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A MIXED PARAMETER FORMULATION WITH APPLICATIONS TO LINEAR VISCOELASTIC SLENDER STRUCTURES

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Abstract. We present the analysis of an abstract parameter-dependent mixed variational formulation based on Volterra integrals of second kind. Adapting the classic mixed theory in the Volterra equations setting, we prove the well posedness of the resulting system. Stability and error estimates are derived, where all the estimates are independent of the perturbation parameter. We provide applications of the developed analysis for a viscoelastic Timoshenko beam and a Reissner-Mindlin plate, together with numerical tests.

1. Introduction

Viscoelasticity is a physical property, present in a wide variety of structures and became important after the popularization of polymers. The study of viscoelastic materials, their damping capabilities and behavior in time due to induced stress or temperature changes, is well established and we refer to the studies of Flugge [18], Christensen [13] and Reddy [31] for a rigorous theoretical development.

There are several mathematical models and numerical methods to approximate the solutions of viscoelastic problems. In particular, the finite difference method and the finite element method are the usual numerical tools that mathematicians and engineers implement in order to compute, with high accuracy, the viscoelastic response of some materials. We refer to [3, 10, 23, 35] as papers that analyze these subjects.

An important subject of study in engineering is the one associated to slender structures. These elements are often modeled by systems of partial differential equations where the thickness is considered on the mode as a parameter (see, for instance, [9]). This parameter takes importance on the elastic response of the structures. Moreover, from the numerical methods point of view, it is well known that when the thickness of some structure is smaller than the rest of the dimensions, difficulties in the convergence of such methods arise, leading to the so-called locking phenomenon. This drawback that arise in elliptic models with no memory terms, can be also found in the context of Volterra type of systems, such as viscoelastic structures.

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ne of the most commonly used mathematical tools to address parameter-dependent problems, are mixed formulations with a perturbation parameter. If λ is this parameter, a mixed formulation of our interest is: Find $(u,p)\in\mathcal{V}\times\mathcal{Q}$ such that

$$\begin{cases} a(u,v)+b(v,p)=\langle f,v\rangle_{\mathcal{V}},\\ b(u,q)-\lambda(p,q)_{\mathcal{Q}}=\langle g,q\rangle_{\mathcal{Q}}, \end{cases}$$

where $a:\mathcal{V}\times\mathcal{V}\to\mathbb{R}$, $b:\mathcal{V}\times\mathcal{Q}\to\mathbb{R}$ are two bilinear forms, $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Q}}$ the duality pairing between the spaces $\mathcal{V}$ and $\mathcal{V}'$ and $\mathcal{Q}$ and $\mathcal{Q}'$, and $(\cdot,\cdot)_{\mathcal{Q}}$ is the corresponding inner product in $\mathcal{Q}$. The theory behind this formulation is rich and extensive for Hilbert spaces, and we refer to [6] as a classic reference about this subject. On the other hand there is the theory of Volterra integrals, which has allowed the study of evolutionary problems where a memory term is present, widely used in viscoelasticity models obtained by means of the Boltzmann superposition principle. In this work we consider both theories in order to study the following mixed formulation with Volterra integral equations:

Problem 1. Given $1\le\ell\le\infty$, $f\in L^{\ell}(\mathcal{J};\mathcal{V}')$ and $g\in L^{\ell}(\mathcal{J};\mathcal{Q}')$, find $(u,p)\in L^{\ell}(\mathcal{J};\mathcal{V}\times\mathcal{Q})$ such that

$$\begin{cases} a(u,v)+b(v,p)=\langle f,v\rangle_{\mathcal{V}}+\displaystyle\int_0^t\left[\widetilde{a}(t,s;(u(s),v))+\widetilde{b}(t,s;(v,p(s)))\right]ds,\\ b(u,q)-\lambda(p,q)_{\mathcal{Q}}=\langle g,q\rangle_{\mathcal{Q}}, \end{cases}$$

for all $(v,q)\in\mathcal{V}\times\mathcal{Q}$.

