
4.6308354	4.6308072	2.9477062	4.6308465	4.6308549
4.7616007	4.7615186	3.0174627	4.7616237	4.7616317
4.7879824	4.7879191	3.4432320	4.7880173	4.7880298

Table 4.1: Computed vibration frequencies for $k = 3$, $\nu = 0.35$ and $N = 8$.

$\mathtt{a}_S = 5$	$\mathtt{a}_S = 10$	$\mathtt{a}_S = 20$	$\mathtt{a}_S = 40$	$\mathtt{a}_S = 80$
0.6805737	0.6805737	0.6805737	0.6805736	0.6805737
1.6990333	1.6990333	1.6990332	1.6990329	1.6990330
1.8222095	1.8222094	1.8222095	1.8222095	1.8222096
2.9476921	2.2970057	2.9476922	2.9476922	2.9476922
3.0176437	2.3909952	3.0176400	3.0176421	3.0176428
3.4432473	2.9476924	3.1845593	3.4432470	3.4432472
4.1417687	3.0176452	3.4392819	4.1417705	4.1417709
4.5534365	3.4432480	3.4432839	4.6309421	4.6309433
4.6309432	4.1417718	4.1417737	4.7615808	4.7615812
4.7195356	4.6309455	4.6309470	4.7882380	4.7882400

Table 4.2: Computed vibration frequencies for $k = 4$, $\nu = 0.35$ and $N = 8$.

$\mathtt{a}_S = 5$	$\mathtt{a}_S = 10$	$\mathtt{a}_S = 20$	$\mathtt{a}_S = 40$	$\mathtt{a}_S = 80$
0.6806522	0.6806522	0.6806522	0.6806522	0.6806522
1.6991254	1.6991254	1.6991255	1.6991250	1.6991253
1.8222137	1.8222137	1.8222138	1.8222137	1.8222137
2.9476935	2.9476935	2.4714299	2.9476935	2.9476935
3.0177848	3.0177848	2.4822317	3.0177827	3.0177844
3.4432656	3.4432656	2.9476935	3.4432652	3.4432656
4.1417853	4.1417852	3.0177862	4.1417845	4.1417852
4.6310201	4.6310201	3.4432657	4.6310172	4.6310196
4.7615803	4.7615803	4.1417853	4.7615800	4.7615802
4.7883889	4.7883889	4.6310208	4.7883835	4.7883878

Table 4.3: Computed vibration frequencies for $k = 5$, $\nu = 0.35$ and $N = 8$.

The observed spurious modes are extremely sensitive with respect to the stabilization parameter $\mathtt{a}_S$. Very small changes of this parameter produce significant changes in the value of the spurious eigenvalues that, in some cases, make them disappear from the table. This fact makes it too difficult to determine a threshold value of $\mathtt{a}_S$ to guarantee avoiding spurious modes. For instance, in spite of the fact that we did not observe in our experiments spurious modes for $\mathtt{a}_S = 40$, we found them for $\mathtt{a}_S = 42$ and $k = 3$.

Next, we present in Table 4.4 different approximations of the first 10 vibration frequencies corresponding to $N = 8, 16, 32, 64$, obtained with $\mathtt{a}_S = 20$ and a polynomial degree $k = 3$. We notice that as the level of refinement increases the lower frequencies are progressively cleaned from spurious modes. We conclude that our method provides a correct approximation of the spectrum as long as N and $\mathtt{a}_S$ are large enough. In the forthcoming tests we will take $\mathtt{a}_S = 100$. We point out that the previous tests have been carried out with a Poisson ratio $\nu = 0.35$, but similar results were obtained for values ranging from 0.35 to 0.5.

$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.6804460	0.6806838	0.6807775	0.6808142
1.6988615	1.6991595	1.6992689	1.6993109
1.8221859	1.8222154	1.8222207	1.8222228
2.3856290	2.9476935	2.9476956	2.9476963
2.3862301	3.0178279	3.0180082	3.0180748
2.5833172	3.2760743	3.4432923	3.4433002
2.5839852	3.2777582	4.1418082	4.1418158
2.9477062	3.4432656	4.4519274	4.6311877
3.0174627	3.5133204	4.4548953	4.7615817
3.4432320	3.5153213	4.6311437	4.7886836

Table 4.4: Computed vibration frequencies for $k = 3$, $\mathtt{a}_S = 20$, $\nu = 0.35$ and different refinement levels.

Table 4.4 shows that the spurious frequencies increase when as the mesh is refined, until disappearing for $N = 64$. In fact, it can be checked from this table that these spurious frequencies blow up as $1/\sqrt{h}$.

As a consequence of our tests, we arrive at the conclusion that it is convenient to take a value of $\mathtt{a}_S$ significantly large to be sure of getting rid of spurious modes. In fact, for all the forthcoming tests, we have used $\mathtt{a}_S = 1000$.

The subsequent numerical tests are aimed to determine the convergence rate of the scheme. With the boundary conditions considered in our model problem, it turns out that (cf. [69] and the references therein) the regularity exponents $\widehat{s}$ defined in Lemma 4.2.2 are given by Table 4.5 for different values of the Poisson ratio ν.

ν	$\widehat{s}$
0.35	0.6797
0.49	0.5999
0.5	0.5946

Table 4.5: Sobolev exponents.

We report in Tables 4.6, 4.7 and 4.8 the smallest vibration frequencies computed on several meshes ($N = 16, 32, 48, 64$) for different Poisson ratios ($\nu = 0.35, 0.49$ and 0.5) and polynomial degrees ($k = 2, 3$ and 4). We also report in these tables the computed order of convergence α and a more accurate value of the vibration frequencies λ_{ex} obtained by means of a least-square fitting of the model $\lambda_{ex} = \lambda_h + Ch^{\alpha}$.

ν	$N=16$	$N=32$	$N=48$	$N=64$	α	λ_{ex}
0.35	0.6806068	0.6807467	0.6807850	0.6808020	1.34	0.6808381
	1.6990672	1.6992327	1.6992773	1.6992969	1.37	1.6993373
0.49	0.6987402	0.6991833	0.6993160	0.6993779	1.19	0.6995295
	1.8359946	1.8366760	1.8368781	1.8369722	1.20	1.8372009
0.5	0.7007298	0.7012091	0.7013534	0.7014210	1.18	0.7015881
	1.8472390	1.8479824	1.8482043	1.8483081	1.19	1.8485623

Table 4.6: Computed lowest vibration frequencies for $k = 2$, $\mathtt{a}_S = 1000$, different values of ν and different meshes.

ν	$N=16$	$N=32$	$N=48$	$N=64$	α	λ_{ex}
0.35	0.6806839	0.6807775	0.6808029	0.6808142	1.35	0.6808379
	1.6991607	1.6992690	1.6992981	1.6993109	1.37	1.6993373
0.49	0.6989872	0.6992929	0.6993836	0.6994258	1.20	0.6995284
	1.8363810	1.8368436	1.8369810	1.8370450	1.20	1.8372002
0.5	0.7009977	0.7013286	0.7014275	0.7014736	1.19	0.7015868
	1.8476611	1.8481669	1.8483181	1.8483888	1.19	1.8485618

Table 4.7: Computed lowest vibration frequencies for $k = 3$, $\mathtt{a}_S = 1000$, different values of ν and different meshes.

ν	$N=16$	$N=32$	$N=48$	$N=64$	α	λ_{ex}
0.35	0.6807342	0.6807973	0.6808144	0.6808219	1.36	0.6808376
	1.6992195	1.6992917	1.6993112	1.6993198	1.36	1.6993377
0.49	0.6991499	0.6993638	0.6994272	0.6994567	1.20	0.6995284
	1.8366280	1.8369510	1.8370470	1.8370917	1.20	1.8372000
0.5	0.7011738	0.7014060	0.7014751	0.7015075	1.19	0.7015869
	1.8479310	1.8482851	1.8483911	1.8484407	1.19	1.8485618

Table 4.8: Computed lowest vibration frequencies for $k = 4$, $\mathtt{a}_S = 1000$, different values of ν and different meshes.

We observe from Tables 4.6–4.8 that the expected order of convergence, $\mathcal{O}(h^{2s})$ for all $s < \widehat{s}$, was attained in all the reported cases.

Finally, Figure 4.2 shows the vibration shapes corresponding to the two lowest frequencies for $\nu = 0.49$.

Figure 4.2: Deformed structures corresponding to the first (left) and second (right) lowest eigenfunctions for $N = 16$, $\nu = 0.49$, $k = 3$ and $\mathtt{a}_S = 1000$. The color reflects the magnitude of the displacement field.

4.8 Appendix. The limit problem

As was shown in the previous section, the proposed method works fine also for the limit problem ($\lambda = +\infty$), namely, for perfectly incompressible elasticity. In this appendix, we will establish a spectral characterization in this case. Also, we will prove that the eigenvalues of the nearly incompressible elasticity problem converge to those of the incompressible elasticity problem as $\lambda \to \infty$.

In the limit case $\lambda = +\infty$, the bilinear forms A and B change in their definitions, since the term where λ appears in (4.2.5) vanishes. Therefore, the limit eigenvalue problem reads as follows: Find $\kappa \in \mathbb{R}$ and $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ such that

$$A_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) = \kappa B_\infty((\boldsymbol{\sigma}, \boldsymbol{r}, (\boldsymbol{\tau}, \boldsymbol{s})) \qquad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} \tag{4.8.64}$$

with

$$B_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) := \frac{1}{2\mu} \int_\Omega \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \boldsymbol{r} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{s} : \boldsymbol{\sigma}$$

and

$$A_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) := \int_\Omega \rho^{-1} \operatorname{\mathbf{div}} \boldsymbol{\sigma} \cdot \operatorname{\mathbf{div}} \boldsymbol{\tau} + B_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s}))$$

for all $(\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$.

It is easy to check that A_∞ is a bounded bilinear form. Moreover, the arguments used in the proofs of Propositions 4.2.1 and 4.2.2 hold true for $\lambda = +\infty$, so that A_∞ satisfies the following inf-sup condition:

$$\sup_{(\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}} \frac{A_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s}))}{\|(\boldsymbol{\tau}, \boldsymbol{s})\|} \geq \alpha \|(\boldsymbol{\sigma}, \boldsymbol{r})\| \qquad \forall (\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}.$$

In consequence, we are in a position to introduce a solution operator for the limit eigenvalue problem. Let $\boldsymbol{T}_\infty : [\mathrm{L}^2(\Omega)^{n \times n}]^2 \to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$ be defined for any $(\boldsymbol{f}, \boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n \times n}]^2$ by

$$A_\infty(\boldsymbol{T}_\infty(\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}, \boldsymbol{s})) = B_\infty((\boldsymbol{f}, \boldsymbol{g}), (\boldsymbol{\tau}, \boldsymbol{s})) \qquad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}. \tag{4.8.65}$$

it is easy to check that μ is a non-zero eigenvalue of $\boldsymbol{T}$ with eigenfunction $(\boldsymbol{\sigma}_\infty, \boldsymbol{r}_\infty)$ if and only of $\kappa = 1/\mu$ is a non-vanishing eigenvalue of problem (4.8.64) with the same eigenfunction.

Our first goal is to prove that the operators $\boldsymbol{T}$ defined by (4.2.10) converges to $\boldsymbol{T}_\infty$ as λ goes to infinity. To recall that $\boldsymbol{T}$ actually depends on λ, in what follows we will denote it by $\boldsymbol{T}_\lambda$.

Before proving the convergence of $\boldsymbol{T}_\lambda$ to $\boldsymbol{T}_\infty$, we will characterize the spectrum of $\boldsymbol{T}_\infty$. Let $\boldsymbol{\mathcal{K}}$ be defined as in (5.2.13) and

$$[\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}]^{\perp_{B_\infty}} := \{(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} : \; B_\infty((\boldsymbol{\sigma}, \boldsymbol{r}), (\boldsymbol{\tau}, \boldsymbol{s})) = 0 \quad \forall (\boldsymbol{\tau}, \boldsymbol{s}) \in \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}\}.$$

We observe that $\boldsymbol{T}_\infty|_{\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{Q}}} : \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$ reduces to the identity, so that $\mu = 1$ is an eigenvalue of $\boldsymbol{T}_\infty$. Moreover, its associated eigenspace is precisely $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$.

Let us introduce the following operator which will play a role similar to thatr of $\boldsymbol{P}$ in the limit problem:

$$\begin{aligned} \boldsymbol{P}_\infty : \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} &\to \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}, \\ (\boldsymbol{\sigma}, \boldsymbol{r}) &\mapsto \boldsymbol{P}_\infty \boldsymbol{\sigma} := (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{r}}). \end{aligned}$$

where $(\widetilde{\boldsymbol{\sigma}}, (\widehat{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \boldsymbol{\mathcal{W}} \times [\mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}]$ is the solution of the following problem:

$$\frac{1}{2\mu} \int_\Omega \widetilde{\boldsymbol{\sigma}}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \widehat{\boldsymbol{u}} \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \widetilde{\boldsymbol{r}} = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{4.8.66}$$

$$\int_\Omega \boldsymbol{v} \cdot \mathbf{div}\, \widetilde{\boldsymbol{\sigma}} + \int_\Omega \widetilde{\boldsymbol{\sigma}} : \boldsymbol{s} = \int_\Omega \boldsymbol{v} \cdot \mathbf{div}\, \boldsymbol{\sigma} \qquad \forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}. \tag{4.8.67}$$

The previous problem is well posed, since the ellipticity of $\int_\Omega \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}}$ in the corresponding kernel is established in Lemma 2.3 of [71] and the following inf-sup condition holds true (see [22]):

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}} \frac{\int_\Omega \boldsymbol{v} \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_\Omega \boldsymbol{s} : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div},\Omega);\Omega}} \geq \beta(\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{s}\|_{0,\Omega}) \qquad \forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}.$$

We observe that problem (4.8.66)–(4.8.67) is a dual mixed formulation with weakly imposed symmetry of the following incompressible elasticity problem with volumetric force density $-\,\mathbf{div}\, \boldsymbol{\sigma}$

$$-\,\mathbf{div}\, \widetilde{\boldsymbol{\sigma}} = -\,\mathbf{div}\, \boldsymbol{\sigma} \qquad \text{in}\, \Omega, \tag{4.8.68}$$

$$\frac{1}{2\mu} \widetilde{\boldsymbol{\sigma}}^{\mathtt{D}} = \varepsilon(\widehat{\boldsymbol{u}}) \qquad \text{in}\, \Omega, \tag{4.8.69}$$

$$\widetilde{\boldsymbol{\sigma}} \nu = 0 \qquad \text{in}\, \Gamma_F, \tag{4.8.70}$$

$$\widehat{\boldsymbol{u}} = 0 \qquad \text{in}\, \Gamma_R. \tag{4.8.71}$$

It is easy to check that $(\widetilde{\boldsymbol{\sigma}}, \widehat{\boldsymbol{u}}) \in \mathrm{H}(\mathbf{div};\Omega) \times \mathrm{H}^1(\Omega)^n$ satisfies (4.8.68)–(4.8.71) if and only if $(\widetilde{\boldsymbol{\sigma}}, (\widehat{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \boldsymbol{\mathcal{W}} \times [\mathrm{L}^2(\Omega)^n \times \boldsymbol{\mathcal{Q}}]$ is the solution of (4.8.66)–(4.8.67) with $\widetilde{\boldsymbol{r}} = \frac{1}{2}[\nabla \widehat{\boldsymbol{u}} - (\nabla \widehat{\boldsymbol{u}})^{\mathtt{t}}]$.

Now, by resorting to the relation between the incompressible elasticity and the Stokes problems, we conclude that there exists $\widehat{s}_\infty \in (0,1)$ depending only on Ω and μ (see for instance

[46], [43] and [84]) such that, for all $s \in (0, \widehat{s}_\infty)$ the solution $\widehat{\boldsymbol{u}}$ of (4.8.68)–(4.8.71) belongs to $\mathrm{H}^{1+s}(\Omega)^n$ and the following estimate hold true

$$\|\widehat{\boldsymbol{u}}\|_{1+s,\Omega} \leq C\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}, \tag{4.8.72}$$

with a constant C independent of $\boldsymbol{\sigma}$.

The following lemma is a consequence of this regularity result.

Lemma 4.8.1 *For all $s \in (0, \widehat{s})$ and $(\boldsymbol{\sigma}, \boldsymbol{r}) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}$, if $(\widetilde{\boldsymbol{\sigma}}, (\widehat{\boldsymbol{u}}, \widetilde{\boldsymbol{r}}))$ is the solution of (4.8.66)–(4.8.67), then $\widetilde{\boldsymbol{\sigma}} \in \mathrm{H}^s(\Omega)^{n\times n}$, $\widetilde{\boldsymbol{u}} \in \mathrm{H}^{1+s}(\Omega)^{n\times n}$, $\widetilde{\boldsymbol{r}} \in \mathrm{H}^s(\Omega)^{n\times n}$ and*

$$\|\widetilde{\boldsymbol{\sigma}}\|_{s,\Omega} + \|\widehat{\boldsymbol{u}}\|_{1+s,\Omega} + \|\widetilde{\boldsymbol{r}}\|_{s,\Omega} \leq C\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega},$$

with a constant C independent of $\boldsymbol{\sigma}$. Consequently, $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \subset \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n}$.

We observe that $\boldsymbol{P}_\infty$ is idempotent and that $\ker(\boldsymbol{P}_\infty) = \boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$. Moreover, being $\boldsymbol{P}_\infty$ a projector, the orthogonal decomposition $\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}} = (\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}) \oplus \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ holds true. On the other hand, $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ is an invariant space of $\boldsymbol{T}_\infty$ (see Proposition A.1 in [69]).

Proposition 4.8.1 *For all $s \in (0, \widehat{s})$*

$$\boldsymbol{T}_\infty(\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})) \subset \{(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) \in \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n} : \mathbf{div}\,\boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega)^n\}, \tag{4.8.73}$$

and there exists $C > 0$ such that for all $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$, if $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) = \boldsymbol{T}_\infty(\boldsymbol{f}, \boldsymbol{g})$, then

$$\|\boldsymbol{\sigma}^*\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}^*\|_{1,\Omega} + \|\boldsymbol{r}^*\|_{s,\Omega} \leq C\|(\boldsymbol{f}, \boldsymbol{g})\|. \tag{4.8.74}$$

Moreover, $\boldsymbol{T}_\infty|_{\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})} : \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}}) \to \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ is a compact operator.

Proof. Let $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ and $(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) = \boldsymbol{T}_\infty(\boldsymbol{f}, \boldsymbol{g})$. Then, testing (4.8.75) with $\boldsymbol{\tau} \in \mathcal{D}(\Omega)^{n\times n} \subset \boldsymbol{\mathcal{W}}$, we have that

$$-\rho^{-1}\nabla(\mathbf{div}\,\boldsymbol{\sigma}^*) + \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{D}} + \boldsymbol{r}^* = \frac{1}{2\mu}\boldsymbol{f}^{\mathtt{D}} + \boldsymbol{g}.$$

Hence, since ρ and μ are constants, we conclude that $\mathbf{div}\,\boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega)^n$.

Since $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$ is invariant with respect to $\boldsymbol{T}_\infty$, applying Lemma 4.8.1 we obtain directly (4.8.73). On the other hand, (4.8.74) is a consequence of Lemma 4.8.1. Finally, the compacteness of $\boldsymbol{T}_\infty|_{\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}}\times\boldsymbol{\mathcal{Q}})}$ is a consequence of the following compact embedding

$$\{(\boldsymbol{\sigma}^*, \boldsymbol{r}^*) \in \mathrm{H}^s(\Omega)^{n\times n} \times \mathrm{H}^s(\Omega)^{n\times n} : \mathbf{div}\,\boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega)^n\} \hookrightarrow \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}},$$

which allow us to conclude the proof. □ Now we are in position to establish a spectral characterization for $\boldsymbol{T}_\infty$.

Theorem 4.8.1 *The spectrum of $\boldsymbol{T}_\infty$ decomposes as follows: $\mathrm{sp}(\boldsymbol{T}_\infty) = \{0, 1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, where:*

(i) $\mu = 1$ is an infinite-multiplicity eigenvalue of $\boldsymbol{T}_\infty$ and its associated eigenspace is $\boldsymbol{\mathcal{K}} \times \boldsymbol{\mathcal{Q}}$.

(ii) $\mu = 0$ *is an eigenvalue of* $\boldsymbol{T}_\infty$ *and its associated eigenspace is* $\mathcal{Z} \times \boldsymbol{\mathcal{Q}}$*, where*

$$\mathcal{Z} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} : \ \boldsymbol{\tau}^{\mathtt{D}} = 0\} = \{q\boldsymbol{I} : \ q \in \mathrm{H}^1(\Omega) \ \textit{and} \ q = 0 \quad \textit{on} \ \Gamma_N\}.$$

(iii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0,1)$ *is a sequence of nondefective finite-multiplicity eigenvalues of* $\boldsymbol{T}_\infty$ *which converge to zero, ; the corresponding eigenspaces lie in* $\boldsymbol{P}_\infty(\boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{Q}})$*.*

Proof. It is enough to follow the steps of Theorem 3.5 from [71]. □

Now we are in position to establish the following convergence result.