For this model, $\mathcal{V}$ and $\mathcal{Q}$ represent suitable spaces satisfying some prescribed boundary conditions, $\mathcal{J}:=[0,T]$, $T\in(0,\infty)$, represents a period of observation. The new introduced bilinear forms $\widetilde{a}:\mathcal{T}\times\mathcal{V}\times\mathcal{V}$ and $\widetilde{b}:\mathcal{T}\times\mathcal{V}\times\mathcal{Q}$, with $\mathcal{T}:=\{s\in\mathcal{J}\quad|\quad 0\le s\le t,\quad t\in\mathcal{J}\}$, take the role of memory term contributions. The specific form of these bilinear forms is directly related to the Volterra kernel, and may acquire a nonlinear nature in some cases. However, our work is based on the application of this model to linear viscoelasticity, so there will be a *similarity* between them and the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ (see for example [33] for a primal formulation and Section 2 below).

In this formulation, the typical features of both theories such as ellipticity of $a(\cdot,\cdot)$ in the kernel of $b(\cdot,\cdot)$, an inf-sup condition of $b(\cdot,\cdot)$, L^1 continuity of the Volterra kernel, or long term behavior, can be considered. Our work will focus on characterizing estimates of this model in order to be applied in slender structures models. Hence, we consider a short-term behavior, together with standard hypotheses of elastic problems.

In our paper we prove that the analysis of a mixed formulation of a viscoelastic system with the proposed abstract framework, is capable of be performed in order to obtain stability as in the classic elasticity approach, but with the addition of ℓ-regularity in time. To make matters precise, our model can be considered as an extension to viscoelasticity of the well-known regular and penalty type cases in elasticity. Moreover, the numerical experiments in this work show that the spatial convergence is not affected by a particular choice of ℓ. Note that since our abstract framework is quasi-static, the initial conditions are not needed.

Also, our formulation is different from those proposed in other works (for example [15, 34, 24]) that consider mixed formulations with memory. This is because the presence of the perturbation parameter, whose presence changes the rules of the game. Here, one of the main difficulties will be to obtain a well-posed problem, such that the estimates do not deteriorate as the perturbation parameter becomes small. The aforementioned problem in viscoelasticity is to be expected, and hence, numerical methods can be affected by the locking phenomenon as well. This will be the major contribution of our work, whose versatility can be extended to the study of systems with dissipative or inertia terms, depending on the phenomenon under study.

The paper is organized as follows: In Section 2 we introduce an abstract setting in which we will operate through our paper. We also provide stability bounds, independent of the perturbation parameter. In Section 3 we present the analysis of a semi-discrete scheme of the continuous problem that takes into account several common discrete spatial assumptions such as conforming finite element spaces or semi-discrete inf-sup conditions. In Section 4 we apply our developed abstract framework to two well-known models of slender structrures: a Timoshenko beam and a Reissner-Mindlin plate. Both models present a parameter associated to the thickness that, as is well established, produces locking for standard numerical schemes, as the finite element method. For the case of the Timoshenko beam, we prove that our abstract framework fits a viscoelastic beam model which, together with spatial regularity assumptions, allows us to derive error estimates, where the involved constants are uniform on the thickness. In the case of the Reissner-Mindlin plate, we prove that under slight modifications of the proposed model, the results for the stability of the viscoelastic plate are similar to those obtained in elasticity. A finite element discretization based on Durán-Liberman elements, allows us to obtain error estimates, independent of the thickness. For each of the models we report numerical experiments in order to confirm the good performance and accuracy of the proposed mixed finite element methods for our viscoelastic models.

2. The abstract setting

We endow the Hilbert spaces $\mathcal{V}, \mathcal{Q}$ with norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{Q}}$, respectively. We denote by $\mathcal{L}(\mathcal{V};\mathcal{Q})$ the space of continuous linear mappings from $\mathcal{V}$ to $\mathcal{Q}$. Also, we denote by $\mathcal{V}'$ and $\mathcal{Q}'$ their corresponding dual spaces endowed with norms $\|\cdot\|_{\mathcal{V}'}$ and $\|\cdot\|_{\mathcal{Q}'}$. For every Banach space $\mathcal{B}$ and every time interval $[0,t]$, we denote by $L^\ell(0,t;\mathcal{B})$ the space of maps $\mathfrak{w}:[0,t]\to\mathcal{B}$ with norm, for $1\leq \ell<\infty$,

$$\|\mathfrak{w}\|_{L^\ell(0,t;\mathcal{B})} := \left(\int_0^t \|\mathfrak{w}\|_{\mathcal{B}}^\ell\right)^{1/\ell},$$

with the obvious modification for $\ell=\infty$.