Lemma 4.8.2 *There exists a constant* $C > 0$ *such that*

$$\|(\boldsymbol{T}_\lambda - \boldsymbol{T}_\infty)((\boldsymbol{f},\boldsymbol{g}))\| \leq \frac{C}{\lambda}\|(\boldsymbol{f},\boldsymbol{g})\| \qquad \forall (\boldsymbol{f},\boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2.$$

Proof. Let $(\boldsymbol{f},\boldsymbol{g}) \in [\mathrm{L}^2(\Omega)^{n\times n}]^2$ and let $(\boldsymbol{\sigma}_\lambda, \boldsymbol{r}_\lambda) := \boldsymbol{T}_\lambda(\boldsymbol{f},\boldsymbol{g})$ and $(\boldsymbol{\sigma}_\infty, \boldsymbol{r}_\infty) := \boldsymbol{T}_\infty(\boldsymbol{f},\boldsymbol{g})$. Then, from (4.2.10) and the definition of $\mathcal{C}$ we have

$$\int_\Omega \rho^{-1}\,\mathbf{div}_\lambda\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}_\lambda^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \frac{1}{n(n\lambda+2\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_\lambda)\operatorname{tr}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{r}_\lambda : \boldsymbol{\tau}$$
$$= \frac{1}{2\mu}\int_\Omega \boldsymbol{f}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \frac{1}{n(n\lambda+2\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{f})\operatorname{tr}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{g} : \boldsymbol{\tau},$$
$$\int_\Omega \boldsymbol{\sigma}_\lambda : \boldsymbol{s} = \int_\Omega \boldsymbol{f} : \boldsymbol{s}.$$

Whereas

$$\int_\Omega \rho^{-1}\,\mathbf{div}\,\boldsymbol{\sigma}_\infty\cdot\mathbf{div}\,\boldsymbol{\tau} + \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}_\infty^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \boldsymbol{r}_\infty : \boldsymbol{\tau} = \frac{1}{2\mu}\int_\Omega \boldsymbol{f}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} + \int_\Omega \boldsymbol{g} : \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{4.8.75}$$

$$\int_\Omega \boldsymbol{\sigma}_\infty : \boldsymbol{s} = \int_\Omega \boldsymbol{f} : \boldsymbol{s} \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{4.8.76}$$

Subtracting the the above equations we have

$$\int_\Omega \rho^{-1}\,\mathbf{div}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\cdot\mathbf{div}\,\boldsymbol{\tau} + \frac{1}{2\mu}\int_\Omega (\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}) : \boldsymbol{\tau}^{\mathtt{D}}$$
$$+ \int_\Omega (\boldsymbol{r}_\lambda - \boldsymbol{r}_\infty) : \boldsymbol{\tau} = \frac{1}{n(n\lambda+2\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{f} - \boldsymbol{\sigma}_\lambda)\operatorname{tr}(\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}, \tag{4.8.77}$$

$$\int_\Omega (\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty) : \boldsymbol{s} = 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}. \tag{4.8.78}$$

Testing this equation with $\boldsymbol{\tau} := \boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty$ and $\boldsymbol{s} := \boldsymbol{r}_\lambda - \boldsymbol{r}_\infty$ we have

$$\begin{aligned}
\rho^{-1}\|\,\mathbf{div}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2 &= \frac{1}{n(n\lambda+2\mu)}\int_\Omega (\operatorname{tr}(\boldsymbol{f}) - \operatorname{tr}(\boldsymbol{\sigma}_\lambda))\operatorname{tr}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty) \\
&\leq \frac{1}{n(n\lambda+2\mu)}\int_\Omega \|\operatorname{tr}(\boldsymbol{f}) - \operatorname{tr}(\boldsymbol{\sigma}_\lambda)\|_{0,\Omega}\,\|\operatorname{tr}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega} \\
&\leq \frac{1}{n\lambda+2\mu}(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{\sigma}_\lambda\|_{0,\Omega})\|\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty\|_{0,\Omega} \\
&\leq \frac{C}{n\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|\|\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty\|_{0,\Omega},
\end{aligned}$$

where we have used (4.2.11) to bound $\|\boldsymbol{\sigma}_\lambda\|_{0,\Omega}$. Moreover

$$\underbrace{\min\left\{\rho^{-1}, \frac{1}{2\mu}\right\}}_{C_{\rho,\mu}} \left(\|\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2 + \|\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2\right) \leq \frac{C}{n\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|\|\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty\|_{0,\Omega}.$$

We observe that $\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty \in \boldsymbol{\mathcal{W}}$ is symmetric due to equation (4.8.78). Then, we resort to the following estimate (see [28] for instance)

$$C\|\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty\|_{0,\Omega}^2 \leq \|\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2 + \|\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2$$

with $C > 0$ to deduce that

$$C\|\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty\|_{\mathbf{div},\Omega} \leq (\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2 + \|\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2)^{1/2}.$$

Hence

$$\begin{aligned}\|\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2 &+ \|\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2 \\ &\leq \frac{C_{\rho,\mu}}{n\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|(\|\boldsymbol{\sigma}^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}^2 + \|\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}^2)^{1/2} \qquad (4.8.79)\end{aligned}$$

and, finally,

$$\|\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty\|_{\mathbf{div},\Omega} \leq \frac{C}{\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|, \qquad (4.8.80)$$

with C a positive constant depending on ρ, μ and n.

On the other hand, taking into account the inf-sup condition (4.2.9), (4.8.77), Cauchy-Schwarz inequality, (4.8.79) and (4.8.80), we have

$$\begin{aligned}&\beta\|\boldsymbol{r}_\lambda - \boldsymbol{r}_\infty\|_{0,\Omega} \\ &\leq \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}} \frac{\frac{1}{n(n\lambda+2\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\operatorname{tr}(\boldsymbol{\tau}) - \int_\Omega \rho^{-1}\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\cdot\operatorname{\mathbf{div}}\boldsymbol{\tau} - \frac{1}{2\mu}\int_\Omega(\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}) : \boldsymbol{\tau}^{\mathtt{D}}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};O)}} \\ &\leq \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}} \frac{\frac{C}{n\lambda+2\mu}\|(\boldsymbol{f},\boldsymbol{g})\|\|\boldsymbol{\tau}\|_{0,\Omega} + \rho^{-1}\|\operatorname{\mathbf{div}}(\boldsymbol{\sigma}_\lambda - \boldsymbol{\sigma}_\infty)\|_{0,\Omega}\|\operatorname{\mathbf{div}}\boldsymbol{\tau}\|_{0,\Omega} + \frac{1}{2\mu}\|\boldsymbol{\sigma}_\lambda^{\mathtt{D}} - \boldsymbol{\sigma}_\infty^{\mathtt{D}}\|_{0,\Omega}\|\boldsymbol{\tau}^{\mathtt{D}}\|_{0,\Omega}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega)}} \\ &\leq \frac{C}{\lambda}\|(\boldsymbol{f},\boldsymbol{g})\|.. \qquad (4.8.81)\end{aligned}$$

Hence, the proof follows by combining (4.8.80) and (4.8.81).

□

Now we are in a position to establish the following result.

Theorem 4.8.2 *Let $\mu_\infty > 0$ be an eigenvalue of $\boldsymbol{T}_\infty$ of multiplicity m. Let D be any disc of the complex plane centered at μ_∞ and containing no other element of the spectrum of $\boldsymbol{T}_\infty$. Then, for λ large enough, D contains exactly m eigenvalues of $\boldsymbol{T}_\lambda$ (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu_\infty > 0$ of $\boldsymbol{T}_\infty$ is a limit of eigenvalues μ of $\boldsymbol{T}_\lambda$, as λ goes to infinity.*

Chapter 5

Quadratic eigenvalue problem for a fluid-structure system

5.1 Introduction

The interaction problem between fluids and structures is one of the most studied topics for engineers, because of the several applications in the industry. The design of the wings of an aircraft, construction of motors of many vehicles and machines, the construction of ships and many other activities, requires to known the vibration frequencies of the interaction of fluids with structures.

To compute these vibration frequencies, the numerical methods that employ finite elements are an important tool for this goal. Articles like [12, 14, 15, 16, 17] among others, deal with interaction problems and different kind of formulations to solve the elastoacustic problem. Nevertheless, this articles neglect the presence of viscosity in the models. Viscosity gives path to the dissipation phenomenon, in particular internal dissipation. For details of this physical phenomenon see for instance [76].

Mixed formulations to study the elastoacustic problem have also been used. In [70] the fluid-structure acoustic interaction problem is analyzed with a mixed formulation written in terms of the Cauchy stress tensor in the solid and the pressure the fluid, respectively. Since the stress tensor is an H(div) function and the pressure is an H^1 function, the finite element method is based into discretize the solid domain with Arnold-Falk-Winther elements and the fluid domain with Lagrange linear elements. With this particular choice of elements the Galerkin method results to be a conforming method.

Recently in [62] a finite element discretization fo the interaction problem between dissipative fluids within a rigid cavity has been analyzed. The presence of viscosity leads to a non-linear eigenvalue problem, a quadratic problem more precisely, where the solution operator is non compact and it is analyzed by adapting the techniques used in [13]. However, since the analysis of the quadratic eigenvalue problem is not direct, an additional unknown is incorporated in order to write a double-size linear problem.

The goal of this paper is to deal with the elastoacustic problem considering a dissipative fluid interacting with an elastic structure. Since the fluid has viscous properties, the eigenvalue problem turns to be quadratic, as it happens in [62]. According to this, it is necessary to introduce additional unknowns for the fluid and the solid, in order to obtain a double-size linear eigenvalue problem. The spectral analysis is similar as the one studied in [62], where we split the solution operator in two operators, where one of them is compact and the other has an essential spectrum. For the spectral characterization we use the thechniques presented in [62]. We introduce a finite element method where the fluid is discretized with Raviart-Thomas elements and the solid with continuous piecewise linear functions. This choice of elements leads to a non-conforming method as happens, for example, in [12], where it is not possible to impose continuity in the contact interface between the degrees of freedom of the Raviart-Thomas and the linear functions. Moreover, a corrected interpolant operator is needed to approximate the solution in the interface. For this reason, we will consider the one connstrcuted in [12].

The paper is organized as follows: in Section 5.2, we introduce the spectral problem and the corresponding variational formulation, which leads to a quadratic eigenvalue problem. This variational spectral problem is written in terms of the displacements of the fluid and the solid. We introduce an auxiliary unknowns to transform the quadratic eigenvalue problem into a linear one. Moreover, we introduce the corresponding solution operator for the spectral problem. In Section 5.3, we provide a rigorous spectral characterization of the solution operator, based on the theory developed in [55]. We also consider the limit problem (i.e., the case when the viscosity vanishes) and the relation between the solutions of the dissipative and non-dissipative problems. In Section 5.4, we introduce a finite element discretization using Raviart-Thomas elements for the fluid and piecewise linear functions for the solid, in order to approximate the eigenfunctions and eigenvalues. We analyze the discrete spectral problem analogously as in the continuous case and introduce the corresponding discrete solution operator. We use the abstract theory from [36] to prove the convergence. We also prove error estimates for our problem by adapting the arguments from [13]. Finally, in Section 5.5, we report some numerical tests which allow us to assess the performance of the proposed method.

Throughout the paper, Ω is generic Lipschitz bounded domains of $\mathbb{R}^d$ ($d = 2, 3$). When corresponds, we will denote by Ω_f and Ω_s the domains of the fluid and the solid, respectively. We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth function compactly supported in Ω. For $r \geq 0$, $\|\cdot\|_{r,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $\mathrm{H}^r(\Omega)$ or $\mathrm{H}^r(\Omega)^d$ with the convention $\mathrm{H}^0(\Omega) := \mathrm{L}^2(\Omega)$. We also define the Hilbert space $\mathrm{H}(\mathrm{div};\Omega) := \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^d : \ \mathrm{div}\,\boldsymbol{v} \in \mathrm{L}^2(\Omega)\}$, whose norm is given by $\|\boldsymbol{v}\|^2_{\mathrm{div},\Omega} := \|\boldsymbol{v}\|^2_{0,\Omega} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Omega}$. We also define the Hilbert space $\mathrm{H}^{s,r}(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \mathrm{H}^s(\Omega) : \ \mathrm{div}\,\boldsymbol{v} \in \mathrm{H}^r(\Omega)\}$ endowed with the norm $\|\boldsymbol{v}\|^2_{\mathrm{H}^s(\mathrm{div},\Omega)} := \|\boldsymbol{v}\|^2_{s,\Omega} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{1,\Omega}$.

Finally, C represents a generic constant independent of the discretization parameters, which may take different values at different places.

5.2 The main problem

In this work we will study the vibration modes of the interaction problem of an homogeneous fluid with an elastic structure. The fluid, as in [62] will be considered as irrotational. Let Ω_f the domain occupied for the fluid and Ω_s the corresponding domain of the solid. Let ρ_f and ρ_S the densities of the fluid and the solid respectively ν is the fluid viscosity and c the acoustic speed of the fluid and λ_S and μ_S the Lamé coefficients. In our analysis, all the previous coefficients will be considered as positive constants.

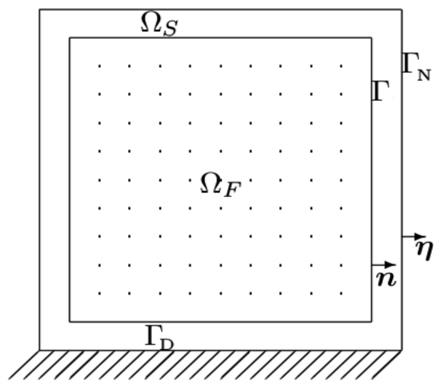

Figure 5.1: Scheme of the problem.

The equation of motion derived from the Stokes equation is

$$\rho_f \ddot{\boldsymbol{U}} = 2\nu \Delta \dot{\boldsymbol{U}} - \nabla P \quad \text{in } \Omega_f,$$

where $\boldsymbol{U}$ denotes the fluid displacement and P the pressure fluctuation on the domain Ω_f. The dot represents derivation with respect to time. Moreover, since the fluid is compressible we consider the isoentropic relation

$$P + \rho_f c^2 \operatorname{div} \boldsymbol{U} = 0 \quad \text{in } \Omega_f,$$

Let us recall the following definitions, given $\varphi : \Omega \to \mathbb{R}$ and $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^2$, let

$$\mathbf{curl}\, \varphi := \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x} \right)^{\mathtt{t}} \quad \text{and} \quad \operatorname{curl} \boldsymbol{\varphi} := \frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y}.$$

Since we are considering irrotational fluids, $\operatorname{curl} \boldsymbol{U} = 0$. Hence, considering the identity $\Delta \dot{\boldsymbol{U}} = \nabla(\operatorname{div} \dot{\boldsymbol{U}}) + \mathbf{curl}(\operatorname{curl} \dot{\boldsymbol{U}})$ we conclude that $\Delta \dot{\boldsymbol{U}} = \nabla(\operatorname{div} \dot{\boldsymbol{U}})$. Then, the equations of our model for an irrotational dissipative fluid and an elastic structure are the following:

$$\rho_f \ddot{\boldsymbol{U}} - 2\nu \nabla(\operatorname{div} \dot{\boldsymbol{U}}) + \nabla P = 0 \quad \text{in } \Omega_f \times (0,T), \tag{5.2.1}$$

$$P + \rho_f c^2 \operatorname{div} \boldsymbol{U} = 0 \quad \text{in } \Omega_f \times (0,T), \tag{5.2.2}$$

$$\rho_s \ddot{\boldsymbol{W}} - \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{W})) = 0 \quad \text{in } \Omega_s \times (0,T), \tag{5.2.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{W}) - \lambda_S \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{W}))I - 2\mu_S \boldsymbol{\varepsilon}(\boldsymbol{W}) = 0 \quad \text{in } \Omega_s \times (0,T), \tag{5.2.4}$$

$$\boldsymbol{U} \cdot \boldsymbol{n} - \boldsymbol{W} \cdot \boldsymbol{n} = 0 \quad \text{in } \Gamma \times [0,T] \tag{5.2.5}$$

$$\boldsymbol{\sigma}(\boldsymbol{W})\boldsymbol{n} - (P + 2\nu \operatorname{div} \boldsymbol{U})\boldsymbol{n} = 0 \quad \text{in } \Gamma \times [0,T] \tag{5.2.6}$$

$$\boldsymbol{\sigma}(\boldsymbol{W})\boldsymbol{\eta} = 0 \quad \text{in } \Gamma_N \times [0,T] \tag{5.2.7}$$

$$\boldsymbol{W} = 0 \quad \text{in } \Gamma_D \times [0,T]. \tag{5.2.8}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}(\cdot)$ is the linear strain tensor defined by $\boldsymbol{\varepsilon}(\boldsymbol{W}) := \frac{1}{2}[\nabla \boldsymbol{W} + (\nabla \boldsymbol{W})^{\mathrm{t}}]$, Γ_{I} is the interface between the solid and the fluid, Γ_D and Γ_N are the Dirichlet and Neumann parts of the boundary of the solid, respectively and $\boldsymbol{n}$ is the outward unit vector to Γ_N. We introduce the following spaces

$$\mathcal{H} := \mathrm{L}^2(\Omega_f)^n \times \mathrm{L}^2(\Omega_s)^2,$$

$$\mathcal{X} := \mathrm{H}(\operatorname{div}; \Omega_f) \times \mathrm{H}^1_{\Gamma_D}(\Omega_s)^2,$$

$$\mathcal{V} := \{(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{X} \ : \ \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{w} \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma\},$$

where $\mathrm{H}^1_{\Gamma_D}(\Omega_s)^2$ is the subspace of $\mathrm{H}^1(\Omega_s)^2$ of functions that vanishes in Γ_D.

Then, multiplying with different test functions, integrating by parts and using the boundary and interface conditions, we obtain the following variational problem:

Find $\boldsymbol{U} : [0,T] \to \mathrm{H}(\operatorname{div}; \Omega_f)$, $P : [0,T] \to \mathrm{L}^2(\Omega_f)^2$ *and* $\boldsymbol{W} : [0,T] \to \mathrm{H}^1_{\Gamma_D}(\Omega_s)^2$ *such that*

$$\int_{\Omega_f} \rho_f \ddot{\boldsymbol{U}} \cdot \overline{\boldsymbol{v}} + \int_{\Omega_s} \rho_s \ddot{\boldsymbol{W}} \cdot \overline{\boldsymbol{\tau}} + 2\int_{\Omega_f} \nu \operatorname{div} \dot{\boldsymbol{U}} \operatorname{div} \overline{\boldsymbol{v}} - \int_{\Omega_f} P \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \sigma(\boldsymbol{W}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = 0, \tag{5.2.9}$$

for all $(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{V}$. Considering the isoentropic relation for the fluid $P = \rho_f c^2 \operatorname{div} \boldsymbol{U}$, we eliminate the pressure in (5.2.9) to obtain

$$\int_{\Omega_f} \rho_f \ddot{\boldsymbol{U}} \cdot \overline{\boldsymbol{v}} + \int_{\Omega_s} \rho_s \ddot{\boldsymbol{W}} \cdot \overline{\boldsymbol{\tau}} + 2\int_{\Omega_f} \nu \operatorname{div} \dot{\boldsymbol{U}} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{U} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \sigma(\boldsymbol{W}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = 0. \tag{5.2.10}$$

The damped vibration modes of the fluid and the structure are complex solutions of the form $\boldsymbol{U}(\boldsymbol{x}, t) = e^{\lambda t}\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{W}(\boldsymbol{x}, t) = e^{\lambda t}\boldsymbol{w}(\boldsymbol{x}, t)$. Then problem (5.2.10) written in the frequency domain reads as follows:

Problem 5.2.1 Find $\lambda \in \mathbb{C}$ and $(\mathbf{0}, \mathbf{0}) \neq (\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{V}$ such that

$$\lambda^2 \left(\int_{\Omega_f} \rho_f \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \int_{\Omega_s} \rho_s \boldsymbol{w} \cdot \overline{\boldsymbol{\tau}} \right) + 2\lambda \int_{\Omega_f} \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = 0 \qquad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{V}. \tag{5.2.11}$$

We remark that in absence of viscosity (i.e $\nu = 0$) we are left only with the classical elastoacustic problem like the studied, for example, in [12]. The eigenvalues λ^2 of this problem are negative real numbers (we will prove below), so those λ are purely imaginary namely $\lambda = \pm\omega$ being called the vibration frequencies, since the corresponding eigenfunctions $\boldsymbol{U}(\boldsymbol{x}, t) = e^{-i\omega t}\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{W}(\boldsymbol{x}, t) = e^{-i\omega t}\boldsymbol{w}(\boldsymbol{x})$. Is for this reason that, for $\nu = 0$ Problem 5.2.1 is usually written as follows: Find $\omega > 0$ and $(\mathbf{0}, \mathbf{0}) \neq (\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{V}$ such that

$$\int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = \omega^2 \left(\int_{\Omega_f} \rho_f \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + \int_{\Omega_s} \rho_s \boldsymbol{w} \cdot \overline{\boldsymbol{\tau}} \right), \tag{5.2.12}$$

for all $(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{V}$.

In the applications, ν is typically small and we will prove below that when $\nu \to 0$, the eigenvalues λ of Problem 5.2.1 lie very close to the imaginary axis ($\pm i\omega$) with ω being the vibration frequency (i.e. a solution of (5.2.12)). For this quadratic eigenvalue problem, we observe that the eigenspace associated to $\lambda = 0$ is

$$\mathcal{K} := \{(\boldsymbol{u}, \mathbf{0}) \in \mathcal{X} \ : \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega_f \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\} \subset \mathcal{V} \tag{5.2.13}$$

and its orthogonal complement, which we denote as $\mathcal{G}$ is defined by

$$\mathcal{G} := \{(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{X} \ : \ \boldsymbol{u} = \nabla\varphi, \quad \varphi \in \mathrm{H}^1_{\Gamma_D}(\Omega_f)^2\}. \tag{5.2.14}$$

Both spaces are closed subspaces of $\mathcal{V}$ endowed with the $\mathrm{H}(\operatorname{div}; \Omega_f) \times \mathrm{H}^1_{\Gamma_D}(\Omega_s)$ norm. We also define the subspace $\mathcal{G}_{\mathcal{V}} := \mathcal{G} \cap \mathcal{V}$. From the physical point of view, the time domain problem (5.2.9) is dissipative in the sense that it solution should decay as t increases. The latter happens if and only if the so called decay rate $\mathrm{Re}(\lambda)$ is negative. The following result shows that this is the case in our formulation.

Lemma 5.2.1 *Let $\lambda \in \mathbb{C}$ and $(\mathbf{0}, \mathbf{0}) \neq (\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{V}$ be the solution of problem* (5.2.10)*. if $\lambda \neq 0$, then $Re(\lambda) < 0$.*

Proof. Testing (5.2.10) with $(\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{\tau})$ we define

$$A = \int_{\Omega_f} \rho_f |\boldsymbol{v}|^2 + \int_{\Omega_s} \rho_s |\boldsymbol{\tau}|^2; \;\; B := 2 \int_{\Omega_f} \nu |\operatorname{div} \boldsymbol{v}|^2,$$

$$C := \int_{\Omega_f} \rho_f c^2 |\operatorname{div} \boldsymbol{v}|^2 + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{\tau}) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}).$$

Since the coefficients ρ_s, ρ_f, ν and c are positive, we deduce that $A > 0$, $B \geq 0$ and $C \geq 0$. We observe that in the definition of C the solid part is no negative because of Korn's inequality. On the other hand, we observe that λ is the solution of the algebraic equation $A\lambda^2 + B\lambda + C = 0$ with the form $\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. From this it is immediate to check that $\mathrm{Re}\lambda < 0$. □

For the theoretical analysis it is convenient to transform problem (5.2.10) into a linear problem. For this reason, we introduce the following auxiliary unknown $\widehat{\boldsymbol{u}} = \lambda \boldsymbol{u}$ for the fluid and $\widehat{\boldsymbol{w}} = \lambda \boldsymbol{w}$ for the solid. Then, replacing this new unknowns in Problem 5.2.1, we obtain the following linear eigenvalue problem:

Problem 5.2.2 Find $\lambda \in \mathbb{C}$ and $((\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{0})) \neq ((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}})) \in \mathcal{V} \times \mathcal{H}$ such that

$$\int_{\Omega_f} \rho c^2 \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = \lambda_h \left(-2 \int_{\Omega_f} \nu \operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} - \int_{\Omega_f} \rho_f \widehat{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} \right.$$
$$\left. - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{w}} \cdot \overline{\boldsymbol{\tau}}_h \right) \qquad \forall (\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{V}, \tag{5.2.15}$$

$$\int_{\Omega_f} \rho_f \widehat{\boldsymbol{u}} \cdot \overline{\widehat{\boldsymbol{v}}} + \int_{\Omega_f} \rho_s \widehat{\boldsymbol{w}} \cdot \overline{\widehat{\boldsymbol{\tau}}}_h = \lambda \left(\int_{\Omega_f} \rho_f \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{v}}} + \int_{\Omega_s} \rho_s \boldsymbol{w} \cdot \overline{\widehat{\boldsymbol{\tau}}} \right) \qquad \forall (\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}) \in \mathcal{V}. \tag{5.2.16}$$

We observe that $\lambda = 0$ is a eigenvalue of problem (5.2.15)–(5.2.16) with associated eigenspace is $\widetilde{\mathcal{K}} = \mathcal{K} \times \{(\mathbf{0},\mathbf{0})\}$. Let $\widetilde{\mathcal{G}}$ the orthogonal complement of $\widetilde{\mathcal{K}}$ defined by $\widetilde{\mathcal{G}} := \mathcal{G}_{\mathcal{V}} \times \mathcal{H}$.

In what follows we will prove additional regularity for the functions in $\mathcal{G}_{\mathcal{V}}$.

Lemma 5.2.2 *There exists $s \in (\frac{1}{2}, 1]$ such that for all $(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{G}_{\mathcal{V}}$ and*

$$\|\boldsymbol{u}\|_{s,\Omega_f} \leq C(\|\operatorname{div} \boldsymbol{u}\|_{0,\Omega_f} + \|\boldsymbol{w}\|_{1,\Omega_s})$$

Proof. Let $(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{G}_{\mathcal{V}}$ and consider the following well posed Neumann problem

$$\begin{aligned} \Delta\varphi &= \operatorname{div} \boldsymbol{u} \quad \text{in}\, \Omega_f, \\ \frac{\partial \varphi}{\partial n} &= \boldsymbol{w} \cdot \boldsymbol{n} \quad \text{on}\, \Gamma, \end{aligned}$$

where $\operatorname{div} \boldsymbol{u} \in \mathrm{L}^2(\Omega_f)$ and $\boldsymbol{w} \cdot \boldsymbol{n}|_{\Gamma_j} \in \mathrm{H}^{1/2}(\Gamma_j)$ with $j = 1, \ldots, N$ when the boundary is polyhedric of N edges and $\Gamma = \cup_{j=1}^N \Gamma_j$.

It is well known that exists $s \in (\frac{1}{2}, 1]$ such that exists an unique $\varphi \in \mathrm{H}^{1+s}(\Omega_f)$ solution of the problem above that satisfies

$$\|\varphi\|_{1+s} \leq C \left(\|\operatorname{div} \boldsymbol{u}\|_{0,\Omega_f} + \sum_{j=1}^{n} \|\boldsymbol{w} \cdot \boldsymbol{n}\|_{\frac{1}{2},\Gamma_j} \right).$$

Moreover, due the classical trace theorem we know that

$$\sum_{j=1}^{n} \|\boldsymbol{w} \cdot \boldsymbol{n}\|_{\frac{1}{2},\Gamma_j} \leq C_T \|\boldsymbol{w}\|_{1,\Omega_s},$$

with C_T the positive constant of the trace Theorem. Then, estimate above is rewritten as follows

$$\|\varphi\|_{1+s,\Omega_f} \leq C_1(\|\operatorname{div} \boldsymbol{u}\|_{0,\Omega_f} + \|\boldsymbol{w}\|_{1,\Omega_s}),$$

where $C_1 = \max\{1, C_T\}$. Since $(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{G}_{\mathcal{V}}$, then $\boldsymbol{u} = \nabla\varphi$ and

$$\|\boldsymbol{u}\|_{s,\Omega_f} = \|\nabla\varphi\|_{s,\Omega_f} \leq C_1(\|\operatorname{div} \boldsymbol{u}\|_{0,\Omega_f} + \|\boldsymbol{w}\|_{1,\Omega_s}), \tag{5.2.17}$$

where s is fixed and depends on the geometry of the domain. Hence, we conclude the proof.

□

Now we introduce the sesquilinear form $a : \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ defined by

$$a((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{\tau})) := \int_{\Omega_f}\rho_f c^2 \operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}).$$

We also define the sesquilinear forms $\widehat{a},\widehat{b} : (\mathcal{V}\times\mathcal{H})\times(\mathcal{V}\times\mathcal{H})\to\mathbb{C}$ as follows

$$\widehat{a}(((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))) = a((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{\tau})) + \int_{\Omega_f}\rho_f\widehat{\boldsymbol{u}}\cdot\overline{\widehat{\boldsymbol{v}}} + \int_{\Omega_s}\rho_s\widehat{\boldsymbol{w}}\cdot\overline{\widehat{\boldsymbol{\tau}}},$$

$$\begin{aligned}\widehat{b}(((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))) = &-2\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} - \int_{\Omega_f}\rho_f\widehat{\boldsymbol{u}}\cdot\overline{\boldsymbol{v}}\\ &+\int_{\Omega_f}\rho_f\boldsymbol{u}\cdot\overline{\widehat{\boldsymbol{v}}} - \int_{\Omega_s}\rho_s\widehat{\boldsymbol{w}}\cdot\overline{\boldsymbol{\tau}} + \int_{\Omega_s}\rho_s\boldsymbol{w}\cdot\overline{\widehat{\boldsymbol{\tau}}}.\end{aligned}$$

The following lemma shows that the sesquilinear forms $a(\cdot,\cdot)$ and $\widehat{a}(\cdot,\cdot)$ are elliptic in the orthogonal complements of $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ respectively.

Lemma 5.2.3 *The sesquilinear form $a : \mathcal{G}_{\mathcal{V}}\times\mathcal{G}_{\mathcal{V}}\to\mathbb{C}$ is $\mathcal{G}_{\mathcal{V}}$-elliptic and consequently $\widehat{a} : \widetilde{\mathcal{G}}\times\widetilde{\mathcal{G}}\to\mathbb{C}$ is $\widetilde{\mathcal{G}}$-elliptic.*

Proof. Let $(\boldsymbol{v},\boldsymbol{\tau})\in\mathcal{G}_{\mathcal{V}}$. Then, using the definition of $a(\cdot,\cdot)$ and the fact that ρ_f, c are constants we observe

$$a((\boldsymbol{v},\boldsymbol{\tau}),(\boldsymbol{v},\boldsymbol{\tau})) = \int_{\Omega_f}\rho_f c^2|\operatorname{div}\boldsymbol{v}|^2 + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{\tau}) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}).$$

We will bound the first term of the right-hand side. From Lemma 5.2.2 we have

$$\|\boldsymbol{v}\|_{0,\Omega_f} \le \|\boldsymbol{v}\|_{s,\Omega_f} \le C(\|\operatorname{div}\boldsymbol{v}\|_{0,\Omega_f} + \|\boldsymbol{\tau}\|_{1,\Omega_s}),$$

which leads to

$$\|\boldsymbol{v}\|_{\operatorname{div},\Omega_f} \le C(\|\operatorname{div}\boldsymbol{v}\|_{0,\Omega_f} + \|\boldsymbol{\tau}\|_{1,\Omega_s}),$$

and applying Young's inequality we obtain $\|\boldsymbol{v}\|^2_{\operatorname{div},\Omega_f} \le C\|(\boldsymbol{v},\boldsymbol{\tau})\|^2_{\mathcal{X}}$. For the second term, we apply Korn's inequality and obtain $\int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{\tau}) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}) \ge C\|\boldsymbol{\tau}\|_{1,\Omega_s}$. Hence, combining the two last estimates we deduce that $a((\boldsymbol{v},\boldsymbol{\tau}),(\boldsymbol{v},\boldsymbol{\tau})) \ge C\|(\boldsymbol{v},\boldsymbol{\tau})\|^2_{\mathcal{X}}$, concluding the $\mathcal{G}_{\mathcal{V}}$-ellipticity of $a(\cdot,\cdot)$.

The $\widetilde{\mathcal{G}}$-ellipticity of $\widehat{a}(\cdot,\cdot)$ is a direct consequence of the $\mathcal{G}_{\mathcal{V}}$-ellipticity of $a(\cdot,\cdot)$. In fact, for $((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))\in\widetilde{\mathcal{G}}$ we write

$$\begin{aligned}\widehat{a}((\boldsymbol{u},\boldsymbol{w},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}),(\boldsymbol{u},\boldsymbol{w},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})) &= a((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{w})) + \rho_s\|\widehat{\boldsymbol{v}}\|^2_{0,\Omega_f} + \rho_f\|\widehat{\boldsymbol{\tau}}\|^2_{0,\Omega_s}\\ &\ge C\|(\boldsymbol{u},\boldsymbol{w})\|^2_{\mathcal{X}} + \rho_s\|\widehat{\boldsymbol{v}}\|^2_{0,\Omega_f} + \rho_f\|\widehat{\boldsymbol{\tau}}\|^2_{0,\Omega_s}\\ &\ge \underbrace{\min\{C,\rho_s,\rho_f\}}_{\alpha}(\|(\boldsymbol{u},\boldsymbol{w})\|^2_{\mathcal{X}} + \|\widehat{\boldsymbol{v}}\|^2_{0,\Omega_f} + \|\widehat{\boldsymbol{\tau}}\|^2_{0,\Omega_s}).\end{aligned}$$

Hence

$$\widehat{a}(((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))) \geq \alpha\|((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))\|^2_{\mathcal{X}\times\mathcal{H}},$$

which completes the proof. □

Now we introduce the following linear and bounded operator $\boldsymbol{T} : (\mathcal{V}\times\mathcal{H}) \to (\mathcal{V}\times\mathcal{H})$ defined by $\boldsymbol{T}((\boldsymbol{f},\boldsymbol{g}),(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}})) = ((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))$, where $((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))$ is the unique solution of the following problem:

$$\widehat{a}(((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))) = \widehat{b}((\boldsymbol{f},\boldsymbol{g},\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}}),(\boldsymbol{v},\boldsymbol{\tau},\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \quad \forall((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \in \widetilde{\mathcal{G}}$$

From this we observe that

$$a((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{\tau})) + \int_{\Omega_f}\rho_f\widehat{\boldsymbol{u}}\cdot\overline{\widehat{\boldsymbol{v}}} + \int_{\Omega_s}\rho_s\widehat{\boldsymbol{w}}\cdot\overline{\widehat{\boldsymbol{\tau}}} = -2\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}} - \int_{\Omega_f}\rho_f\widehat{\boldsymbol{f}}\cdot\overline{\boldsymbol{v}}$$
$$- \int_{\Omega_s}\rho_s\widehat{\boldsymbol{g}}\cdot\overline{\boldsymbol{\tau}} + \int_{\Omega_f}\rho_f\boldsymbol{f}\cdot\overline{\widehat{\boldsymbol{v}}} + \int_{\Omega_s}\rho_s\boldsymbol{g}\cdot\overline{\widehat{\boldsymbol{\tau}}},$$

$\forall(\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{G}$ and $\forall(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}) \in \mathcal{H}$, which implies that

$$(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}) = (\boldsymbol{f},\boldsymbol{g}). \tag{5.2.18}$$

Hence, for $(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{G}_{\mathcal{V}}$ we have

$$a((\boldsymbol{u},\boldsymbol{w}),(\boldsymbol{v},\boldsymbol{\tau})) = -2\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}} - \int_{\Omega_f}\rho_f\widehat{\boldsymbol{f}}\cdot\overline{\boldsymbol{v}} - \int_{\Omega_s}\rho_s\widehat{\boldsymbol{g}}\cdot\overline{\boldsymbol{\tau}}. \tag{5.2.19}$$

The following lemma shows that the non-zero eigenvalues of $\boldsymbol{T}$ are exactly the inverses of the non-zero eigenvalues of problem (5.2.15)–(5.2.16) with the same corresponding eigenfunctions.

Lemma 5.2.4 *$(\mu,(\boldsymbol{u},\boldsymbol{w},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))$ is an eigenpair of $\boldsymbol{T}$ if and only if $\mu \neq 0$ and $(\lambda,((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})))$ is a solution of Problem 5.2.2 with $\lambda = 1/\mu$.*

Proof. First, it is easy to check that $\mu = 0$ is not an eigenvalue of $\boldsymbol{T}$. Let $(\mu,((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})))$ be an eigenpair of $\boldsymbol{T}$, with $\mu \neq 0$. Hence

$$\widehat{a}((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))) = \frac{1}{\mu}\widehat{b}((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))), \tag{5.2.20}$$

for all $((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \in \widetilde{\mathcal{G}}$. Then, according to (5.2.18) we have that $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}) = \frac{1}{\mu}(\boldsymbol{u},\boldsymbol{w}) \in \mathcal{G}$. Then, for $((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \in \widetilde{\mathcal{K}}$ and the fact that $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}) \in \mathcal{G}$ we have that $\widehat{b}((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))) = 0$ and hence

$$\widehat{a}((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})),((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))) = 0.$$

This implies that (5.2.20) holds true for all $((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \in \mathcal{V}\times\mathcal{X}$. Conversely, let $(\lambda,((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})))$ be a solution of Problem 5.2.2. Taking $(\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{K}$ in (5.2.15) and using the orthogonality between $\mathcal{K}$ and $\mathcal{G}_{\mathcal{V}}$ we obtain that

$$\int_{\Omega_f}\rho_f\widehat{\boldsymbol{u}}\cdot\overline{\boldsymbol{v}} + \int_{\Omega_s}\rho_s\widehat{\boldsymbol{w}}\cdot\overline{\boldsymbol{\tau}} = 0 \quad (\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{K},$$

which implies that $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}) \in \mathcal{G}$. On the other hand, (5.2.16) implies that $\lambda(\boldsymbol{u},\boldsymbol{w}) = (\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}) \in \mathcal{G}$. Hence, it is easy to check that $\boldsymbol{T}((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})) = \mu((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}}))$ with $\mu = 1/\lambda$. □

5.3 Spectral Characterization

In this section we will characterize the spectrum of the operator $\boldsymbol{T}$. With this aim, we will use the theory described in [55] to decompose appropriately $\boldsymbol{T}$. We will not be able to prove that $\boldsymbol{T}$ has only discrete spectrum, although we will show that the essential spectrum, if it exists, has to lie in a region of the complex plane, well separated from the isolated eigenvalues.

Let $\boldsymbol{T}_1, \boldsymbol{T}_2 : \mathcal{G}_{\mathcal{V}} \to \mathcal{G}_{\mathcal{V}}$ operators given by

$$\boldsymbol{T}_1(\boldsymbol{f},\boldsymbol{g}) = (\boldsymbol{u}_1,\boldsymbol{w}_1) \in \mathcal{G}_{\mathcal{V}} : \quad a((\boldsymbol{u}_1,\boldsymbol{w}_1),(\boldsymbol{v},\boldsymbol{\tau})) = 2\int_{\Omega_f} \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \overline{\boldsymbol{v}}, \tag{5.3.21}$$

$$\boldsymbol{T}_2(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}}) = (\boldsymbol{u}_2,\boldsymbol{w}_2) \in \mathcal{G}_{\mathcal{V}} : \quad a((\boldsymbol{u}_2,\boldsymbol{w}_2),(\boldsymbol{v},\boldsymbol{\tau})) = \int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}} \cdot \overline{\boldsymbol{v}} + \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}} \cdot \overline{\boldsymbol{\tau}}. \tag{5.3.22}$$

Both operators are well defined because of Lemma 5.2.3 and Lax–Milgram's lemma. Moreover $\|(\boldsymbol{u}_1,\boldsymbol{w}_1)\|_{\mathcal{X}} \le C\|\operatorname{div}\boldsymbol{f}\|_{0,\Omega_f}$ and $\|(\boldsymbol{u}_2,\boldsymbol{w}_2)\|_{\mathcal{X}} \le C(\|\widehat{\boldsymbol{f}}\|_{0,\Omega_f} + \|\widehat{\boldsymbol{g}}\|_{0,\Omega_s})$. Notice that $\boldsymbol{T}_2$ is the solution operator corresponding to the non-dissipative eigenvalue problem (5.2.12).

The following lemma proves additional regularity for the eigenfunctions of $\boldsymbol{T}_2$.

Lemma 5.3.1 *Given $(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}}) \in \mathcal{G}_{\mathcal{V}}$, let $(\boldsymbol{u}_2,\boldsymbol{w}_2) = \boldsymbol{T}_2(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}})$ a solution of problem* (5.3.22). *Then, there exists $s \in (0,\frac{1}{2}]$ and $\beta \in (0,1]$ such that*

$$\|\boldsymbol{u}_2\|_{s,\Omega_f} + \|\operatorname{div}\boldsymbol{u}_2\|_{1,\Omega} + \|\boldsymbol{w}_2\|_{1+\beta,\Omega_s} \le C(\|\widehat{\boldsymbol{f}}\|_{0,\Omega_f} + \|\widehat{\boldsymbol{g}}\|_{0,\Omega_s}).$$

Proof. Since $(\boldsymbol{u}_2,\boldsymbol{w}_2) \in \mathcal{G}_{\mathcal{V}}$, applying Lemma 5.2.2 we have that $\boldsymbol{u}_2 \in \mathrm{H}^s(\Omega_f)^2$ and $\|\boldsymbol{u}_2\|_{s,\Omega_f} \le C(\|\operatorname{div}\boldsymbol{u}_2\|_{0,\Omega_f} + \|\boldsymbol{w}_2\|_{1,\Omega_s})$. Hence, from Lax-Milgram's lemma we obtain$\|\boldsymbol{u}_2\|_{s,\Omega_f} \le C(\|\widehat{\boldsymbol{f}}\|_{0,\Omega_f} + \|\widehat{\boldsymbol{g}}\|_{0,\Omega_s})$.

On the other hand, testing (5.3.22) with $\boldsymbol{v} \in \mathcal{D}(\Omega_f)$ and $\boldsymbol{\tau} = 0$ we obtain that $-\nabla(\rho_f c^2 \operatorname{div}\boldsymbol{u}_2) = \rho_f \boldsymbol{f} \in \mathrm{L}^2(\Omega_f)^2$ which implies that $\operatorname{div}\boldsymbol{u}_2 \in \mathrm{H}^1(\Omega_f)^2$ and $\|\operatorname{div}\boldsymbol{u}_2\|_{1,\Omega_f} \le C\|\widehat{\boldsymbol{f}}\|_{0,\Omega}$.

For the additional regularity for function $\boldsymbol{w}$, we consider the following elasticity problem as the one considered in [12]:

$$\begin{aligned} -L(\boldsymbol{w}) + \rho_s \boldsymbol{w} = \rho_s \widehat{\boldsymbol{g}} &\quad \text{in } \Omega_s, \\ \boldsymbol{\sigma}(\boldsymbol{w}_2)\boldsymbol{n} = (\rho c^2 \operatorname{div}\boldsymbol{u}_2)\boldsymbol{n} &\quad \text{on } \Gamma, \\ \boldsymbol{\sigma}(\boldsymbol{w}_2)\eta = 0 &\quad \text{on } \Gamma_N, \\ \boldsymbol{w}_2 = 0 &\quad \text{on } \Gamma_D, \end{aligned}$$

with $L(\boldsymbol{w}_2)$ is the elasticity operator defined by

$$L(\boldsymbol{w}_2) := (\lambda_s + \mu_s)\nabla(\operatorname{div}\boldsymbol{w}_2) + \mu_s \Delta \boldsymbol{w}_2,$$

where λ_s and μ_s are the Lammé constants. From the a priori estimates for this problem (see for instance [47]) we obtain that $\boldsymbol{w}_2 \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ and satisfies

$$\|\boldsymbol{w}_2\|_{1+\beta.\Omega_s} \le C(\|\widehat{\boldsymbol{g}}\|_{0,\Omega_s} + \|\operatorname{div}\boldsymbol{u}_2\|_{\frac{1}{2},\Gamma}),$$

where $\beta \in (0,1]$ and depends on the geometry of Ω_s and the Lammé coefficients. □

Now we are in position to establish the following compactness result for the operator $\boldsymbol{T}_2$.