The integral equations are formulated in such a way that the spatial components are analyzed using results of mixed formulations. Here, and in the forthcoming sections, we will omit the time dependence of the solutions and test functions outside the time integral unless necessary in the arguments.

We recall that λ is a small parameter such that $0<\lambda\leq\lambda_{\max}$. It is important to remark that in real applications this parameter is allowed to be significantly small.

Let $\mathfrak{R}_{\mathcal{Q}}:\mathcal{Q}\to\mathcal{Q}'$ be the Riesz operator. Let us denote by $\mathbb{A}:\mathcal{V}\to\mathcal{V}'$ and $\mathbb{B}:\mathcal{V}\to\mathcal{Q}'$ the corresponding induced linear operators from the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively and $\mathbb{B}^*$ the adjoint operator of $\mathbb{B}$. In the forthcoming analysis we denote the induced linear of $\widetilde{a}(\cdot,\cdot)$ and $\widetilde{b}(\cdot,\cdot)$ by $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{B}}$, respectively.

Moreover, we assume the existence of functions $\phi_a : \mathcal{J} \to \mathbb{R}$ and $\phi_b : \mathcal{J} \to \mathbb{R}$, such that (see for example [33, Section 1])

$$\widetilde{\mathbb{A}}(t,s) := \phi_a(t-s)\mathbb{A}, \qquad \widetilde{\mathbb{B}}(t,s) := \phi_b(t-s)\mathbb{B}, \qquad \widetilde{\mathbb{B}}^*(t,s) := \phi_b(t-s)\mathbb{B}^*. \tag{2.1}$$

The system of operator equations associated to Problem 1 is the following:

Problem 2. Given $f \in L^\ell(\mathcal{J};\mathcal{V}')$ and $g \in L^\ell(\mathcal{J};\mathcal{Q}')$, find $(u,p) \in L^\ell(\mathcal{J};\mathcal{V}\times\mathcal{Q})$ such that

$$\begin{cases} \mathbb{A}u(t) + \mathbb{B}^*p(t) = f(t) + \displaystyle\int_0^t \left[\widetilde{\mathbb{A}}(t,s)u(s) + \widetilde{\mathbb{B}}^*(t,s)p(s)\right] ds, \\ \mathbb{B}u(t) - \lambda\mathfrak{R}_{\mathcal{Q}}p(t) = g(t). \end{cases}$$

Now we will introduce a series of hypotheses that are required to prove our results. Some of the results are classic in the mixed formulations literature [6].

Assumption 2.1. Given $\ell \in [1,\infty]$, we assume that:

i.) The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric, positive semi-definite and continuous on $\mathcal{V}$ and $\mathcal{V}\times\mathcal{Q}$, respectively, i.e.,
$$|a(v,w)| \le \|\mathbb{A}\|_{\mathcal{L}(\mathcal{V};\mathcal{V}')}\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}} \equiv C\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}},$$
$$|b(v,q)| \le \|\mathbb{B}\|_{\mathcal{L}(\mathcal{V};\mathcal{Q}')}\|v\|_{\mathcal{V}}\|q\|_{\mathcal{Q}} \equiv C\|v\|_{\mathcal{V}}\|q\|_{\mathcal{Q}},$$
for all $w,v \in \mathcal{V}$. Also, we have that
$$\langle\mathbb{A}w,v\rangle_{\mathcal{V}} = \langle w,\mathbb{A}v\rangle_{\mathcal{V}} = a(w,v) \quad \forall w,v \in \mathcal{V}.$$
$$\langle\mathbb{B}v,q\rangle_{\mathcal{Q}} = \langle v,\mathbb{B}^*q\rangle_{\mathcal{V}} = b(v,q) \quad \forall v \in \mathcal{V}, \forall q \in \mathcal{Q}.$$
For the operator $\mathbb{B}$, we set $\mathcal{K} := \ker\mathbb{B} \subset \mathcal{V}$.

ii.) The operators $\widetilde{\mathbb{A}}(t,s)$ and $\widetilde{\mathbb{B}}^*(t,s)$ are similar to $\mathbb{A}$ and $\mathbb{B}^*$, respectively, in the sense that
$$|\widetilde{a}(t,s,(w,v))| \le C\phi_a(t-s)\,\|w\|_{\mathcal{V}}\|v\|_{\mathcal{V}},$$
$$|\widetilde{b}(t,s,(v,q))| \le C\phi_b(t-s)\,\|v\|_{\mathcal{V}}\|q\|_{\mathcal{Q}},$$
a.e in $\mathcal{T}$, for all $v,w \in \mathcal{V}$ and for all $q \in \mathcal{Q}$, where $\phi_a,\phi_b \in L^1(\mathcal{J};[0,\infty))$ are given functions.