Corollary 5.3.1 *The operator $\boldsymbol{T}_2 : \mathcal{G}_{\mathcal{V}} \to \mathcal{G}_{\mathcal{V}}$ is compact.*

Proof. The compactness of $\boldsymbol{T}_2$ follows from Lemma 5.3.1 and the compact inclusions of $\mathrm{H}^s(\Omega_f)^2$, $\mathrm{H}^1(\Omega_f)^2$ and $\mathrm{H}^\beta(\Omega_s)^2$ into $\mathrm{L}^2(\Omega_f)^2$ and $\mathrm{L}^2(\Omega_s)^2$. □

Now, using the ideas developed in [55] the operator $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2 \\ \boldsymbol{I} & 0 \end{pmatrix}. \tag{5.3.23}$$

Moreover, we define as in [55]

$$\boldsymbol{S} := \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{T}_2^{1/2} \end{pmatrix}, \quad \boldsymbol{U} := \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{I} & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{H} := \begin{pmatrix} -\boldsymbol{T}_1 & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & 0 \end{pmatrix}$$

which satisfies the following identities

$$\boldsymbol{S}\boldsymbol{T} = \boldsymbol{H}\boldsymbol{S}, \quad \boldsymbol{T} = \boldsymbol{U}\boldsymbol{S}, \quad \boldsymbol{H} = \boldsymbol{S}\boldsymbol{U}, \quad \text{and } \boldsymbol{U}\boldsymbol{H} = \boldsymbol{T}\boldsymbol{U}.$$

We note that the eigenvalues of $\boldsymbol{T}$ and $\boldsymbol{H}$ and their algebraic multiplicities coincide and the corresponding Jordan chains have the same length. In fact, let $\{x_k\}_{k=1}^r$ a Jordan chain associated to the eigenvalue μ of $\boldsymbol{T}$. Then, using the identities above and the definition of a Jordan chain we obtain

$$\boldsymbol{H}\boldsymbol{S}x_k = \boldsymbol{S}\boldsymbol{T}x_k = \boldsymbol{S}(\mu x_k + x_{k-1}) = \mu \boldsymbol{S}x_k + \boldsymbol{S}x_{k-1}, \quad k = 1, \ldots, r.$$

This shows that $\{\boldsymbol{S}x_k\}_{k=1}^r$ is a Jordan chain of $\boldsymbol{H}$ of the same length. The following lemma shows that the spectra of $\boldsymbol{T}$ and $\boldsymbol{H}$ coincide, which proof can be found on Lemma 3.2 of [12].

Lemma 5.3.2 *There holds*

$$\mathrm{sp}(\boldsymbol{T}) = \mathrm{sp}(\boldsymbol{H}).$$

Proof. The proof runs identically as in Lemma 3.2 of [12]. □

The operator $\boldsymbol{H}$ can be written as the sum of a self-adjoint operator $\boldsymbol{B}$ and a compact one $\boldsymbol{C}$. Then, using the identities in [55]:

$$\boldsymbol{H} = \boldsymbol{B} + \boldsymbol{C}, \quad \text{with } \boldsymbol{B} := \begin{pmatrix} -\boldsymbol{T}_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \boldsymbol{C} := \begin{pmatrix} 0 & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & 0 \end{pmatrix}$$

Then, applying classical Weyl's Theorem (see [83]) we have that $\mathrm{Sp}_{\mathrm{e}}(\boldsymbol{H}) = \mathrm{Sp}_{\mathrm{e}}(\boldsymbol{B})$ and the rest of the spectrum $\mathrm{Sp}_{\mathrm{d}}(\boldsymbol{H}) = \mathrm{sp}(\boldsymbol{H}) \setminus \mathrm{Sp}_{\mathrm{e}}(\boldsymbol{H})$ consists of isolated eigenvalues with finite algebraic multiplicity. Moreover, $\mathrm{Sp}_{\mathrm{e}}(\boldsymbol{B}) = \mathrm{Sp}_{\mathrm{e}}(-\boldsymbol{T}_1) \cup \{0\}$.

Our next goal is to show that the essential spectrum of $\boldsymbol{T}_1$ lies in a region of the complex plane. With this aim, we need to determine the values of μ for which the operator $(\mu \boldsymbol{I} - \boldsymbol{T}_1) : \mathcal{G}_{\mathcal{V}} \to \mathcal{G}_{\mathcal{V}}$ is not invertible. Then, we proceed using the technics used in [62] in order to determine this.

- $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ *is not one to one.* If $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ is not one-to-one, then there exists $(\boldsymbol{f},\boldsymbol{g}) \neq (\boldsymbol{0},\boldsymbol{0})$ such that $(\mu\boldsymbol{I}-\boldsymbol{T}_1)(\boldsymbol{f},\boldsymbol{g}) = 0$, which is equivalent to

$$\mu\left(\int_{\Omega_f}\rho_f c^2 \operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}} + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{g}):\varepsilon(\overline{\boldsymbol{\tau}})\right) = 2\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}}$$

In particular, for $\boldsymbol{v}=\boldsymbol{f}$ and $\boldsymbol{\tau}=\boldsymbol{g}$ we obtain

$$\mu = \frac{2\displaystyle\int_{\Omega_f}\nu|\operatorname{div}\boldsymbol{f}|^2}{\displaystyle\int_{\Omega_f}\rho_f c^2|\operatorname{div}\boldsymbol{f}|^2 + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{g}):\varepsilon(\boldsymbol{g})} \leq \frac{2\displaystyle\int_{\Omega_f}\nu|\operatorname{div}\boldsymbol{f}|^2}{\displaystyle\int_{\Omega_f}\rho_f c^2|\operatorname{div}\boldsymbol{f}|^2} = \frac{2\nu}{\rho_f c^2} \in \mathbb{R}$$

From here we observe that $\mu \geq 0$. Moreover, $\mu \leq \dfrac{2\nu}{\rho_f c^2}$.

- On the other hand, $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ is onto if and only if for $(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}}) \in \mathcal{G}_{\mathcal{V}}$ there exists $(\boldsymbol{f},\boldsymbol{g}) \in \mathcal{G}_{\mathcal{V}}$ such that $\boldsymbol{T}_1(\boldsymbol{f},\boldsymbol{g}) = \mu(\boldsymbol{f},\boldsymbol{g}) - (\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}})$. Hence,

$$\int_{\Omega_f}\rho_f c^2 \operatorname{div}(\mu\boldsymbol{f}-\widehat{\boldsymbol{f}})\operatorname{div}\overline{\boldsymbol{v}} + \int_{\Omega_s}\boldsymbol{\sigma}(\mu\boldsymbol{g}-\widehat{\boldsymbol{g}}):\varepsilon(\overline{\boldsymbol{\tau}}) = 2\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}}$$

which leads to

$$\begin{aligned}\int_{\Omega_f}(\mu\rho_f c^2 - 2\nu)\operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}} + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{g}):\varepsilon(\overline{\boldsymbol{\tau}}) = &\int_{\Omega_f}\rho c^2\operatorname{div}\widehat{\boldsymbol{f}}\operatorname{div}\overline{\boldsymbol{v}} \\ &+ \int_{\Omega_s}\boldsymbol{\sigma}(\widehat{\boldsymbol{g}}):\varepsilon(\overline{\boldsymbol{\tau}})\end{aligned} \tag{5.3.24}$$

for all $(\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{G}_{\mathcal{V}}$. By writing $\mu = a+bi$ with $a,b\in\mathbb{R}$, the equation above reads:

$$\begin{aligned}\int_{\Omega_f}((a+bi)\rho_f c^2 - 2\nu)\operatorname{div}\boldsymbol{f}\operatorname{div}\overline{\boldsymbol{v}} + (a+bi)\int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{g}):\varepsilon(\overline{\boldsymbol{\tau}}) = &\int_{\Omega_f}\rho c^2\operatorname{div}\widehat{\boldsymbol{f}}\operatorname{div}\overline{\boldsymbol{v}} \\ &+ \int_{\Omega_s}\boldsymbol{\sigma}(\widehat{\boldsymbol{g}}):\varepsilon(\overline{\boldsymbol{\tau}})\end{aligned}$$

for all $(\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{G}_{\mathcal{V}}$. From here we observe that for $b\neq 0$ problem above has a solution and hence the operator $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ is onto. On the other hand, if $b=0$ we conclude that $\mu=a$ and hence $\mu\in\mathbb{R}$. Now we have the following cases:

(i) If $\mu>0$ and $(\rho_f c^2\mu - 2\nu)\geq 0$ where $\mu \geq \dfrac{2\nu}{\rho_f c^2}$ implies that $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ is onto.

(ii) If $\mu<0$ and $(\rho_f c^2\mu - 2\nu)\geq 0$ where $\mu \leq \dfrac{2\nu}{\rho_f c^2}$ implies that $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ is onto.

Therefore, if $(\mu\boldsymbol{I}-\boldsymbol{T}_1)$ is not onto, then $\mu \in \left[0, \dfrac{2\nu}{\rho_f c^2}\right[$.

The previous calculations are summarized in the following result which gives a spectral characterization of the solution operator $\boldsymbol{T}$.

Theorem 5.3.1 *The spectrum of* $\boldsymbol{T}$ *consists of*

$$\mathrm{Sp_e}(\boldsymbol{T}) = \mathrm{sp}(-\boldsymbol{T}_1) \cup \{0\}$$

with

$$\mathrm{sp}(\boldsymbol{T}_1) \subset \left[0, \frac{2\nu}{\rho_f c^2}\right[,$$

and $\mathrm{Sp_d}(\boldsymbol{T}) = \mathrm{sp}(\boldsymbol{T}) \diagdown \mathrm{Sp_e}(\boldsymbol{T})$ *consists in a set of isolated eigenvalues of finite algebraic multiplicity.*

Proof. As a consequence of the classical Weyl's Theorem (see [83]) and Lemma 5.3.2,

$$\mathrm{Sp_e}(\boldsymbol{T}) = \mathrm{Sp_e}(\boldsymbol{H}) = \mathrm{Sp_e}(\boldsymbol{B}) = \mathrm{Sp_e}(-\boldsymbol{T}_1) \cup \{0\},$$

whereas the inclusion follows from the above analysis. □

Our next goal is to show that for ν small enough, some of the eigenvalues of $\boldsymbol{T}$ are well separated from the essential spectrum. With this end, given $\boldsymbol{f} \in \mathcal{G}_{\mathcal{V}}$ and testing (5.3.21) with $(\boldsymbol{u}_1, \boldsymbol{w}_1) \in \mathcal{G}_{\mathcal{V}}$ and using the $\mathcal{G}_{\mathcal{V}}$-ellipticity of $a(\cdot,\cdot)$, we have that

$$\begin{aligned}\alpha \|(\boldsymbol{u}_1, \boldsymbol{w}_1)\|_{\mathcal{X}}^2 \leq & a((\boldsymbol{u}_1, \boldsymbol{w}_1), (\boldsymbol{u}_1, \boldsymbol{w}_1)) \\ = & 2 \int_\Omega \nu \operatorname{div} \boldsymbol{f} \operatorname{div} \boldsymbol{u}_1 \leq 2\nu \| \operatorname{div} \boldsymbol{f} \|_{0,\Omega} \|\boldsymbol{u}_1\|_{\operatorname{div},\Omega},\end{aligned}$$

which implies that $\|\boldsymbol{T}_1\|_{\mathcal{L}(\mathcal{G}_{\mathcal{V}} \times \mathcal{G}_{\mathcal{V}})} \to 0$ as $\nu \to 0$. Consequently, $\boldsymbol{H}$ converges in norm to the operator

$$\boldsymbol{H}_0 := \begin{pmatrix} 0 & -\boldsymbol{T}_2^{1/2} \\ \boldsymbol{T}_2^{1/2} & 0 \end{pmatrix},$$

as ν goes to zero. therefore, form the classical spectral approximation theory, the isolated eigenvalues of $\boldsymbol{H}$ converge to those of $\boldsymbol{H}_0$. Since the isolates eigenvalues of $\boldsymbol{H}$ and $\boldsymbol{T}$ coincide (cf. Lemma 5.3.2), to localize those of $\boldsymbol{T}$, it is enough to characterize those of $\boldsymbol{H}_0$. Let μ be an isolated eigenvalue of $\boldsymbol{H}_0$ and $((\boldsymbol{u}, \boldsymbol{w}), (\boldsymbol{v}, \boldsymbol{\tau})) \in \mathcal{G}_{\mathcal{V}} \times \mathcal{G}_{\mathcal{V}}$ the corresponding eigenfunction. It is easy to check that

$$\boldsymbol{H}_0 \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \\ \boldsymbol{v} \\ \boldsymbol{\tau} \end{pmatrix} = \mu \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \\ \boldsymbol{v} \\ \boldsymbol{\tau} \end{pmatrix} \Longleftrightarrow \boldsymbol{T}_2(\boldsymbol{u}, \boldsymbol{w}) = -\mu^2 (\boldsymbol{u}, \boldsymbol{w}).$$

Since $\boldsymbol{T}_2$ is the solution operator corresponding to problem (5.2.12), we have that its eigenvalues are related with those of (5.2.12) by $-\mu^2 = \omega^2$, or equivalently , $\mu = \pm i/\omega$. Moreover, because of the symmetry and ellipticity of the bilinear forms in (5.2.12), we have that $\omega > 0$.

Finally, from the compactness of $\boldsymbol{T}_2$, we know that the spectrum of $\boldsymbol{T}_2$ consists of a sequence of finite multiplicity eigenvalues $\mu_k = \pm i/\omega_k$ which converges to zero. Therefore we have that the eigenvalues of $\boldsymbol{H}_0$ are also given by $\pm i/\omega_k$ with ω_k and $\omega_k \to \infty$, which let to conclude that this eigenvalues are purely imaginary and converge to zero. Now we are in position to establish the following result.

Theorem 5.3.2 *For each eigenvalue i/ω_k of $\boldsymbol{T}_2$ of multiplicity m, there exists $r > 0$ such that the disc $D_r := \{z \in \mathbb{C} : \ |z - i/\omega_k| < r\}$, intersects $\mathrm{sp}(\boldsymbol{T}_2)$ only in i/ω_k. Then, there exists $\delta > 0$ such that if $\nu < \delta$, there exists m eigenvalues of $\boldsymbol{T}$, $\mu_1, \dots, \mu_m$ lying in the disc D_r. Moreover, $\mu_1, \dots, \mu_m \to \frac{i}{\omega_k}$ as ν goes to zero.*

The previous theorem shows that the eigenvalues of $\boldsymbol{T}$, which are relevant in the applications, are well separated from the real axis and hence, from the essential spectrum of $\boldsymbol{T}$.

5.4 Spectral Approximation

In the following section we will study the finite element method to approximate the solutions of problem (5.2.11). With this aim, we introduce a triangulation of the domains $\{\mathcal{T}_h\}_{h>0} \subset \Omega_f \cup \Omega_s$ and the following finite element spaces. First, for the solid we introduce the piecewise linear space

$$\mathcal{M}_h := \{\boldsymbol{v} \in \mathrm{H}^1(\Omega_s)^n \ : \ \boldsymbol{v}_h|_T \in \mathcal{P}_1(T)^n, \ T \subset \mathcal{T}(\Omega_s)\}$$

and for the fluid, the lowest order Raviart–Thomas space

$$\mathcal{W}_h := \{\boldsymbol{\tau} \in \mathrm{H}(\mathrm{div}, \Omega_f) \ : \ \boldsymbol{\tau}_h|_T \in \mathrm{RT}_0(T), \ T \subset \mathcal{T}(\Omega_f)\},$$

and the following piecewise constant space

$$\mathcal{U}_h := \{v_h \in \mathrm{L}^2(\Omega_f)^2 \ : \ v_h|_T \in \mathcal{P}_0(T)^2 \quad \forall T \subset \mathcal{T}(\Omega_f)\}.$$

The degrees of freedom in $\mathcal{W}_h$ are the values of the normal components of $\boldsymbol{\tau}$ along each edge of the triangulation. We observe that these normal components values are constant in each edge. Then, the discrete space of $\mathcal{X}$ is

$$\mathcal{X}_h := \{(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{M}_h \times \mathcal{W}_h \ : \ \boldsymbol{\tau}|_{\Gamma_D} = 0\}.$$

Then, as in [12], we need to impose a weaker condition than (5.2.5) for the discrete space. Then, we introduce the following finite element space

$$\mathcal{V}_h := \left\{(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{X}_h \quad : \quad \int_\ell (\boldsymbol{v} - \boldsymbol{\tau}) \cdot \boldsymbol{n} = 0 \ \ \forall \ell \subset \Gamma \right\}.$$

Notice that if $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in \mathcal{V}_h$ then $\boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{w} \cdot \boldsymbol{n}$ coincide at the midpoint of each edge in the interface. In general, our method is non-conforming since $\mathcal{V}_h \not\subseteq \mathcal{V}$.

We recall some well-known approximation for this finite element spaces. Given $s > 0$, let $\mathbf{\Pi}_h : \mathrm{H}^s(\mathrm{div};\Omega_f) \to \mathcal{W}_h$ the classic global lowest order Raviart-Thomas interpolant. Moreover, for $\boldsymbol{v} \in \mathrm{H}^s(\mathrm{div};\Omega_f)$ we have the following well known result

$$\|\boldsymbol{v} - \mathbf{\Pi}_h\boldsymbol{v}\|_{0,\Omega_f} \leq Ch^s(\|\boldsymbol{v}\|_{s,\Omega_f} + \|\operatorname{div}\boldsymbol{v}\|_{0,\Omega_f}) \tag{5.4.25}$$

and the following commuting diagram property holds true

$$\operatorname{div}(\mathbf{\Pi}_h\boldsymbol{v}) = \boldsymbol{\mathcal{P}}_h(\operatorname{div}\boldsymbol{v}) \quad \forall\boldsymbol{v} \in \mathrm{H}^s(\Omega_f) \cap \mathrm{H}(\mathrm{div};\Omega_f). \tag{5.4.26}$$

where $\mathcal{P}_h : \mathrm{L}^2(\Omega_f)^2 \to \mathcal{U}_h := \{v_h \in \mathrm{L}^2(\Omega_f) : \in \mathcal{P}_0(T)\ \forall T \in \mathcal{T}_h\}$ is the $\mathrm{L}^2(\Omega_f)$-orthogonal projection. Then, for any $r \in (0,1]$ we have

$$\|q - \mathcal{P}_hq\|_{0,\Omega_f} \leq Ch^r\|q\|_{r,\Omega_f} \quad \forall q \in \mathrm{H}^r(\Omega_f). \tag{5.4.27}$$

In the part of the solid we introduce the classical Lagrange interpolant $\mathcal{L}_h : \mathrm{H}^1(\Omega_s)^2 \to \mathcal{M}_h$. Then, for any $s \in (0,1]$ and $\boldsymbol{\tau} \in \mathrm{H}^{1+s}(\Omega_s)^2$ we have

$$\|\boldsymbol{\tau} - \mathcal{L}_h\boldsymbol{\tau}\|_{1,\Omega_s} \leq Ch^s\|\boldsymbol{\tau}\|_{1+s,\Omega_s}. \tag{5.4.28}$$

The discrete version of Problem 5.2.1 reads as follows.

Problem 5.4.1 *Find $\lambda_h \in \mathbb{C}$ and $(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{u}_h,\boldsymbol{w}_h) \in \mathcal{V}_h$ such that*

$$\lambda_h^2\left(\int_{\Omega_f}\rho_f\boldsymbol{u}_h\cdot\overline{\boldsymbol{v}}_h + \int_{\Omega_s}\rho_s\boldsymbol{w}_h\cdot\overline{\boldsymbol{\tau}}_h\right) + 2\lambda_h\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{u}_h\operatorname{div}\overline{\boldsymbol{v}}_h + \int_{\Omega_f}\rho_fc^2\operatorname{div}\boldsymbol{u}_h\operatorname{div}\overline{\boldsymbol{v}}_h$$
$$+ \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{w}_h) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}_h) = 0 \qquad \forall(\boldsymbol{v},\boldsymbol{\tau}) \in \mathcal{V}_h. \tag{5.4.29}$$

Proceeding as we did in the continuous case, we introduce the following discrete auxiliary unknowns $\widehat{\boldsymbol{u}}_h = \lambda\boldsymbol{u}_h$ and $\widehat{\boldsymbol{w}}_h = \lambda_h\boldsymbol{w}_h$ and we obtain the following linear discrete eigenvalue problem:

Problem 5.4.2 *Find $\lambda_h \in \mathbb{C}$ and $((\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{0})) \neq ((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h)) \in \mathcal{V}_h \times \mathcal{V}_h$ such that*

$$\int_{\Omega_f}\rho c^2\operatorname{div}\boldsymbol{u}_h\operatorname{div}\overline{\boldsymbol{v}}_h + \int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{w}_h) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}_h) = \lambda_h\left(-2\int_{\Omega_f}\nu\operatorname{div}\boldsymbol{u}_h\operatorname{div}\overline{\boldsymbol{v}}_h - \int_{\Omega_f}\rho_f\widehat{\boldsymbol{u}}_h\cdot\overline{\boldsymbol{v}}_h\right.$$
$$\left. - \int_{\Omega_s}\rho_s\widehat{\boldsymbol{w}}_h\cdot\overline{\boldsymbol{\tau}}_h\right) \qquad \forall(\boldsymbol{v}_h,\boldsymbol{\tau}_h) \in \mathcal{V}_h, \tag{5.4.30}$$

$$\int_{\Omega_f}\rho_f\widehat{\boldsymbol{u}}_h\cdot\overline{\widehat{\boldsymbol{v}}}_h + \int_{\Omega_f}\rho_s\widehat{\boldsymbol{w}}_h\cdot\overline{\widehat{\boldsymbol{\tau}}}_h = \lambda_h\left(\int_{\Omega_f}\rho_f\boldsymbol{u}_h\cdot\overline{\widehat{\boldsymbol{v}}}_h + \int_{\Omega_s}\rho_s\boldsymbol{w}_h\cdot\overline{\widehat{\boldsymbol{\tau}}}_h\right) \qquad \forall(\widehat{\boldsymbol{v}}_h,\widehat{\boldsymbol{\tau}}_h) \in \mathcal{V}_h. \tag{5.4.31}$$

We observe that $\lambda_h = 0$ is an eigenvalue of the problem above with associated eigenspace $\widetilde{\mathcal{K}}_h := \mathcal{K}_h \times \{0\}$ where $\mathcal{K}_h := \mathcal{K} \cap \mathcal{V}_h$. Moreover, let $\mathcal{G}_h$ be the orthogonal complement of $\mathcal{K}_h$ and $\widetilde{\mathcal{G}}_h := \mathcal{G}_h \times \mathcal{G}_h \subset \mathcal{V}_h$. Note $\mathcal{G}_h \not\subseteq \mathcal{G}$ and hence $\widetilde{\mathcal{G}}_h \not\subseteq \widetilde{\mathcal{G}}$.

In order to prove the ellipticity of the sesquilinear forms $a(\cdot,\cdot)$ and $\widehat{a}(\cdot,\cdot)$ in the spaces $\mathcal{G}_h$ and $\widetilde{\mathcal{G}}_h$ respectively, we require a Helmoltz decomposition for discrete elements. With this aim we prove the following result.

Lemma 5.4.1 *For* $(\boldsymbol{u}_h, \boldsymbol{w}_h) \in \mathcal{G}_h$ *we have the following decomposition* $(\boldsymbol{u}_h, \boldsymbol{w}_h) = (\nabla\xi, \boldsymbol{w}_h) + (\chi, \boldsymbol{0})$ *with* $(\nabla\xi, \boldsymbol{w}_h) \in \mathcal{G}$ *and* $\chi \in \mathcal{K}$ *which satisfies*

$$\|\nabla\xi\|_{s,\Omega} \leq C(\|\operatorname{div}\boldsymbol{u}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s})$$

and

$$\|\chi\|_{0,\Omega} \leq C h^s(\|\operatorname{div}\boldsymbol{u}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s}).$$

Proof. Consider the following Neumann problem

$$\begin{aligned} \Delta\xi &= \operatorname{div}\boldsymbol{u}_h \quad \text{in}\,\Omega_f \\ \frac{\partial\xi}{\partial n} &= \boldsymbol{w}_h\cdot\boldsymbol{n} \quad \text{on}\,\Gamma. \end{aligned}$$

Clearly from Lax-Milgram's lemma the problem above is well posed, i.e. there exists unique $\xi \in \mathrm{H}^1(\Omega_f)^2/\mathbb{C}$ solution of this problem. Moreover, due (5.2.14) and Lemma 5.2.2 we have that there exists $s \in [0,1)$ such that $\nabla\xi \in \mathrm{H}^s(\Omega_f)^2$ which satisfies

It is clear that $\nabla\xi$ is a gradient. On the other hand we can observe that $\nabla\xi = \frac{\partial\xi}{\partial\boldsymbol{n}} = \boldsymbol{w}_h\cdot\boldsymbol{n}$, which implies that $(\nabla\xi, \boldsymbol{w}_h) \in \mathcal{G}$.

We observe that $\chi = \boldsymbol{u}_h - \nabla\xi$. It is immediate from the Neumann problem that $\operatorname{div}\chi = 0$ and $\chi\cdot\boldsymbol{n} = 0$. Hence $\chi \in \mathcal{K}$.

Since $\nabla\xi \in \mathrm{H}^s(\Omega_f)^2$, it is clear that $\boldsymbol{\Pi}_h(\nabla\xi)$ is well defined. Then

$$\begin{aligned} \|\chi\|^2_{0,\Omega_f} = \int_{\Omega_f}\chi\cdot\chi = \int_{\Omega_f}\chi\cdot(\boldsymbol{u}_h - \nabla\xi) \\ = \underbrace{\int_{\Omega_f}\chi\cdot(\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi))}_{(I)} + \underbrace{\int_{\Omega_f}\chi\cdot(\Pi_h(\nabla\xi) - \nabla\xi)}_{(II)}. \end{aligned}$$

For (I) we observe that $\operatorname{div}(\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi)) = \operatorname{div}\boldsymbol{u}_h - \mathcal{P}_h(\Delta\xi) = 0$, where we have used (5.4.26). Therefore $\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi) \in \mathcal{K}_h \subset \mathcal{K}$. Since $(\nabla\xi, 0) \in \mathcal{G}$ and $(\boldsymbol{u}_h, 0) \in \mathcal{G}_h$ we obtain

$$\begin{aligned} \int_{\Omega_f}\chi\cdot(\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi)) &= \int_{\Omega_f}(\boldsymbol{u}_h - \nabla\xi)\cdot(\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi)) \\ &= \int_{\Omega_f}\boldsymbol{u}_h\cdot(\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi)) - \int_{\Omega_f}\nabla\xi\cdot(\boldsymbol{u}_h - \boldsymbol{\Pi}_h(\nabla\xi)) = 0. \end{aligned}$$

On the other hand, applying the Cauchy-Schwarz inequality in (II) we obtain

$$\begin{aligned}\int_{\Omega_f} \chi \cdot (\mathbf{\Pi}_h(\nabla\xi) - \nabla\xi) \leq &\|\chi\|_{0,\Omega_f} \|\mathbf{\Pi}_h(\nabla\xi) - \nabla\xi)\|_{0,\Omega_f} \\ \leq &\|\chi\|_{0,\Omega_f} \|\mathbf{\Pi}_h(\nabla\xi) - \nabla\xi)\|_{\mathrm{div},\Omega_f} \\ \leq &Ch^s \|\chi\|_{0,\Omega_f} (\|\nabla\xi\|_{s,\Omega_f} + \|\Delta\xi\|_{0,\Omega_f}) \\ \leq &Ch^s \|\chi\|_{0,\Omega_f} (2\|\operatorname{div} \boldsymbol{u}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s}).\end{aligned}$$

Finally we obtain $\|\chi\|_{0,\Omega_f} \leq Ch^s(\|\operatorname{div} \boldsymbol{u}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s})$, concluding the proof. □

Now we are in position to prove that the sesquilinear forms $a(\cdot,\cdot)$ and $\widehat{a}(\cdot,\cdot)$ are elliptic in $\mathcal{G}_h$ and $\widetilde{\mathcal{G}}_h$ respectively.

Lemma 5.4.2 *The sesquilinear form $a : \mathcal{G}_h \times \mathcal{G}_h \to \mathbb{C}$ is $\mathcal{G}_h$-elliptic, with ellipticity constant not depending on h and consequently, $\widehat{a} : \widetilde{\mathcal{G}}_h \times \widetilde{\mathcal{G}}_h \to \mathbb{C}$ is $\widetilde{\mathcal{G}}$-elliptic uniformly on h.*

Proof. Let $(\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h$. Then

$$a((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) = \int_{\Omega_f} \rho_f c^2 |\operatorname{div} \boldsymbol{v}_h|^2 + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{\tau}_h) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}_h).$$

Since $\boldsymbol{\tau}_h|_\Gamma = 0$, applying Korn's inequality in the part of the solid we have

$$a((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) \geq \min\{\rho_f c^2, C_K\}(\|\operatorname{div} \boldsymbol{v}_h\|^2_{0,\Omega_f} + \|\boldsymbol{\tau}_h\|^2_{1,\Omega_s}). \tag{5.4.32}$$

From Lemma 5.4.1 we write $\boldsymbol{v}_h = \nabla\xi + \chi$, where $\|\boldsymbol{v}_h\|^2_{0,\Omega_f} \leq \|\nabla\xi\|^2_{0,\Omega_f} + \|\chi\|^2_{0,\Omega_f}$.Thus, using Lemma 5.4.1 once again we obtain

$$\begin{aligned}\|\boldsymbol{v}_h\|_{0.\Omega_f} \leq &C(\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s}) + Ch^s(\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s}), \\ \leq &C(\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s}).\end{aligned}$$

Adding $\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega_f}$ in both sides of the last inequality we obtain $\|\boldsymbol{v}_h\|_{\mathrm{div},\Omega_f} \leq C(\|\operatorname{div} \boldsymbol{v}_h\|_{0,\Omega_f} + \|\boldsymbol{w}_h\|_{1,\Omega_s}$, and replacing this on (5.4.32) we have

$$a((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) \geq C(\|\boldsymbol{v}_h\|^2_{\mathrm{div},\Omega_f} + \|\boldsymbol{\tau}_h\|^2_{1,\Omega_s}) = \|(\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|^2_{\mathcal{X}},$$

concluding the $\mathcal{G}_\mathcal{V}$-ellipticity of $a(\cdot,\cdot)$. On the other hand, the $\widetilde{\mathcal{G}}$-ellipticity of $\widehat{a}(\cdot,\cdot)$ is a direct consequence of the $\mathcal{G}_\mathcal{V}$-ellipticity of $a(\cdot,\cdot)$. Indeed: let $((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\tau}}_h)) \in \widetilde{\mathcal{G}}_h$. Then

$$\begin{aligned}\widehat{a}((\boldsymbol{v}_h, \boldsymbol{\tau}_h, \widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\tau}}_h), (\boldsymbol{v}_h, \boldsymbol{\tau}_h, \widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\tau}}_h)) &= a((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) + \int_{\Omega_f} \rho_f |\widehat{\boldsymbol{v}}_h|^2 + \int_{\Omega_s} \rho_s |\widehat{\boldsymbol{\tau}}_h|^2 \\ &\geq C\|(\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|^2_{\mathcal{X}} + \rho_f \|\widehat{\boldsymbol{v}}_h\|^2_{0,\Omega_f} + \rho_s \|\widehat{\boldsymbol{\tau}}_h\|^2_{0,\Omega_s} \\ &\geq \widetilde{C}(\|(\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|^2_{\mathcal{X}} + \|\widehat{\boldsymbol{v}}_h\|^2_{0,\Omega_f} + \|\widehat{\boldsymbol{\tau}}_h\|^2_{0,\Omega_s}), \\ &= \widetilde{C}\|((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\tau}}_h))\|_{\mathcal{V}\times\mathcal{H}},\end{aligned}$$

where $\widetilde{C} := \min\{C, \rho_f, \rho_s\}$. Hence we conclude the proof. □

Now we introduce the discrete operator $\boldsymbol{T}_h : (\mathcal{V}\times\mathcal{H}) \to (\mathcal{V}\times\mathcal{H})$ defined by $\boldsymbol{T}_h((\boldsymbol{f},\boldsymbol{g}),(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}})) = ((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h))$, where for $((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h)) \in \widetilde{\mathcal{G}}_h$

$$\widehat{a}((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h),((\boldsymbol{v}_h,\boldsymbol{\tau}_h),(\widehat{\boldsymbol{w}}_h,\widehat{\boldsymbol{\tau}}_h)) = \widehat{b}((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h),((\boldsymbol{v}_h,\boldsymbol{\tau}_h),(\widehat{\boldsymbol{w}}_h,\widehat{\boldsymbol{\tau}}_h))$$

for all $((\boldsymbol{v}_h,\boldsymbol{\tau}_h),(\widehat{\boldsymbol{w}}_h,\widehat{\boldsymbol{\tau}}_h)) \in \widetilde{\mathcal{G}}_h$.

We define $\mathcal{P}_{\mathcal{G}_h} : \mathcal{H} \to \mathcal{H}$ the L^2-projection onto $\mathcal{G}_h$. This leads to

$$(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h) = \mathcal{P}_{\mathcal{G}_h}(\boldsymbol{f},\boldsymbol{g}), \tag{5.4.33}$$

and hence, for $(\boldsymbol{u}_h,\boldsymbol{w}_h) \in \mathcal{G}_h$

$$a((\boldsymbol{u}_h,\boldsymbol{w}_h),(\boldsymbol{v}_h,\boldsymbol{\tau}_h)) = -2\int_{\Omega_f} \nu \operatorname{div}\boldsymbol{f} \operatorname{div}\overline{\boldsymbol{v}}_h - \int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \overline{\boldsymbol{v}}_h - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}}_h \cdot \overline{\boldsymbol{\tau}}_h. \tag{5.4.34}$$

Since $\boldsymbol{T}_h(\mathcal{V}\times\mathcal{H}) \subset \widetilde{\mathcal{G}}_h$, there holds $\operatorname{sp}(\boldsymbol{T}_h) = \operatorname{sp}(\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}) \cup \{0\}$ (cf. Lemma 4.1 of [12]), is that we will restrict our attention to $\boldsymbol{T}_h|_{\widetilde{\mathcal{G}}_h}$.

The following lemma holds true

Lemma 5.4.3 *For $\mu_h \neq 0$, $(\mu_h,((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h))$ is an eigenpair of $\boldsymbol{T}_h$ if and only if $(\frac{1}{\mu_h},((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h)))$ is a solution of problem* (5.4.30)–(5.4.31).

Proof. The proof runs identically as in Lemma 5.2.4, but considering the decomposition $\mathcal{V}_h\times\mathcal{V}_h = \widetilde{\mathcal{G}} \oplus (\mathcal{K}_h \times \mathcal{K}_h)$. □

Our next goal is to prove that any isolated eigenvalue of $\boldsymbol{T}$ with algebraic multiplicity m, is approximated by exactly m eigenvalues of $\boldsymbol{T}_h$ (repeated according to their respective algebraic multiplicities) and that spurious eigenvalues do not arise.

In order to prove convergence of the discrete solution operator to the continuous one, we will apply the theory of [36]. From now and on, let $\mu \in \operatorname{Sp_d}(\boldsymbol{T})$, $\mu \neq 0$ be a fixed isolated eigenvalue of finite algebraic multiplicity m and let $\mathcal{E}$ be the invariant subspace of $\boldsymbol{T}$ associated to μ. Our analysis will be based on the following properties:

P1. $$\|\boldsymbol{T}-\boldsymbol{T}_h\|_h := \sup_{0\neq((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{f}}_h,\widehat{\boldsymbol{g}}_h))\in\widetilde{\mathcal{G}}_h} \frac{\|(\boldsymbol{T}-\boldsymbol{T}_h)((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{f}}_h,\widehat{\boldsymbol{g}}_h))\|_{\widetilde{\mathcal{V}}}}{\|(\boldsymbol{f}_h,\boldsymbol{g}_h,\widehat{\boldsymbol{f}}_h,\widehat{\boldsymbol{g}}_h)\|_{\widetilde{\mathcal{V}}}} \to 0 \text{ as } h \to 0,$$

P2. $\forall((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \in \mathcal{E}$,

$$\inf_{(\boldsymbol{v}_h,\widehat{\boldsymbol{v}}_h)\in\widetilde{\mathcal{G}}_h} \|((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) - ((\boldsymbol{v}_h,\boldsymbol{\tau}_h),(\widehat{\boldsymbol{v}}_h,\widehat{\boldsymbol{\tau}}_h))\|_{\mathcal{X}\times\mathcal{H}} \to 0$$

as $h \to 0$

Let $((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)) \in \widetilde{\mathcal{G}}_h$ and $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) := \boldsymbol{T}((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h))$. From (5.2.19), we can write $(\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{u}_1, \boldsymbol{w}_1) + (\boldsymbol{u}_2, \boldsymbol{w}_2)$, with $(\boldsymbol{u}_1, \boldsymbol{w}_1), (\boldsymbol{u}_2, \boldsymbol{w}_2) \in \mathcal{G}_{\mathcal{V}}$ satisfying

$$(\boldsymbol{u}_1, \boldsymbol{w}_1) \in \mathcal{G}_{\mathcal{V}} : \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_1 \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_1) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = -2 \int_{\Omega_f} \nu \operatorname{div} \boldsymbol{f}_h \operatorname{div} \overline{\boldsymbol{v}} \qquad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{G}_{\mathcal{V}}, \tag{5.4.35}$$

and

$$(\boldsymbol{u}_2, \boldsymbol{w}_2) \in \mathcal{G}_{\mathcal{V}} : \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_2 \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_2) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = -\int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \overline{\boldsymbol{v}} - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}}_h \cdot \overline{\boldsymbol{\tau}} \qquad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{G}_{\mathcal{V}}. \tag{5.4.36}$$

The following result states some properties of the solutions of problems (5.4.35) and (5.4.36).

Lemma 5.4.4 *For* $((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)) \in \widetilde{\mathcal{G}}_h$ *let* $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) = \boldsymbol{T}((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h))$ *and consider the decomposition* $(\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{u}_1, \boldsymbol{w}_1) + (\boldsymbol{u}_2, \boldsymbol{w}_2)$*. Then* $\operatorname{div} \boldsymbol{u}_1 \in \mathcal{U}_h$ *and the following estimates holds*

$$\|\boldsymbol{u}_1\|_{s,\Omega_f} + \|\boldsymbol{w}_1\|_{1+\beta,\Omega_s} \leq C\|\boldsymbol{f}_h\|_{div,\Omega}, \tag{5.4.37}$$

Proof. First we prove (5.4.37). Since $(\boldsymbol{u}_1, \boldsymbol{w}_1) \in \mathcal{G}_{\mathcal{V}}$ is the unique solution of (5.4.35), by Lax–Milgram's lemma $\|\boldsymbol{u}_1\|_{\mathrm{div},\Omega_f} + \|\boldsymbol{w}_1\|_{1,\Omega_s} \leq C(\|\operatorname{div} \boldsymbol{f}_h\|_{0,\Omega})$. Hence, because of Lemma 5.2.2, $\boldsymbol{u}_1 \in \mathrm{H}^s(\Omega_f)^2$ and $\|\boldsymbol{u}_1\|_{s,\Omega_f} \leq C(\|\operatorname{div} \boldsymbol{u}_1\|_{0,\Omega_f} + \|\boldsymbol{w}\|_{1,\Omega_s})$. Thus, we conclude that $\|\boldsymbol{u}_1\|_{s,\Omega} \leq C\|\boldsymbol{f}_h\|_{\mathrm{div},\Omega}$. On the other hand, we observe that (5.4.35) also holds for $(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{K}$. Hence we write

$$\int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_1 \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_1) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = -2 \int_{\Omega_f} \nu \operatorname{div} \boldsymbol{f}_h \operatorname{div} \overline{\boldsymbol{v}} \quad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{V},$$

or equivalently

$$\int_{\Omega_f} (\rho_f c^2 \operatorname{div} \boldsymbol{u}_1 + 2\nu \operatorname{div} \boldsymbol{f}_h) \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_1) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = 0 \quad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{V}.$$

Then, testing with $\boldsymbol{v} \in \mathcal{D}(\Omega_f)^2$ and $\boldsymbol{\tau} \equiv \boldsymbol{0}$, we have

$$\nabla(\rho_f c^2 \operatorname{div} \boldsymbol{u}_1 + 2\nu \operatorname{div} \boldsymbol{f}_h) = 0. \tag{5.4.38}$$

Hence, since Ω_f is connected and ρ_f, c^2, ν and $\operatorname{div} \boldsymbol{f}_h$ are constants, we conclude that $\operatorname{div} \boldsymbol{u}_1 \in \mathcal{U}_h$.

On the other hand, considering $\boldsymbol{v} \in \mathcal{D}(\Omega)^2$, where $\Omega = \Omega_f \cup \Omega_s$, with $(\boldsymbol{v}|_{\Omega_f}, \boldsymbol{v}|_{\Omega_s}) \in \mathcal{V}$, integrating by parts in (5.4.35) and using (5.4.45) and the fact that $\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{w}_1)) = 0$, we obtain that

$$\int_\Gamma (\rho_f c^2 \operatorname{div} \boldsymbol{u}_1 + 2\nu \operatorname{div} \boldsymbol{f}_h) \boldsymbol{n} \cdot \boldsymbol{v} - \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{w}_1) \boldsymbol{n}) \cdot \boldsymbol{v} = 0,$$

which implies that $\boldsymbol{\sigma}(\boldsymbol{w}_1)\boldsymbol{n} = (\rho_f c^2 + 2\nu \operatorname{div} \boldsymbol{f}_h)\boldsymbol{n}$ on Γ. Then, we consider the following well posed elasticity problem

$$\begin{aligned}
-\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{w}_1)) &= 0 \quad \text{in } \Omega_s,\\
\boldsymbol{\sigma}(\boldsymbol{w}_1)\boldsymbol{n} - (\rho c^2 \operatorname{div} \boldsymbol{u}_1 + 2\nu \operatorname{div} \boldsymbol{f}_h)\boldsymbol{n} &= 0 \quad \text{on } \Gamma,\\
\boldsymbol{\sigma}(\boldsymbol{w}_1)\eta &= 0 \quad \text{on } \Gamma_N,\\
\boldsymbol{w}_1 &= 0 \quad \text{on } \Gamma_D,
\end{aligned}$$

where $\boldsymbol{w}_1 \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ and satisfies

$$\begin{aligned}
\|\boldsymbol{w}_1\|_{1+\beta,\Omega_s} \leq & C \sum_{j=1}^{N} \|\rho c^2 \operatorname{div} \boldsymbol{u}_1 + 2\nu \operatorname{div} \boldsymbol{f}_h\|_{\frac{1}{2},\Gamma_j},\\
\leq & C \|\rho c^2 \operatorname{div} \boldsymbol{u}_1 + 2\nu \operatorname{div} \boldsymbol{f}_h\|_{1,\Omega_s}\\
\leq & C \|\boldsymbol{f}_h\|_{\operatorname{div},\Omega_f},
\end{aligned}$$

where we have used Trace Theorem. Hence, we complete the proof. □

Now we consider a similar decomposition for the discrete case. For $((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)) \in \widetilde{\mathcal{G}}_h$, let $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) = \boldsymbol{T}_h((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h))$. We write $(\boldsymbol{u}_h, \boldsymbol{w}_h) = (\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h}) + (\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h})$, such that $(\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h})$ and $(\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h})$ satisfies, respectively, the following problems:

$$\begin{aligned}
(\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h}) \in \mathcal{G}_h : \quad & \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_{1h} \operatorname{div} \overline{\boldsymbol{v}}_h + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_{1h}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}_h)\\
&= -2 \int_{\Omega_f} \nu \operatorname{div} \boldsymbol{f}_h \operatorname{div} \overline{\boldsymbol{v}}_h \qquad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{G}_h, \qquad (5.4.39)
\end{aligned}$$

and

$$\begin{aligned}
(\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h}) \in \mathcal{G}_h : \quad & \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_{2h} \operatorname{div} \overline{\boldsymbol{v}}_h + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_{2h}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}_h)\\
&= -\int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \overline{\boldsymbol{v}}_h - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}}_h \cdot \overline{\boldsymbol{\tau}}_h \qquad \forall (\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{G}_h. \qquad (5.4.40)
\end{aligned}$$

These are the finite element discretization of problems (5.4.35) and (5.4.36) respectively. The next result gives approximation properties between the solutions of problems (5.4.35)–(5.4.36) and (5.4.39)–(5.4.40).