iii.) We consider that $f \in L^\ell(\mathcal{J};\mathcal{V}')$ and $g \in L^\ell(\mathcal{J};\mathcal{Q}')$ are continuous in the sense that
$$|\langle f,v\rangle_{\mathcal{V}}| \le \|f\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} \ \ \forall v \in \mathcal{V},$$
$$|\langle g,q\rangle_{\mathcal{Q}}| \le \|g\|_{\mathcal{Q}'}\|q\|_{\mathcal{Q}} \ \ \forall q \in \mathcal{Q},$$
a.e. in $\mathcal{J}$, for all $v \in \mathcal{V}$ and for all $q \in \mathcal{Q}$.

Through all our paper, C denotes a strictly positive constant, depending on the spatial stability constants such as the continuity, ellipticity or inf-sup constants, and the observation time T, but always independent of the perturbation parameter λ and the given functions f and g.

In order to derive the main result of the present section, we recall some properties on Volterra equations which are necessary to obtain the stability of the solutions of Problem 1 (see [19, 20] for instance).

Definition 1 (Laplace-type convolution)**.** Let m and n be two integrable functions over $\mathcal{J}$. We denote the Laplace-type convolution between m and n by

$$(m * n)(t) := \int_0^t m(t-\tau)n(\tau)d\tau.$$

In the following we provide a lemma that will be widely used through our work since it will allow us to establish estimates between L^1 and L^ℓ functions, for $\ell \in [1,\infty]$. The proof can be found in [33].

Lemma 2.2. Let $m \in L^1(\mathcal{J})$ and $n \in L^\ell(\mathcal{J})$ for some $\ell \in [1,\infty]$. Then, the convolution $m * n$ belongs to $\in L^\ell(\mathcal{J})$ and the estimate

$$\|m * n\|_{L^\ell(0,t)} \leq \|m\|_{L^1(0,t)}\|n\|_{L^\ell(0,t)},$$

holds, for all $t \in \mathcal{J}$. If $\ell = \infty$, then $m * n$ is a bounded uniformly continuous function on $\mathcal{J}$.

From [19] we have that viscoelastic solids are characterized according to the long time behavior. This is contained in the following definition [33, Definition 6].

Definition 2 (Viscoelastic solid)**.** We say that a function $\phi \in L^1(\mathcal{J};[0,\infty))$ characterize a viscoelastic solid if there exist a constant $C > 0$ such that

$$0 \leq \left(1 - \int_0^t \phi(\tau)d\tau\right)^{-1} \leq C.$$

Observe that, since ϕ is non-negative a.e. in $\mathcal{J}$, we also have $\|\phi\|_{L^1(0,t)} \leq 1$.

Let us define the semi-norm $|v|_a^2 := a(v,v)$, which implies that

$$|v|_a^2 \leq \|a\|\,\|v\|_{\mathcal{V}}^2, \tag{2.2}$$

due to the continuity of $a(\cdot,\cdot)$. Also, from [6, Lemma 4.2.1] we have

$$a(u,v) \leq |u|_a\,|v|_a, \qquad \|\mathbb{A}u\|_{\mathcal{V}'}^2 \leq \|a\|\,|u|_a^2. \tag{2.3}$$

We are now in position to prove the main result of this section.

Theorem 2.3. *Assume that $\mathbb{B}$ is surjective and that Assumption 2.1 holds. Also assume $a(\cdot,\cdot)$ is strongly coercive in $\mathcal{K}$, i.e., there exist α_0 such that*

$$a(v_0,v_0) \geq \alpha_0\|v_0\|_{\mathcal{V}}^2 \quad \forall v_0 \in \mathcal{K}.$$