The first step is to construct an adequate interpolant for the correct approximation in the domain occupied by the fluid, the domain of the solid and the interface. With this aim, we introduce the following *corrected* operator $\boldsymbol{M}_h$ defined as follows:

We introduce the operator $\boldsymbol{M}_h : \left(\mathrm{H}^{s,1}(\operatorname{div}, \Omega_f) \times \mathrm{H}^{1+\beta}(\Omega_s)^2\right) \cap \mathcal{V} \to \mathcal{V}_h$ defined as follows

$$\boldsymbol{M}_h(\boldsymbol{u}, \boldsymbol{w})|_T := \begin{cases} \boldsymbol{\Pi}_h \boldsymbol{u}|_T & \text{if } T \subset \Omega_f \ \text{ and } \partial T \cap \Gamma = \emptyset,\\ \mathcal{L}_h \boldsymbol{w}|_T & \text{if } \subset \Omega_s,\\ \widehat{\boldsymbol{\Pi}}_h \boldsymbol{u}|_T & \text{if } T \subset \Omega_f \ \text{ and } \partial T \cap \Gamma \neq \emptyset, \end{cases}$$

where $\mathcal{L}_h$ is the Lagrange interpolant, $\mathbf{\Pi}_h$ is the Raviart-Thomas interpolant and $\widehat{\mathbf{\Pi}}_h$ is the function in $\mathcal{W}_h$ with degrees of freedom given by

$$(\widehat{\mathbf{\Pi}}_h \boldsymbol{u})|_\ell \cdot \boldsymbol{n} := \begin{cases} (\mathbf{\Pi}_h \boldsymbol{u})|_\ell & \text{if } \ell \not\subset \Gamma, \\ \frac{1}{|\ell|} \int_\ell (\mathcal{L}_h \boldsymbol{w})|_{T_\ell} & \text{if } \ell \subset \Gamma, \end{cases}$$

with T_ℓ the triangle contained in Ω_s such that $\partial T \cap \partial T_\ell = \ell$.

The following lemma gives an approximation property for $\|\mathbf{\Pi}_h - \widehat{\mathbf{\Pi}}_h\|_{\mathrm{div},\Omega_f}$.

Lemma 5.4.5 *There exists $C > 0$ depending only on the regularity of T and T_ℓ, such that for each $(\boldsymbol{u}, \boldsymbol{w}) \in (\mathrm{H}^{s,1}(\mathrm{div}, \Omega_f) \times \mathrm{H}^{1+\beta}(\Omega_s)^2) \cap \mathcal{V}$ the following estimate holds true*

$$\|\mathbf{\Pi}_h - \widehat{\mathbf{\Pi}}_h\|_{\mathrm{div},\Omega_f} \leq C h^\beta \|\boldsymbol{w}\|_{1+\beta, T_\ell}.$$

Proof. See Lemma 5.1 of [12]. □

Now we will prove that operator $\boldsymbol{M}_h$ satisfies the following approximation property.

Lemma 5.4.6 *There exists a positive constant C such that, for each $(\boldsymbol{u}, \boldsymbol{w}) \in \left(\mathrm{H}^{s,1}(\mathrm{div}, \Omega_f) \times \mathrm{H}^{1+\beta}(\Omega_s)^2\right) \cap \mathcal{V}$ there holds*

$$\|(\boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{M}_h(\boldsymbol{u}, \boldsymbol{w})\|_{\mathcal{X}} \leq C h^r (\|\boldsymbol{u}\|_{s,\Omega_f} + \|\operatorname{div} \boldsymbol{u}\|_{1,\Omega_f} + \|\boldsymbol{w}\|_{1+\beta,\Omega_s}).$$

Proof. See Theorem 5.2 in [12]. □

The next result gives error estimates between the solutions of problems

Lemma 5.4.7 *Let $((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)) \in \widetilde{\mathcal{G}}_h$. Let $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) = \boldsymbol{T}((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h))$ and $((\boldsymbol{u}_h, \boldsymbol{w}_h), (\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{w}}_h)) = \boldsymbol{T}((\boldsymbol{f}_h, \boldsymbol{g}_h), (\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h))$. Let $(\boldsymbol{u}_1, \boldsymbol{w}_1), (\boldsymbol{u}_2, \boldsymbol{w}_2)$ be the solutions of problems* (5.4.35) *and* (5.4.36) *and $(\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h}), (\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h})$ be the solutions of problems* (5.4.39) *and* (5.4.40) *respectively. Then, the following estimates hold*

$$\|(\boldsymbol{u}_1, \boldsymbol{w}_1) - (\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h})\|_{\mathcal{X}} \leq C h^r \|\boldsymbol{f}_h\|_{\mathrm{div},\Omega_f} \tag{5.4.41}$$

and

$$\|(\boldsymbol{u}_2, \boldsymbol{w}_2) - (\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h})\|_{\mathcal{X}} \leq C h^r \|(\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h))\|_{\mathcal{H}}. \tag{5.4.42}$$

Proof. First we show (5.4.41). Since $\mathcal{G}_h \not\subseteq \mathcal{G}_{\mathcal{V}}$ we need to apply the second Strang's lemma for each approximation error. For problems (5.4.35) and (5.4.39) the Strang estimate reads as follows:

$$\|(\boldsymbol{u}_1, \boldsymbol{w}_1) - (\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h})\|_{\mathcal{X}} \leq \left\{ C \inf_{(\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h} \|(\boldsymbol{u}_1, \boldsymbol{w}_1) - (\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|_{\mathcal{X}} \right. \\ \left. + \sup_{(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h} \frac{a((\boldsymbol{u}_1, \boldsymbol{w}_1) - (\boldsymbol{u}_{1h}, \boldsymbol{w}_{1h}), (\boldsymbol{v}_h, \boldsymbol{\tau}_h))}{\|(\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|_{\mathcal{X}}} \right\}. \tag{5.4.43}$$

Since $\boldsymbol{M}_h(\boldsymbol{u}_1,\boldsymbol{w}_1)\in\mathcal{V}_h$ and $\mathcal{V}_h=\mathcal{G}_h\oplus\mathcal{K}_h$, there exists $(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{w}}_h)\in\mathcal{G}_h$ and $(\boldsymbol{u}_{\mathcal{K}_h},\boldsymbol{w}_{\mathcal{K}_h})\in\mathcal{K}_h$ such that $\boldsymbol{M}_h(\boldsymbol{u}_1,\boldsymbol{w}_1)=(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{w}}_h)+(\boldsymbol{u}_{\mathcal{K}_h},\boldsymbol{w}_{\mathcal{K}_h})$. Then, since $(\boldsymbol{u}_1,\boldsymbol{w}_1)\in\mathcal{G}_{\mathcal{V}}$ is orthogonal to $(\boldsymbol{u}_{\mathcal{K}_h},\mathbf{0})\in\mathcal{K}_h$, we observe that

$$
\begin{aligned}
\|(\boldsymbol{u}_1,\boldsymbol{w}_1)-(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{w}}_h)\|^2_{\mathcal{X}} \le& \|(\boldsymbol{u}_1,\boldsymbol{w}_1)-(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{w}}_h)\|^2_{\mathcal{X}}+\|(\boldsymbol{u}_{\mathcal{K}_h},\mathbf{0})\|^2_{\mathcal{X}},\\
=&\|((\boldsymbol{u}_1,\boldsymbol{w}_1)-(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{w}}_h))+(\boldsymbol{u}_{\mathcal{K}_h},\mathbf{0})\|^2_{\mathcal{X}},\\
=&\|(\boldsymbol{u}_1,\boldsymbol{w}_1)-\boldsymbol{M}_h(\boldsymbol{u}_1,\boldsymbol{w}_1)\|^2_{\mathcal{X}},\\
=&\|\boldsymbol{u}_1-\boldsymbol{\Pi}_h\boldsymbol{u}_1\|^2_{\operatorname{div},\Omega_f}+\|\boldsymbol{w}_1-\mathcal{L}_h\boldsymbol{w}_1\|^2_{1,\Omega_s}+\|\boldsymbol{u}_1-\widehat{\boldsymbol{\Pi}}_h\boldsymbol{u}_1\|^2_{\mathcal{V}},
\end{aligned}
$$

Using (5.4.25) in the first term on the right-hand side we obtain

$$
\begin{aligned}
\|\boldsymbol{u}_1-\boldsymbol{\Pi}_h\boldsymbol{u}_1\|_{0,\Omega_f}\le& Ch^s(\|\boldsymbol{u}_1\|_{s,\Omega_f}+\|\operatorname{div}\boldsymbol{u}_1\|_{0,\Omega_f}),\\
\le& Ch^s(\|\boldsymbol{u}_1\|_{s,\Omega_f}+\|\operatorname{div}\boldsymbol{f}_h\|_{0,\Omega})\le Ch^s\|\boldsymbol{f}_h\|_{\operatorname{div},\Omega_f}.
\end{aligned}
\tag{5.4.44}
$$

On the other hand, applying (5.4.26) and using the fact that $\operatorname{div}\boldsymbol{u}_1\in\mathcal{U}$ (cf. Lemma 5.4.4) we obtain

$$
\|\operatorname{div}\boldsymbol{u}_1-\operatorname{div}(\boldsymbol{\Pi}_h\boldsymbol{u}_1)\|_{0,\Omega_f}=\|\operatorname{div}\boldsymbol{u}_1-\mathcal{P}_h(\operatorname{div}\boldsymbol{u}_1)\|_{0,\Omega_f}=0. \tag{5.4.45}
$$

For the second term we have

$$
\|\boldsymbol{w}_1-\mathcal{L}_h\boldsymbol{w}_1\|_{1,\Omega_s}\le Ch^{\beta}\|\boldsymbol{w}_1\|_{1+\beta,\Omega_s}\le Ch^{\beta}\|\boldsymbol{f}_h\|_{\operatorname{div},\Omega_f}, \tag{5.4.46}
$$

where we have used (5.4.28) and Lemma 5.4.4.

For the third term we proceed as in (5.4.45) and Theorem 5.2 of [12] obtaining for $T\subset\Omega_f$, such that $\partial T\cap\Gamma\neq\emptyset$, the following estimate

$$
\begin{aligned}
\|\boldsymbol{u}_1-\widehat{\boldsymbol{\Pi}}_h\boldsymbol{u}_1\|_{\operatorname{div},T}\le&\|\boldsymbol{u}_1-\boldsymbol{\Pi}_h\boldsymbol{u}_1\|_{\operatorname{div},T}+\|\boldsymbol{u}_1-\widehat{\boldsymbol{\Pi}}_h\boldsymbol{u}_1\|_{\operatorname{div},T}\\
\le& C\left(\|\boldsymbol{u}_1\|_{s,T}+\|\operatorname{div}\boldsymbol{u}_1\|_{0,T}+\sum_{T'\subset\Omega_s,T'\cap T\neq\emptyset}h^{\beta}\|\boldsymbol{w}_1\|_{1+\beta,\Omega_s}\right)
\end{aligned}
\tag{5.4.47}
$$

Then, combining (5.4.44), (5.4.46) and (5.4.47) we obtain

$$
\|(\boldsymbol{u}_1,\boldsymbol{w}_1)-(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{w}}_h)\|_{\mathcal{X}}\le Ch^r\|\boldsymbol{f}_h\|_{\operatorname{div},\Omega_f}.
$$

with $r=\min\{s,\beta\}$. Thus, we have bound the infimum in (5.4.43).

For the consistency term is enough to observe that (5.4.39) holds for all $(\boldsymbol{v}_h,\boldsymbol{\tau}_h)\in\mathcal{K}_h$. Then it is easy to check that $a((\boldsymbol{u}_1,\boldsymbol{w}_1)-(\boldsymbol{u}_{1h},\boldsymbol{w}_{1h}),(\boldsymbol{v}_h,\boldsymbol{\tau}_h))=0$.

On the other hand we will prove the following Strang's estimate is bounded

$$
\begin{aligned}
\|(\boldsymbol{u}_2,\boldsymbol{w}_2)-(\boldsymbol{u}_{2h},\boldsymbol{w}_{2h})\|_{\mathcal{X}}\le C\Bigg\{&\inf_{(\boldsymbol{v}_h,\boldsymbol{\tau}_h)\in\mathcal{G}_h}\|(\boldsymbol{u}_2,\boldsymbol{w}_2)-(\boldsymbol{v}_h,\boldsymbol{\tau}_h)\|_{\mathcal{X}}\\
&+\sup_{(0,0)\neq(\boldsymbol{v}_h,\boldsymbol{\tau}_h)\in\mathcal{G}_h}\frac{a((\boldsymbol{u}_2,\boldsymbol{w}_2)-(\boldsymbol{u}_{2h},\boldsymbol{w}_{2h}),(\boldsymbol{v}_h,\boldsymbol{\tau}_h))}{\|(\boldsymbol{v}_h,\boldsymbol{\tau}_h)\|_{\mathcal{X}}}\Bigg\}.
\end{aligned}
\tag{5.4.48}
$$

Since $(\boldsymbol{u}_2, \boldsymbol{w}_2) \in \mathcal{G}_{\mathcal{V}}$, $\boldsymbol{u}_2 \in \mathrm{H}^s(\Omega_f)^2 \cap \mathrm{H}(\mathrm{div}; \Omega_f)$ due Lemma 5.2.2. On the other hand, $\operatorname{div} \boldsymbol{u}_2 \in \mathrm{H}^1(\Omega_f)$ and $\boldsymbol{w}_2 \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ due Lemma 5.3.1. Hence, $\boldsymbol{M}_h(\boldsymbol{u}_2, \boldsymbol{w}_2)$ is well defined. Then, as in the previous case, there exists $(\widetilde{\boldsymbol{u}}_{2h}, \widetilde{\boldsymbol{w}}_{2h}) \in \mathcal{G}_h$ and $(\boldsymbol{u}_{\mathcal{K}_h} . \boldsymbol{w}_{\mathcal{K}_h}) \in \mathcal{K}_h$ such that $\boldsymbol{M}_h(\boldsymbol{u}_2, \boldsymbol{w}_2) = (\widetilde{\boldsymbol{u}}_{2h}, \widetilde{\boldsymbol{w}}_{2h}) + (\boldsymbol{u}_{\mathcal{K}_h} . \boldsymbol{w}_{\mathcal{K}_h})$.

From the first term of the right hand side we use (5.4.25) to obtain

$$\|\boldsymbol{u}_2 - \boldsymbol{\Pi}_h \boldsymbol{u}_2\|_{0,\Omega_f} \leq C h^s (\|\boldsymbol{u}_2\|_{s,\Omega_f} + \|\operatorname{div} \boldsymbol{u}_2\|_{0,\Omega_f}) \leq C h^s \|(\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)\|_{\mathcal{H}}.$$

On the other hand, using (5.4.28)

$$\|\boldsymbol{w}_2 - \widetilde{\boldsymbol{w}}_{2h}\|_{1,\Omega_s} \leq C h^\beta \|\boldsymbol{w}_2\|_{1+\beta,\Omega_s} \leq C h^\beta \|(\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)\|_{\mathcal{H}}.$$

Hence

$$\inf_{(\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h} \|(\boldsymbol{u}_2, \boldsymbol{w}_2) - (\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|_{\mathcal{X}} \leq C h^r \|(\widehat{\boldsymbol{f}}_h, \widehat{\boldsymbol{g}}_h)\|_{\mathcal{H}}, \tag{5.4.49}$$

with $r = \min\{s, \beta\}$

Now we observe the consistency term in (5.4.48). As in the previous estimate, for $(\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h$, we consider the decomposition $(\boldsymbol{v}_h, \boldsymbol{\tau}_h) = (\nabla \xi, \boldsymbol{\tau}_h) + (\chi, 0)$ of Lemma 5.4.1. Then

$$\begin{aligned}
a((\boldsymbol{u}_2, \boldsymbol{w}_2), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) &= \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_2 \operatorname{div} \boldsymbol{v}_h + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{\tau}_2) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}_h) \\
&= \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_2 \operatorname{div}(\nabla \xi + \chi) + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_2) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}_h) \\
&= \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_2 \operatorname{div} \nabla \xi + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_2) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}_h) \\
&= -\int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \nabla \xi - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}}_h \cdot \boldsymbol{\tau}_h.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a((\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h}), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) &= \int_{\Omega_f} \rho_f c^2 \operatorname{div} \boldsymbol{u}_{2h} \operatorname{div} \boldsymbol{v}_h + \int_{\Omega_s} \boldsymbol{\sigma}(\boldsymbol{w}_{2h}) : \boldsymbol{\varepsilon}(\boldsymbol{\tau}_h) \\
&= -\int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot (\nabla \xi + \chi) - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}}_h \boldsymbol{\tau}_h \\
&= -\int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \nabla \xi + \int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \chi - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{g}}_h \boldsymbol{\tau}_h
\end{aligned}$$

Hence

$$a((\boldsymbol{u}_2, \boldsymbol{w}_2) - (\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h}), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) = \int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \chi \leq C \|\widehat{\boldsymbol{f}}_h\|_{0,\Omega_f} \|\chi\|_{0,\Omega_f},$$

and applying Lemma 5.2.2 we obtain

$$\begin{aligned}
a((\boldsymbol{u}_2, \boldsymbol{w}_2) - (\boldsymbol{u}_{2h}, \boldsymbol{w}_{2h}), (\boldsymbol{v}_h, \boldsymbol{\tau}_h)) &\leq C h^s \|\widehat{\boldsymbol{f}}_h\|_{0,\Omega_f} (\|\operatorname{div} \boldsymbol{v}\|_{0,\Omega_f} + \|\boldsymbol{\tau}\|_{0,\Omega_s}) \\
&\leq C h^s \|\widehat{\boldsymbol{f}}_h\|_{0,\Omega_f} (\|\boldsymbol{v}\|_{\mathrm{div},\Omega_f} + \|\boldsymbol{\tau}\|_{0,\Omega_s})
\end{aligned}$$

Therefore, taking supremum

$$\sup_{(0,0)\neq(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h)\in\mathcal{G}_h} \frac{a((\boldsymbol{u}_2,\boldsymbol{w}_2)-(\boldsymbol{u}_{2h},\boldsymbol{w}_{2h}),(\boldsymbol{v}_h,\boldsymbol{\tau}_h))}{\|(\boldsymbol{v}_h,\boldsymbol{\tau}_h)\|_{\mathcal{X}}} \leq Ch^s\|\widehat{\boldsymbol{f}}_h\|_{0,\Omega_f}.$$

Finally, combining (5.4.49) and the previous estimation, we conclude (5.4.48), concluding thus the proof of the lemma. □

Lemma 5.4.8 *The following properties holds true*

$$\|\boldsymbol{T}-\boldsymbol{T}_h\|_h \leq Ch^r.$$

Proof. Let $((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{f}}_h,\widehat{\boldsymbol{g}}_h))\in\widetilde{\mathcal{G}}_h$ such that be such that $((\boldsymbol{u},\boldsymbol{w}),(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})) := \boldsymbol{T}((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{f}}_h,\widehat{\boldsymbol{g}}_h))$ and $((\boldsymbol{u}_h,\boldsymbol{w}_h),(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h)) := \boldsymbol{T}_h((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{f}}_h,\widehat{\boldsymbol{g}}_h))$. Using Lemma 5.4.7 and the fact that (5.2.18) and (5.4.33) implies $(\boldsymbol{f}_h,\boldsymbol{g}_h)=\mathcal{P}_{\mathcal{G}}(\boldsymbol{f}_h,\boldsymbol{g}_h)$, we have

$$\begin{aligned}
\|\boldsymbol{T}-\boldsymbol{T}_h\|_h &= \sup_{0\neq(\boldsymbol{g}_h,\boldsymbol{f}_h,\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h)\in\widetilde{\mathcal{G}}_h} \frac{\|(\boldsymbol{T}-\boldsymbol{T}_h)((\boldsymbol{g}_h,\boldsymbol{f}_h),(\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h))\|_{\mathcal{X}\times\mathcal{H}}}{\|((\boldsymbol{g}_h,\boldsymbol{f}_h),(\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h))\|_{\mathcal{X}\times\mathcal{H}}} \\
&= \sup_{0\neq(\boldsymbol{g}_h,\boldsymbol{f}_h,\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h))\in\widetilde{\mathcal{G}}_h} \frac{\|(\boldsymbol{u},\boldsymbol{w})-(\boldsymbol{u}_h,\boldsymbol{w}_h)\|_{\mathcal{X}}+\|(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{w}})-(\widehat{\boldsymbol{u}}_h,\widehat{\boldsymbol{w}}_h)\|_{\mathcal{H}}}{\|((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h))\|_{\mathcal{X}\times\mathcal{H}}} \\
&\leq \sup_{\boldsymbol{0}\neq((\boldsymbol{g}_h,\boldsymbol{f}_h),(\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h))\in\widetilde{\mathcal{G}}_h} \frac{\|(\boldsymbol{u}_1,\boldsymbol{w}_1)-(\boldsymbol{u}_{1h},\boldsymbol{w}_{1h})\|_{\mathcal{X}}+\|(\boldsymbol{u}_2,\boldsymbol{w}_2)-(\boldsymbol{u}_{2h},\boldsymbol{w}_{2h})\|_{\mathcal{X}}}{\|((\boldsymbol{f}_h,\boldsymbol{g}_h),(\widehat{\boldsymbol{g}}_h,\widehat{\boldsymbol{f}}_h))\|_{\mathcal{X}\times\mathcal{H}}} \\
&\leq Ch^r.
\end{aligned}$$

Thus we conclude the proof. □

Our next goal is to prove property P2. The proof of this property is based in the following regularity results. The next result gives us a necessary condition for the regularity of $\operatorname{div}\boldsymbol{u}$ and $\boldsymbol{w}$.

Lemma 5.4.9 *If* $(2\mu\nu+\rho_f c^2)\neq 0$, *then* $\operatorname{div}\boldsymbol{u}\in \mathrm{H}^1(\Omega_f)$. *Moreover,* $\boldsymbol{w}\in\mathrm{H}^{1+\beta}(\Omega_s)^2$.

Proof. From Problem 5.2.2, testing with $\boldsymbol{v}\in\mathcal{D}(\Omega_f)^2$ and $\boldsymbol{\tau}\equiv 0$ we obtain

$$\nabla((\rho c^2+2\lambda\nu)\operatorname{div}\boldsymbol{u}) = -\lambda\rho_f\widehat{\boldsymbol{u}}\in\mathrm{L}^2(\Omega). \tag{5.4.50}$$

We observe that if $(\rho_f c^2+2\lambda\nu)=0$ implies that $\widehat{\boldsymbol{u}}=0$. this means that $\lambda\boldsymbol{u}=0$. From this, since $\lambda=0$ is not the eigenvalue of interest, remains the case when $\boldsymbol{u}=0$. If $\boldsymbol{u}=0$, the problem to solve is

$$\int_{\Omega_s}\boldsymbol{\sigma}(\boldsymbol{w}):\boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = -\lambda^2\int_{\Omega_s}\rho_s\boldsymbol{w}\cdot\overline{\boldsymbol{\tau}} \quad \forall\boldsymbol{\tau}\in\mathrm{H}^1_{\Gamma_D}(\Omega_s)^2.$$

testing the equation above with $\boldsymbol{\tau}=\boldsymbol{w}$ and since both sides of the equality are non-negative, we deduce that $\lambda=i\omega$, $\omega\in\mathbb{R}$. Hence, in (5.4.50), $(\rho_f c^2+2\lambda\nu)\neq 0$ which implies that $\operatorname{div}\boldsymbol{u}\in\mathrm{H}^1(\Omega_f)$.

On the other hand, testing with $\boldsymbol{\tau} \in \mathcal{D}(\Omega_s)^2$ and $\boldsymbol{v} = 0$ we obtain the following well posed elasticity problem

$$\begin{aligned} \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{w})) &= \lambda^2 \rho_s \boldsymbol{w} && \text{in } \Omega_s, \\ \boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{n} &= (\rho_f c^2 \operatorname{div} \boldsymbol{u})\boldsymbol{n} && \text{on } \Gamma, \\ \boldsymbol{\sigma}(\boldsymbol{w})\eta &= 0 && \text{on } \Gamma_N, \\ \boldsymbol{w} &= 0 && \text{on } \Gamma_D, \end{aligned}$$

where $\boldsymbol{w} \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ and satisfies

$$\|\boldsymbol{w}\|_{1+\beta,\Omega_s} \leq C\left(\|\boldsymbol{w}\|_{0,\Omega_s} \sum_{j=1}^{N} \|\operatorname{div} \boldsymbol{u}\|_{\frac{1}{2},\Gamma_j}\right) \leq C(\|\boldsymbol{w}\|_{0,\Omega_s} \|\operatorname{div} \boldsymbol{u}\|_{1,\Omega_f}),$$

where we have used the trace theorem. Hence we conclude the proof □

From now and on, let $\mu \in \mathrm{Sp_d}(\boldsymbol{T})$ be a fixed isolated eigenvalue of finite algebraic multiplicity m. Therefore, according to Lemma 5.2.4, $\mu \neq 0$. Let $\mathcal{E}$ be the invariant subspace $\boldsymbol{T}$ corresponding to μ. With this space, we prove the following regularity result for the eigenfunctions which will be useful to prove spectral approximation properties.

Lemma 5.4.10 *Let* $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) \in \mathcal{E}$. *Then* $(\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) \in \mathcal{G}_{\mathcal{V}}$, $\operatorname{div} \boldsymbol{u}, \operatorname{div} \widehat{\boldsymbol{u}} \in \mathrm{H}^{1+s}(\Omega_f)$ *and*

$$\|\boldsymbol{u}\|_{s,\Omega_f} + \|\operatorname{div} \boldsymbol{u}\|_{1+s,\Omega_f} + \|\boldsymbol{w}\|_{1+\beta,\Omega_s} \leq C\|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X}\times\mathcal{H}}, \tag{5.4.51}$$

$$\|\widehat{\boldsymbol{u}}\|_{s,\Omega_f} + \|\operatorname{div} \widehat{\boldsymbol{u}}\|_{1+s,\Omega_f} + \|\widehat{\boldsymbol{w}}\|_{1+\beta,\Omega_s} \leq C\|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X}\times\mathcal{H}}. \tag{5.4.52}$$

Proof. We will prove the estimates for all the generalized eigenfunctions

$$\{((\boldsymbol{u}_k, \boldsymbol{w}_k), (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k))\}_{k=1}^{p},$$

of a Jordan chain of the operator $\boldsymbol{T}$ associated to the eigenvalue μ. the proof is inductive. Assume that $(\boldsymbol{u}_{k-1}, \boldsymbol{w}_{k-1})$ and $(\widehat{\boldsymbol{u}}_{k-1}, \widehat{\boldsymbol{w}}_{k-1})$ belong to $\mathcal{G}_{\mathcal{V}}$ and satisfies (5.4.51) and (5.4.52) respectively- We will prove that $(\boldsymbol{u}_k, \boldsymbol{w}_k)$ and $(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k)$ satisfies it as well. (For $k = 1$, i.e., $((\boldsymbol{u}, \boldsymbol{w}_k), (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k))$ an eigenfunction, consider $((\boldsymbol{u}, \boldsymbol{w}_k), (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k)) = ((\mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{0}))$).

Since

$$\boldsymbol{T}((\boldsymbol{u}_k, \boldsymbol{w}_k), (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k)) = \mu((\boldsymbol{u}_k, \boldsymbol{w}_k), (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k)) + ((\boldsymbol{u}_{k-1}, \boldsymbol{w}_{k-1}), (\widehat{\boldsymbol{u}}_{k-1}, \widehat{\boldsymbol{w}}_{k-1})).$$

Then, due the boundedness of $\boldsymbol{T}$, we get

$$\|((\boldsymbol{u}_{k-1}, \boldsymbol{w}_{k-1}), (\widehat{\boldsymbol{u}}_{k-1}, \widehat{\boldsymbol{w}}_{k-1}))\|_{\mathcal{X}\times\mathcal{H}} \leq C\|((\boldsymbol{u}_k, \boldsymbol{w}_k), (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k))\|_{\mathcal{X}\times\mathcal{H}}. \tag{5.4.53}$$

On the other hand, since (5.2.18) and (5.2.19) we obtain

$$(\boldsymbol{u}_k, \boldsymbol{w}_k) = \mu(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k) + (\widehat{\boldsymbol{u}}_{k-1}, \widehat{\boldsymbol{w}}_{k-1}) \tag{5.4.54}$$

and that $\mu(\boldsymbol{u}_k, \boldsymbol{w}_k) + (\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k) \in \mathcal{G}_{\mathcal{V}}$ satisfies

$$\int_{\Omega_f} \rho c^2 \operatorname{div}(\mu \boldsymbol{u}_k + \boldsymbol{u}_{k-1}) \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_s} \boldsymbol{\sigma}(\mu \boldsymbol{w}_k + \boldsymbol{w}_{k-1}) : \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}}) = -2 \int_{\Omega_f} \nu \operatorname{div} \boldsymbol{u}_k \operatorname{div} \overline{\boldsymbol{v}} + \int_{\Omega_f} \rho_f \widehat{\boldsymbol{u}}_k \cdot \overline{\boldsymbol{v}} - \int_{\Omega_s} \rho_s \widehat{\boldsymbol{w}}_k \cdot \overline{\boldsymbol{\tau}}, \quad (5.4.55)$$

for all $(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{G}_{\mathcal{V}}$. Hence $(\boldsymbol{u}_k, \boldsymbol{w}_k)$, $(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{w}}_k) \in \mathcal{G}_{\mathcal{V}}$. On the other hand, we observe that the equation above also holds for $(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{K}$, then, $(\boldsymbol{v}, \boldsymbol{\tau}) \in \mathcal{V}$. Hence, testing with $\boldsymbol{v} \in \mathcal{D}(\Omega_f)^2$ and $\boldsymbol{\tau} = 0$ in the equation above, using the same arguments of Lemma 4.7 of [62], Lemma 5.4.9 and Lemma 5.2.2, we can prove that $\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k \in \mathcal{G}_{\mathcal{V}}$ and

$$\|\boldsymbol{u}\|_{s,\Omega_f} + \|\operatorname{div} \boldsymbol{u}\|_{1+s,\Omega_f} \leq C \|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X} \times \mathcal{H}},$$

and

$$\|\widehat{\boldsymbol{u}}\|_{s,\Omega_f} + \|\operatorname{div} \widehat{\boldsymbol{u}}\|_{1+s,\Omega_f} \leq C \|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X} \times \mathcal{H}}.$$

Remains to prove that $\boldsymbol{w}_k, \widehat{\boldsymbol{w}}_k \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ and the analogous estimates above for $\boldsymbol{w}_k$ and $\widehat{\boldsymbol{w}}_k$. We proceed as follows: testing (5.4.55) with $\boldsymbol{\tau} \in \mathcal{D}(\Omega_s)^2$ and $\boldsymbol{v} = 0$, we obtain

$$\begin{aligned} -\operatorname{div}(\boldsymbol{\sigma}(\mu \boldsymbol{w}_k + \boldsymbol{w}_{k-1})) &= \rho_s \widehat{\boldsymbol{w}}_k \in \mathrm{L}^2(\Omega_s)^2, \\ \boldsymbol{\sigma}(\mu \boldsymbol{w}_k + \boldsymbol{w}_{k-1}) \boldsymbol{n} &= [(\rho c^2 \operatorname{div}(\mu \boldsymbol{u}_k + \boldsymbol{u}_{k-1}) + 2\nu \operatorname{div} \boldsymbol{u}_k] \boldsymbol{n} \quad \text{on } \Gamma, \\ \boldsymbol{\sigma}(\mu \boldsymbol{w}_k + \boldsymbol{w}_{k-1}) \boldsymbol{n} &= 0 \quad \text{on } \Gamma_N, \\ \mu \boldsymbol{w}_k + \boldsymbol{w}_{k-1} &= 0 \quad \text{on } \Gamma_D. \end{aligned}$$

We observe that $[(\rho c^2 \operatorname{div}(\mu \boldsymbol{u}_k + \boldsymbol{u}_{k-1}) + 2\nu \operatorname{div} \boldsymbol{u}_k] \boldsymbol{n} \in \mathrm{H}^{1/2}(\Gamma_j)$, for $j = 1, \ldots, N$. Hence, $\mu \boldsymbol{w}_k + \boldsymbol{w}_{k_1} \in \mathrm{H}^{1+\beta}(\Omega_s)^2$. Since $\boldsymbol{w}_{k-1} \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ because of the inductive hypothesis, then $\boldsymbol{w}_k \in \mathrm{H}^{1+\beta}(\Omega_s)^2$. Moreover, from (5.4.54) we have that $\boldsymbol{w}_k = \mu \widehat{\boldsymbol{w}}_k + \boldsymbol{w}_{k-1}$. Then, since $\boldsymbol{w}_{k-1} \in \mathrm{H}^{1+\beta}(\Omega_s)^2$ because of the inductive hypothesis and we have shown that $\boldsymbol{w}_k \in \mathrm{H}^{1+\beta}(\Omega_s)^2$, we conclude that $\widehat{\boldsymbol{w}}_k \in \mathrm{H}^{1+\beta}(\Omega_s)^2$. □

Lemma 5.4.11 *Let* $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) \in \mathcal{E}$. *Then, there exists* $(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h), (\widetilde{\widehat{\boldsymbol{u}}}_h, \widetilde{\widehat{\boldsymbol{w}}}_h) \in \mathcal{G}_h$ *such that*

$$\|(\boldsymbol{u}, \boldsymbol{w}) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h)\|_{\mathcal{X}} + \|(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) - (\widetilde{\widehat{\boldsymbol{u}}}_h, \widetilde{\widehat{\boldsymbol{w}}}_h)\|_{\mathcal{X}} \leq C h^r \|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X} \times \mathcal{H}}.$$

Proof. Consider the decomposition $\mathcal{V}_h = \mathcal{G}_h \oplus \mathcal{K}_h$. Since $\boldsymbol{M}_h(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{V}_h$, there exists $(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h) \in \mathcal{G}_h$ and $(\boldsymbol{u}_{\mathcal{K}_h}, \boldsymbol{w}_{\mathcal{K}_h}) \in \mathcal{K}_h$ such that $\boldsymbol{M}_h(\boldsymbol{u}, \boldsymbol{w}) = (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h) + (\boldsymbol{u}_{\mathcal{K}_h}, \boldsymbol{w}_{\mathcal{K}_h})$. Since $(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{G}_{\mathcal{V}}$ is orthogonal to the elements of $\mathcal{K}_h$, we have

$$\begin{aligned} \|(\boldsymbol{u}, \boldsymbol{w}) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h)\|_{\mathcal{X}}^2 &\leq \|(\boldsymbol{u}, \boldsymbol{w}) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h)\|_{\mathcal{X}}^2 + \|(\boldsymbol{u}_{\mathcal{K}_h}, \boldsymbol{w}_{\mathcal{K}_h})\|_{\mathcal{X}}^2, \\ &= \|((\boldsymbol{u}, \boldsymbol{w}) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h)) - (\boldsymbol{u}_{\mathcal{K}_h}, \boldsymbol{w}_{\mathcal{K}_h})\|_{\mathcal{X}}^2, \\ &= \|(\boldsymbol{u}, \boldsymbol{w}) - \boldsymbol{M}_h(\boldsymbol{u}, \boldsymbol{w})\|_{\mathcal{X}}^2. \end{aligned}$$

Then, according to Lemma 5.4.10 and the definition of $\boldsymbol{M}_h$ we obtain

$$\|(\boldsymbol{u}, \boldsymbol{w}) - (\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h)\|_{\mathcal{X}} \leq C h^r \|(\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})\|_{\mathcal{X} \times \mathcal{H}},$$

with $r = \min\{s, \beta\}$. On the other hand, since $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) \in \mathcal{G}_{\mathcal{V}}$ due Lemma 5.4.10, the construction of $(\widetilde{\widehat{\boldsymbol{u}}}_h, \widetilde{\widehat{\boldsymbol{w}}}_h)$ is analogous to the case of $(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{w}}_h)$. Hence we conclude the proof. □

The following result provides convergence of continuous and discrete solutions considering sources in $\mathcal{E}$.

Lemma 5.4.12 *Let* $((\boldsymbol{f}, \boldsymbol{g}), (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{g}})) \in \mathcal{E}$ *such that* $((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}})) := \boldsymbol{T}((\boldsymbol{f}, \boldsymbol{g}), (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{g}}))$ *and* $((\boldsymbol{u}_h, \boldsymbol{w}_h), (\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{w}}_h)) := \boldsymbol{T}_h((\boldsymbol{f}, \boldsymbol{g}), (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{g}}))$. *Then*

$$\|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)\|_{\mathcal{X}} + \|(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) - (\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{w}}_h)\|_{\mathcal{H}} \leq C h^r \|((\boldsymbol{f}, \boldsymbol{g}), (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{g}}))\|_{\mathcal{X}\times\mathcal{H}}.$$

Proof. Since $\mathcal{G}_h \not\subseteq \mathcal{G}_{\mathcal{V}}$, we apply the second Strang's Lemma, which for this case, reads as follows

$$\|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)\|_{\mathcal{X}} \leq C \left\{ \inf_{(\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h} \|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|_{\mathcal{X}} \right.$$
$$\left. + \sup_{(\mathbf{0},\mathbf{0}) \neq (\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h} \frac{a((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{v}_h, \boldsymbol{\tau}_h))}{\|(\boldsymbol{v}_h, \boldsymbol{\tau}_h)\|_{\mathcal{X}}} \right\}.$$

Immediately from Lemma 5.4.11 we have

$$\|(\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h)\|_{\mathcal{X}} \leq C h^r \|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X}\times\mathcal{H}} \leq C h^r \|((\boldsymbol{f}, \boldsymbol{g}), (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{g}}))\|_{\mathcal{X}\times\mathcal{H}}.$$

For the consistency term we proceed as follows: let $(\boldsymbol{v}_h, \boldsymbol{\tau}_h) \in \mathcal{G}_h$ and consider the decomposition $(\boldsymbol{v}_h, \boldsymbol{\tau}_h) = (\nabla\xi, \boldsymbol{\tau}_h) + (\chi, 0)$ as in Lemma 5.4.1. Using the same arguments as in the proof of Lemma 5.4.7, we prove that

$$a((\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_h, \boldsymbol{w}_h), (\boldsymbol{v}, \boldsymbol{\tau})) = \int_{\Omega_f} \rho_f \widehat{\boldsymbol{f}}_h \cdot \chi = 0,$$

where the equality holds since $\widehat{\boldsymbol{f}} \in \mathcal{G}_{\mathcal{V}}$ and $\chi \in \mathcal{K}$.

On the other hand, since $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) = (\boldsymbol{f}, \boldsymbol{g})$ and $(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{w}}_h) = \mathcal{P}_{\mathcal{G}_h}(\boldsymbol{f}, \boldsymbol{g})$ because (5.2.18) and (5.4.33) respectively. Then, since $\mathcal{P}_{\mathcal{G}_h}$ is the L^2-projection onto $\mathcal{G}_h$, we have that

$$\|(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) - (\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{w}}_h)\|_{\mathcal{X}\times\mathcal{H}} \leq \|(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) - (\widetilde{\widehat{\boldsymbol{u}}}_h, \widetilde{\widehat{\boldsymbol{w}}}_h)\|_{\mathcal{X}\times\mathcal{H}},$$

with $(\widetilde{\widehat{\boldsymbol{u}}}_h, \widetilde{\widehat{\boldsymbol{w}}}_h) \in \mathcal{G}_h$ as in Lemma 5.4.11. Hence, we obtain

$$\|(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}) - (\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{w}}_h)\|_{\mathcal{X}\times\mathcal{H}} \leq C h^r \|((\boldsymbol{u}, \boldsymbol{w}), (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}))\|_{\mathcal{X}\times\mathcal{H}} \leq C h^r \|((\boldsymbol{f}, \boldsymbol{g}), (\widehat{\boldsymbol{f}}, \widehat{\boldsymbol{g}}))\|_{\mathcal{X}\times\mathcal{H}}.$$

Hence, the proof is complete. □

Now we are in position to establish property P2.

Lemma 5.4.13 *For all* $((\boldsymbol{v}, \boldsymbol{\tau}), (\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{\tau}})) \in \mathcal{E}$, *there holds*

$$\inf_{(\boldsymbol{v}_h, \widehat{\boldsymbol{v}}_h) \in \widetilde{\mathcal{G}}_h} \|((\boldsymbol{v}, \boldsymbol{\tau}), (\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{\tau}})) - ((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (\widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\tau}}_h))\|_{\mathcal{X}\times\mathcal{H}} \to 0$$

as $h \to 0$

Proof. We observe that for all $((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}})) \in \mathcal{E}$ there exists $((\boldsymbol{f},\boldsymbol{g}),(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}})) \in \mathcal{E}$ such that $\boldsymbol{T}((\boldsymbol{f},\boldsymbol{g}),(\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}})) = ((\boldsymbol{v},\boldsymbol{\tau}),(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{\tau}}))$. Hence, P2 is a direct consequence of Lemma 5.4.12. □

A direct consequence of P1 is the following result which has been proved in [36] and shows that our method does not introduce spurious modes.

Theorem 5.4.1 *Let $K \subset \mathbb{C}$ be a compact set such that $K \cap \mathrm{sp}(\boldsymbol{T}) = \emptyset$. There exists $h_0 > 0$ such that,if $h \leq h_0$, then $K \cap \mathrm{sp}(\boldsymbol{T}_h) = \emptyset$.*

Proof. The proof is exactly as in Theorem 1 of [36]. □

Let $D \subset \mathbb{C}$ be a closed disk centered at μ, such that $0 \notin D$ and $D \cap \mathrm{sp}(\boldsymbol{T}) = \{\mu\}$. Let $\mu_{1h}, \ldots, \mu_{m(h)_h}$ be the eigenvalues of $\boldsymbol{T}_h$ contained in D (repeated according to their algebraic multiplicities). Under assumptions P1 and P2, it is proved in [36] that $m(h) = m$, for h small enough and that $\lim_{h\to 0} \mu_{kh} = \mu$, for $k = 1, \ldots, m$.

On the other hand the arguments used in Section 5 of [13] can be readily adapted to our problem, to obtain error estimates.