If ϕ_a and ϕ_b characterize a viscoelastic solid, then, for every $f \in L^\ell(\mathcal{J};\mathcal{V}')$ and for every $g \in L^\ell(\mathcal{J};\mathcal{Q}')$, there exists a unique solution to Problem 1. Moreover, there exists a positive constant C, uniform respect to λ, such that

$$\|u\|_{L^\ell(0,t;\mathcal{V})} + \|p\|_{L^\ell(0,t;\mathcal{Q})} \leq C\big(\|f\|_{L^\ell(0,t;\mathcal{V}')} + \|g\|_{L^\ell(0,t;\mathcal{Q}')}\big).$$

Proof. Since $f,g \in L^\ell(\mathcal{J};\mathcal{V}' \times \mathcal{Q}')$, the existence an uniqueness of a solution to Problem 1 follows from the application of the Volterra integral equations theory on Hilbert spaces (see, for instance [26, 25, 16]).

We divide the rest of the proof in two cases, and then sum the estimates by linearity. The first case corresponds when $(u,p) \in L^\ell(\mathcal{J};\mathcal{V}\times\mathcal{Q})$ solves

$$\begin{cases} a(u,v) + b(v,p) = \langle f,v\rangle_{\mathcal{V}} + \displaystyle\int_0^t \left[\widetilde{a}(t,s;(u(s),v)) + \widetilde{b}(t,s;(v,p(s)))\right] ds, \\ b(u,q) - \lambda(p,q)_{\mathcal{Q}} = 0, \end{cases} \tag{2.4}$$

for all $(v,q) \in \mathcal{V}\times\mathcal{Q}$, or equivalently in its operator form,

$$\begin{cases} \mathbb{A}u + \mathbb{B}^*p = f + \displaystyle\int_0^t \left[\widetilde{\mathbb{A}}(t,s)u(s) + \widetilde{\mathbb{B}}^*(t,s)p(s)\right] ds, \\ \mathbb{B}u - \lambda\mathfrak{R}_{\mathcal{Q}}p = 0, \end{cases} \tag{2.5}$$

a.e. in $\mathcal{J}$. Following [6, Chapter 4], we set $\widetilde{u} := \mathbb{L}_{\mathbb{B}}\lambda\mathfrak{R}_{\mathcal{Q}}p$, so that $\mathbb{B}u = \mathbb{B}\widetilde{u} = \lambda\mathfrak{R}_{\mathcal{Q}}p$. Defining $u_0 := u-\widetilde{u}$, we have that $u_0 \in \mathcal{K}$. Now we set $v = \widetilde{u}(t)$ in the first equation of (2.4) and since $p = \mathfrak{R}_{\mathcal{Q}}^{-1}\lambda^{-1}\mathbb{B}u$ we have

$$\begin{aligned} a(u,\widetilde{u}) + b\big(\widetilde{u},\mathfrak{R}_{\mathcal{Q}}^{-1}\lambda^{-1}\mathbb{B}u\big) &= \langle f,\widetilde{u}\rangle_{\mathcal{V}} \\ &+ \int_0^t \left[\widetilde{a}(t,s;\big((u(s),\widetilde{u})\big) + \widetilde{b}\big(t,s;(\widetilde{u},\mathfrak{R}_{\mathcal{Q}}^{-1}\lambda^{-1}\mathbb{B}u(s))\big)\right] ds. \end{aligned}$$

Since $\mathbb{B}\widetilde{u} = \mathbb{B}u$, from the equation above and (2.1) it follows that

$$\begin{aligned} \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'}^2 &= \langle f,\widetilde{u}\rangle_{\mathcal{V}} - a(u,\widetilde{u}) \\ &+ \int_0^t \left[\widetilde{a}\big(t,s;(u(s),\widetilde{u})\big) + \phi_b(t-s)\lambda^{-1}\langle\mathbb{B}u(s),\mathfrak{R}_{\mathcal{Q}}^{-1}\mathbb{B}u\rangle_{\mathcal{Q}}\right] ds. \end{aligned} \tag{2.6}$$

Now we estimate $-a(u,\widetilde{u})$. To do this task, first we observe that (2.2), (2.3), and the splitting $u = \widetilde{u} + u_0$ yields to

$$-a(u,\widetilde{u}) = a(\widetilde{u}+u_0,\widetilde{u}) = -a(\widetilde{u},\widetilde{u}) - a(u_0,\widetilde{u}) \le -|\widetilde{u}|_a^2 + |\widetilde{u}|_a|u_0|_a. \tag{2.7}$$