We recall the definition of the gap between two closed subspaces $\mathcal{W}$ and $\mathcal{Y}$ of $\widetilde{\mathcal{V}}$:

$$\widehat{\delta}(\mathcal{W},\mathcal{Y}) := \max\{\delta(\mathcal{W},\mathcal{Y}), \delta(\mathcal{Y},\mathcal{W})\},$$

with

$$\delta(\mathcal{W},\mathcal{Y}) := \sup_{(\boldsymbol{\phi},\boldsymbol{\psi})\in\mathcal{W}} \left[\inf_{(\widehat{\boldsymbol{\phi}},\widehat{\boldsymbol{\psi}})\in\mathcal{Y}} \|(\boldsymbol{\phi},\boldsymbol{\psi}) - (\widehat{\boldsymbol{\phi}},\widehat{\boldsymbol{\psi}})\| \right].$$

Let $\mathcal{E}_h$ be the invariant subspace of $\boldsymbol{T}_h$ relative to the eigenvalues $\mu_{1h}, \ldots, \mu_{mh}$ converging to μ. We have the following results for which we do not include proofs to avoid repeating step by step [13, Section 5].

Theorem 5.4.2 *There exists constants $h_0 > 0$ and $C > 0$ such that for all $h \leq h_0$ we have*

$$\widehat{\delta}\,(\mathcal{E}_h, \mathcal{E}) \leq C h^r.$$

Theorem 5.4.3 *There exists constants $h_0 > 0$ and $C > 0$ such that for all $h \leq h_0$,*

$$\left| \mu - \frac{1}{m}\sum_{k=1}^{m} \mu_{kh} \right| \leq C h^{2r},$$

$$\left| \frac{1}{\mu} - \frac{1}{m}\sum_{k=1}^{m} \frac{1}{\mu_{kh}} \right| \leq C h^{2r},$$

$$\max_{k=1,\ldots,m} |\mu - \mu_{kh}| \leq C h^{2r/p},$$

where p is the ascent of the eigenvalue μ of $\boldsymbol{T}$.

5.5 Numerical Experiments

In this section we present some numerical experiments to asses the accuracy and performance of our finite element method. All the present were computed with a MATLAB code. With this goal we introduce a convenient matrix form of the discrete quadratic eigenvalue problem, to use standard eigensolvers.

Let $\{\phi_j, \psi_j\}_{j=1}^N$ be a nodal basis of $\mathcal{V}_h$. we define the matrices $\mathbf{K}_f := (\mathbf{K}_{ij}^{(f)})$, $\mathbf{M}_f := (\mathbf{M}_{ij}^{(f)})$, $\mathbf{M}_s := (\mathbf{M}_{ij}^{(s)})$ and $\mathbf{K}_s := (\mathbf{K}_{(ij)}^{(s)})$ as follows:

$$\mathbf{K}_{(ij)}^{(f)} := \int_\Omega \operatorname{div}\phi_i \operatorname{div}\phi_j, \quad \mathbf{M}_{ij}^{(f)} := \int_\Omega \rho_f \phi_i \cdot \phi_j,$$

$$\mathbf{M}_{ij}^{s} := \int_\Omega \rho_s \psi_i \cdot \psi_j \quad \text{and} \quad \mathbf{K}_{ij}^{(s)} := \int_\Omega \boldsymbol{\sigma}(\psi_i) : \boldsymbol{\varepsilon}(\psi_j).$$

We also define $\boldsymbol{V}_h := (\vec{\boldsymbol{u}}_h \, \vec{\boldsymbol{w}}_h)^t$. Hence, the matrix form of Problem 5.4.1 is the following

$$(\lambda_h^2 \mathbf{M} + \lambda_h \mathbf{K} + \mathbf{K})\boldsymbol{V}_h = \mathbf{0}. \tag{5.5.56}$$

Introducing $\boldsymbol{Z}_h := \lambda_h \boldsymbol{V}$, where $\boldsymbol{Z} = (\widehat{\boldsymbol{u}} \; \widehat{\boldsymbol{w}})^t$, the matrix form of Problem 5.4.2 reads

$$\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \boldsymbol{Z}_h \end{pmatrix} = \lambda_h \begin{pmatrix} -\mathbf{K} & -\mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \boldsymbol{Z}_h \end{pmatrix}.$$

We observe that the right-hand side matrix of the previous problem is not positive definite. This fact leads to problems when we use standard eigensolvers. This problem is avoided as follows: let $\lambda_h \neq 0$ such that $\mu_h := 1/\lambda_h$. Then, we rewrite problem (5.5.56) as follows

$$(\mathbf{M} + \mu_h \mathbf{K} + \mu_h^2 \mathbf{K})\boldsymbol{V}_h = \mathbf{0}.$$

Introducing $\widetilde{\boldsymbol{Z}}_h := \mu_h \boldsymbol{V}_h$, the problem above is equivalent to

$$\begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \widetilde{\boldsymbol{Z}}_h \end{pmatrix} = \lambda_h \begin{pmatrix} -\mathbf{K} & -\mathbf{K} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \widetilde{\boldsymbol{Z}}_h \end{pmatrix},$$

which in turn is equivalent to

$$\begin{pmatrix} -\mathbf{K} & -\mathbf{K} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \widetilde{\boldsymbol{Z}}_h \end{pmatrix} = \lambda_h \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \widetilde{\boldsymbol{Z}}_h \end{pmatrix}.$$

Thus, the last problem is equivalent to Problem 5.4.1 except for $\lambda_h = 0$ and the matrix in its right-hand side is Hermitian and positive definite. Hence, it is well posed and can be safely solved by standard eigensolvers.

The geometry for the experiments is represented in Figure 5.2

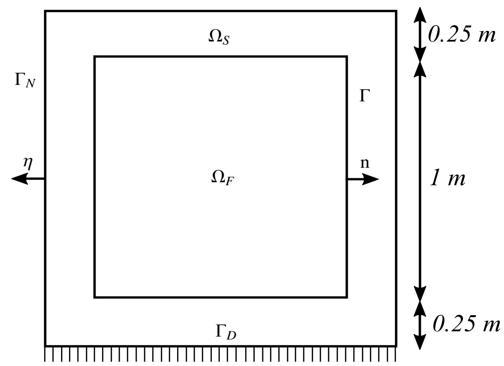

Figure 5.2: Geometry of the experimental domain. Fuente: Elaboración propia.

In what follows we will consider for the structure the physical parameters of steel

- $\rho_s = 7.71 \cdot 10^3$ kg/m^3,
- Young's modulus: $1.44 \cdot 10^{11}$ Pa,
- Poisson's coefficient: 0.35,

and the parameters of water

- $\rho_f = 1000$ kg/m^3,
- Sound speed : 1430 m/s.

In Figure 5.3 is represented the type of meshes that we will consider for our experiments.

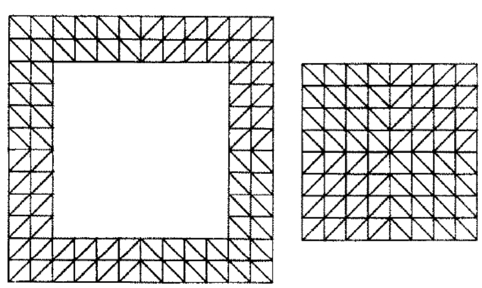

Figure 5.3: Mesh with refinement level $N = 2$ for the solid domain (left) and the fluid domain (right).

For the first experiment, we will consider the fluid-structure interaction for an inviscid fluid, i.e., $\nu = 0$ N/ms^2. In this case, the eigenvalues λ are purely imaginary. Table 5.1 shows the first eight eigenvalues computed for different refinement levels.

$N=2$	$N=4$	$N=8$	Order	Extrap.
$877.390\,i$	$734.427\,i$	$676.261\,i$	1.30	636.573 i
$2641.618\,i$	$2333.957\,i$	$2203.240\,i$	1.23	2105.714 i
$4005.082\,i$	$3714.027\,i$	$3524.047\,i$	0.62	3171.041 i
$4314.219\,i$	$4092.834\,i$	$3965.676\,i$	0.80	3794.102 i
$4478.747\,i$	$4291.236\,i$	$4234.947\,i$	1.74	4211.005 i
$4921.540\,i$	$4791.398\,i$	$4733.408\,i$	1.17	4687.158 i
$5522.426\,i$	$5271.407\,i$	$5191.402\,i$	1.65	5154.002 i
$6275.156\,i$	$5734.827\,i$	$5519.017\,i$	1.32	5374.313 i

Table 5.1: Computed eigenvalues for the non-dissipative fluid-structure problem.

As the theory predicted, the eigenvalues are purely imaginary. The 'Order' of convergence and the 'Extrapolated' eigenvalues has been computed with a least square fitting.

For the next experiment, we will consider viscosity in the fluid. In order to observe the viscosity effects in the method, we will take an unrealistic value of ν, which in this case is $\nu = 10^4$ N/ms^2. In Table 5.2 we show the eigenvalues of the dissipative fluid-structure problem where we observe that the computed eigenvalues of the dissipative problem.

$N=2$	$N=4$	$N=8$	Order	Extrap.
$-0.0037+877.390\,i$	$-0.0018+734.427\,i$	$-0.0013+676.261\,i$	1.30	$-0.0007+636.573\,i$
$-0.7208+2641.630\,i$	$-0.3327+2333.962\,i$	$-0.2405+2203.243\,i$	1.23	$-0.1081+2105.715\,i$
$-20.2757+4005.425\,i$	$-9.8763+3714.287\,i$	$-6.2886+3524.213\,i$	0.62	$-4.9439+3171.090\,i$
$-25.1284+4314.861\,i$	$-15.8249+4093.266\,i$	$-12.5853+3965.966\,i$	0.80	$-5.7427+3794.227\,i$
$-2.1351+4478.829\,i$	$-5.3674+4291.357\,i$	$-6.7213+4235.145\,i$	1.74	$-7.0241+4211.190\,i$
$-27.0326+4920.743\,i$	$-36.3487+4790.824\,i$	$-39.2902+4732.990\,i$	1.17	$-42.7221+4686.813\,i$
$-34.7515+5521.818\,i$	$-40.3646+5270.850\,i$	$-41.9127+5190.882\,i$	1.65	$-42.8131+5153.484\,i$
$-3.5917+6275.198\,i$	$-1.9258+5734.849\,i$	$-1.3568+5519.035\,i$	1.32	$-0.8947+5374.326\,i$

Table 5.2: Computed eigenvalues for the dissipative fluid-structure problem.

We observe from Table 5.2 that the computed eigenvalues are perturbations of the non-dissipative one, where this perturbation is represented showing in the fact that the complex eigenvalues have a real part, which is negative as the theory predicted.

Finally, the following plots represent the pressure of the fluid and the displacement of the solid for the first four eigenvalues of the dissipative problem.

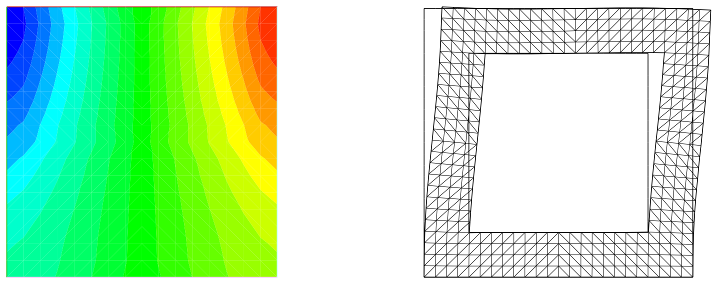

Figure 5.4: Pressure of the fluid (left) and displacement of the structure (right) for the first eigenvalue.

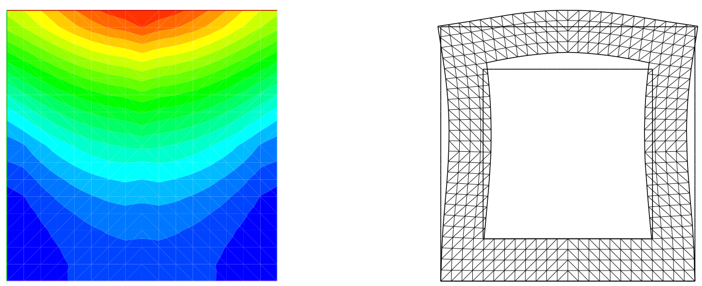

Figure 5.5: Pressure of the fluid (left) and displacement of the structure (right) for the second eigenvalue.

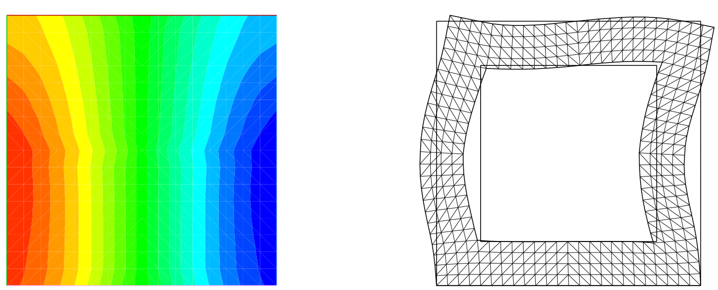

Figure 5.6: Pressure of the fluid (left) and displacement of the structure (right) for the third eigenvalue.

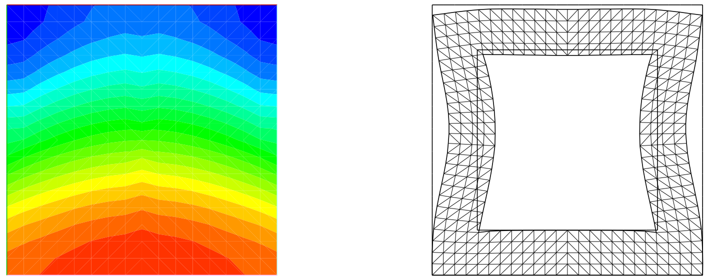

Figure 5.7: Pressure of the fluid (left) and displacement of the structure (right) for the fourth eigenvalue.

Figures 5.4, 5.5, 5.6 and 5.7 represent the real parts of the pressure of the fluid and the displacements of the solid for first four eigenvalues, since there is no significant differences between the real and imaginary parts.

Chapter 6

Conclusiones y trabajo futuro

6.1 Conclusiones

En esta sección resumiremos los principales resultados obtenidos en esta tesis para el problema de vibraciones de una viga de Timoshenko no homogénea, la interacción acústica de dos fluidos heterogéneos, un método de Galerkin discontinuo para el problema de elasticidad lineal en formulación dual-mixta y un método de elementos finitos para el problema de elastoacústica disipativa.

- En el Capítulo 2 se estudió un método de elementos finitos para el problema de vibraciones de una viga de Timoshenko empotrada, con la geometría de su sección transversal y composisicón física variable. Se obtuvo una caracterización espectral del operador solución aplicando la teoría espectral de operadores compactos. Se demostró que la solución del problema límite de Timoshenko, i.e cuando el parámetro de espesor t tiende a cero, converge a la solución del problema de Euler-Bernoulli. Se discretizó la rotación y el desplazamiento con constantes a trozos, y el esfuerzo de corte y el momento flector con lineales a trozos. Se demostró la convergencia de operador discreto al continuo usando la teoría de operadores no compactos de [36]. Se demostró que nuestro método no introduce modos espurios y es convergente independientemente del parámetro de espesor t con orden $\mathcal{O}(h^2)$. Se presentaron resultados numéricos que confirman los resultados teóricos.

- En el Capítulo 3 se estudió el problema de vibraciones acústicas de dos fluidos irrotacionales y disipativos en una cavidad rígida. La formulación estudiada corresponde a un problema de valores propios cuadrático, que se reescribió como un sistema ampliado de doble tamaño, incorporando una incógnita auxiliar. Se demostró que el problema continuo está bien planteado en el espacio ortogonal al autoespacio del valor propio $\lambda = 0$. Se obtuvo una caracterización del espectro, y se analizó un método de elementos finitos basado en elementos de Raviart-Thomas. Se demostró que el problema discreto no introduce modos de vibración espurios y se demostró que el método converge con orden $\mathcal{O}(h^{2s})$, siendo $s > 1/2$ la regularidad adicional de las autofunciones. Se obtuvo una solución analítica para una cavidad rectangular, con la cual se compararon los resultados obtenidos por el método numérico, validando el análisis realizado.

- En el Capítulo 4 se analizó un método de Galerkin discontinuo para el problema de elasticidad lineal en formulación mixta. A partir del buen planteo del problema continuo, se obtuvo un análisis del espectro del operador solución respectivo de acuerdo a los resultados obtenidos en [69]. Se demostró que el problema espectral discreto estaba bien plenteado, y que este no introduce modos espurios para un paso de malla suficientemente pequeño y para un valor del parámetro de estabilización suficientemente grande. Se demostró convergencia del método y se obtuvieron estimaciones del error, adaptando los resultados de la teoría de operadores no compactos a normas dependientes del tamaño de la malla. Los experimentos computacionales corroboraron los resultados, mostrando el orden de aproximación esperado para las autofunciones y valores propios y además, entregaron información acerca del comportamiento del método respecto al parámetro de estabilización, el grado polinomial, el tamaño de la malla. Además, se verificó computacionalmente que el método es aplicable para el caso del problema de elasticidad incompresible.

- En el Capítulo 5 se estudió un método de elementos finitos para el problema elastoacústico entre un fluido dissipativo y una estructura elástica. Se introdujo una formulación continua del problema de valores propio, el cual es cuadrático debido a la presencia de la viscosidad en el fluido. Se introdujeron variables auxiliares en el fluido y en el sólido con el fin de linearizar el problema cuadrático, convirtiéndolo en un problema lineal de doble tamaño. Se demostró que el problema continuo está bien planteado. Se estudió rigurosamente el espectro del operador solución asociado al problema espectral. El método numérico estudiado se basa en discretizar, por un lado, el fluido con elementos de Raviart-Thomas de primer orden y por otro lado el sólido con funciones lineales a trozos. Esto implica que el método numérico propuesto es no conforme. Se demostró que el método de elementos finitos no introduce modos espurios, es convergente. También de demostraron estimaciones del error para las autofunciones y valores propios. Los experimentos numéricos presentados confirman los resultados teóricos.

6.2 Trabajo futuro

1. Estudiar la extensión del problema de interacción entre un fluido disipativo y una estructura elástica implementando los elementos de Brezzi-Douglas-Marini (BDM) en el fluido.

2. Analizar problemas de interacción entre fluidos disipativos y estructuras delgadas, como vigas o placas.

3. Estudiar estimadores a posteriori para problemas de acústica disipativa.

Conclusions and future work

6.3 Conclusions

In this section we summarize the main results obtained in this thesis for the vibration problem of a non-homogeneous Timoshenko beam, the acoustic interaction between two dissipative fluids, a discontinuous Galerkin method for the dual-mixed formulation of the linear elasticity problem and a finite element method for the dissipative elastoacustic problem.

- In Chapter 2 we studied a finite element method for the vibration problem of a clamped non-homogeneous Timoshenko beam, where the geometry of the cross section, the material properties, or simultaneously both can be discontinuous along the axis. We obtained a spectral characterization of the solution operator applying the spectral theory for compact operators. We showed that the solution of the limit problem (i.e., when the thickness parameter t goes to zero) converges to the solution of the Euler-Bernoulli vibration problem. The rotation and the displacement were approximated using piecewise constants, while the bending moment and the shear stress were approximated using piecewise linear elements. We proved the convergence between the discrete solution operator and the continuous one using the non-compact operator theory of [36]. It was showed that our method does not introduce spurious modes, and converges independently of the thickness parameter with order $\mathcal{O}(h^2)$. The numerical experiments confirm the theoretical results.

- In Chapter 3 the acoustic vibration problem for two irrotational dissipative fluids in a rigid cavity was studied. The proposed formulation leads to a quadratic eigenvalue problem which, for the purpose of the analysis, is rewritten as a double-size system, by introducing an additional unknown. We proved that the problem is well posed in the orthogonal complement of the eigenspace associated to $\lambda = 0$. The spectrum of the solution operator was characterized. The finite element scheme using Raviart-Thomas element was introduced, proving that is well posed. We also proved that the method does not introduce spurious modes, and is convergent with optimal order $\mathcal{O}(h^{2s})$, with $s > 1/2$ being the Sobolev regularity exponent of the eigenfunction. We obtained an analytical solution for the rectangular cavity that we use to compare with the numerical results obtained with the method. The numerical experiments confirm the theoretical results.

- In Chapter 4 we analyzed a discontinuous Galerkin method for the linear elasticity problem, considering a dual-mixed formulation where the main unknowns are the Cauchy stress tensor and the rotation tensor. The continuous problem is shown to be well posed, and the spectrum of the corresponding solution operator was characterized in [69]. We proved the well posedness of the discrete problem by means of a global inf-sup condition which holds for sufficiently large value of the stabilization parameter. We proved that the numerical method does not introduce spurious modes for meshsize sufficiently small and for a stabilization parameter large enough, convergence and error estimates for the numerical method, by adapting the spectral theory of non-compact operators for mesh-depending norms. We report numerical experiments to asses the performance of the method, with respect to the stabilization parameter, the polynomial degree and the meshsize. Also, the experiments show that the method works for the incompressible elasticity problem, too.

- In Chapter 5 we studied the interaction problem between a dissipative fluid and an elastic structure. We introduce the continuous formulation of the eigenvalue problem, which is quadratic since the presence of viscosity. We introduced additional unknowns for the fluid and the solid in order to obtain a linear double-size problem. We proved that the continuous problem is well posed. The spectral characterization of the solution operator is given. The numerical method is based in the discretization with Raviart-Thomas for the fluid and piecewise linear functions for the solid, leading to a non-conforming method. We proved that the method does not introduce spurious modes, convergence and error estimates. The numerical experiments confirm the theoretical results.

6.4 Future work

1. To extend the analysis of interaction problem between a dissipative fluid and an elastic structure, introducing the Brezzi-Douglas-Marini (BDM) finite element to discretize the fluid domain.
2. Study the interaction between dissipative fluids and slender structures like beams or plates.
3. To perform an posteriori analysis of dissipative acoustics problems.

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