On the other hand, testing the first equation in (2.4) with $v = u_0(t)$ gives

$$a(u,u_0) = \langle f,u_0\rangle_{\mathcal{V}} + \int_0^t \widetilde{a}(t,s;(u(s),u_0))ds,$$

and then, from (2.2) and (2.3) we have that

$$\begin{aligned} |u_0|_a^2 = a(u_0,u_0) &= a(u,u_0) - a(\widetilde{u},u_0) \\ &\le \|f\|_{\mathcal{V}'}\|u_0\|_{\mathcal{V}} + |u_0|_a|\widetilde{u}|_a + (\phi_a * |u|_a)(t)\,|u_0|_a, \end{aligned} \tag{2.8}$$

where we have used the ellipticity in the kernel of $\mathbb{B}$. Then, (2.8) is reduced to

$$|u_0|_a \le C\|f\|_{\mathcal{V}'} + |\widetilde{u}|_a + (\phi_a * |u|_a)(t). \tag{2.9}$$

Inserting (2.9) in (2.7) yields to

$$-a(u,\widetilde{u}) \le C\|f\|_{\mathcal{V}'}|\widetilde{u}|_a + (\phi_a * |u|_a)(t)\,|\widetilde{u}|_a,$$

and replacing this inequality in (2.6) we obtain

$$\begin{aligned} \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'}^2 &\le C\|f\|_{\mathcal{V}'}\|\widetilde{u}\|_{\mathcal{V}} \\ &+ C(\phi_a * \|u\|_{\mathcal{V}})(t)\|\widetilde{u}\|_{\mathcal{V}} + (\phi_b * \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'})(t)\|\mathbb{B}u\|_{\mathcal{Q}'}. \end{aligned} \tag{2.10}$$

From the inf-sup condition of $\mathbb{B}$, we have that $C\|\widetilde{u}\|_{\mathcal{V}} \le \|\mathbb{B}\widetilde{u}\|_{\mathcal{Q}'} = \|\mathbb{B}u\|_{\mathcal{Q}'}$. Hence, by using the split $u = \tilde{u} + u_0$ and the inf-sup condition of $\mathbb{B}$, inequality (2.10) becomes

$$\begin{aligned} \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'} &\le C\|f\|_{\mathcal{V}'} + C(\phi_a * \|u_0\|_{\mathcal{V}})(t) \\ &+ C(\phi_a * \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'})(t) + (\phi_b * \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'})(t). \end{aligned} \tag{2.11}$$

Now we will estimate $(\phi_a * \|u_0\|_{\mathcal{V}})(t)$. Using the $\mathcal{K}$-ellipticity of $a(\cdot,\cdot)$ in (2.9), along with (2.2), and the split $u = \widetilde{u} + u_0$ we obtain

$$\|u_0\|_{\mathcal{V}} \le C\left[\|f\|_{\mathcal{V}'} + \|\widetilde{u}\|_{\mathcal{V}} + (\phi_a * \|\widetilde{u}\|_{\mathcal{V}})(t)\right] + C(\phi_a * \|u_0\|_{\mathcal{V}})(t).$$

From Gronwall's lemma we have that

$$\|u_0\|_{\mathcal{V}} \le \widetilde{m}(t) + \int_0^t \chi(s)\widetilde{m}(s)\exp\left(\int_s^t \chi(\tau)d\tau\right) ds, \tag{2.12}$$

where

$$\widetilde{m}(t) := C\left[\|f\|_{\mathcal{V}'} + \|\widetilde{u}\|_{\mathcal{V}} + (\phi_a * \|\widetilde{u}\|_{\mathcal{V}})(t)\right],$$

and

$$\chi(s) = C\phi_a(t-s).$$

Since $\|\phi_a\|_{L^1(0,t)} \leq 1$ and ϕ_a is non-negative a.e. in $\mathcal{J}$, we obtain

$$\exp\left(\int_s^t \chi(\tau)d\tau\right) = \exp\left(C\int_s^t \phi_a(t-\tau)d\tau\right) \leq \exp\left(C\int_0^t \phi_a(t-\tau)d\tau\right) \leq C.$$

Hence, it follows that

$$\int_0^t \chi(s)\widetilde{m}(s)\exp\left(\int_s^t \chi(\tau)d\tau\right)\,ds \leq C(\phi_a * \widetilde{m})(t).$$

On the other hand, observe that

$$(\phi_a * \widetilde{m})(t) = C\Big[(\phi_a * \|f\|_{\mathcal{V}'})(t) + (\phi_a * \|\widetilde{u}\|_{\mathcal{V}})(t) + (\phi_a * (\phi_a * \|\widetilde{u}\|_{\mathcal{V}}))(t)\Big].$$

Since $\phi_a \in L^1(\mathcal{J};[0,\infty))$, from Fubini's theorem and Lemma 2.2 we have that $(\phi_a * \phi_a)(t) \leq \|\phi_a\|^2_{L^1(0,t)}$. Hence, we have

$$(\phi_a * \widetilde{m})(t) \leq C(\phi_a * \|f\|_{\mathcal{V}'})(t) + C\left[(\phi_a * \|\widetilde{u}\|_{\mathcal{V}})(t) + \int_0^t \|\widetilde{u}(s)\|_{\mathcal{V}}\,ds\right].$$

Then, from (2.12) we obtain that

(2.13)
$$\|u_0\|_{\mathcal{V}} \leq C\left\{\widetilde{m}(t) + C(\phi_a * \|f\|_{\mathcal{V}'})(t) + C\left[(\phi_a * \|\widetilde{u}\|_{\mathcal{V}})(t) + \int_0^t \|\widetilde{u}(s)\|_{\mathcal{V}}\,ds\right]\right\}.$$

Taking the convolution with ϕ_a in (2.13) and using the inf-sup condition of $\mathbb{B}$ together with Lemma 2.2, yields to

(2.14)
$$(\phi_a * \|u_0\|_{\mathcal{V}}) \leq C\left[(\phi_a * \|f\|_{\mathcal{V}'})(t) + (\phi_a * \|\mathbb{B}u\|_{\mathcal{Q}'})(t) + \int_0^t \|[f(s)\|_{\mathcal{V}'} + \|\mathbb{B}u(s)\|_{\mathcal{Q}'}]ds\right].$$

Inserting this inequality in (2.11) gives

$$\lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'} \leq C\big[\|f\|_{\mathcal{V}'} + ([1+\phi_a] * \|f\|_{\mathcal{V}'})(t)\big] + C([1+\phi_a+\phi_b] * \lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'})(t).$$

Hence, observing that ϕ_b also characterizes a viscoelastic solid we obtain

$$\begin{aligned}\lambda^{-1}\|\mathbb{B}u\|_{\mathcal{Q}'} &\leq C\big[\|f\|_{\mathcal{V}'} + ([1+\phi_a] * \|f\|_{\mathcal{V}'})(t)\big] \\ &\quad + C\big([1+\phi_a+\phi_b] * [\|f\|_{\mathcal{V}'} + ([1+\phi_a] * \|f\|_{\mathcal{V}'})]\big)(t).\end{aligned} \tag{2.15}$$

From the fact that $C\|\widetilde{u}\|_{\mathcal{V}} \leq \|\mathbb{B}u\|_{\mathcal{Q}'}$, we take the $L^\ell(0,t)$ norm in (2.15) and apply Lemma 2.2 to obtain the following estimate for $\widetilde{u}$

$$\|\widetilde{u}\|_{L^\ell(0,t;\mathcal{V})} \leq C\lambda\|f\|_{L^\ell(0,t;\mathcal{V}')}. \tag{2.16}$$

To estimate u_0, we take the $L^\ell(0,t)$ norm in (2.13) and use Lemma 2.2 to obtain

$$\|u_0\|_{L^\ell(0,t;\mathcal{V})} \leq C(1+\lambda)\|f\|_{L^\ell(0,t;\mathcal{V}')}. \tag{2.17}$$

The estimate for u follows directly from the triangle inequality

$$\|u\|_{L^\ell(0,t;\mathcal{V})} \leq C\|f\|_{L^\ell(0,t;\mathcal{V}')}. \tag{2.18}$$

On the other hand, from the second equation in (2.5) together with (2.15) and Lemma 2.2, we derive the following estimate for p,

$$\|p\|_{L^\ell(0,t;\mathcal{Q})} = \lambda^{-1}\|\mathbb{B}u\|_{L^\ell(0,t;\mathcal{Q}')} \leq C\|f\|_{L^\ell(0,t;\mathcal{V}')}.$$

For the second case, we assume that u, p and g are such that the following system is satisfied

$$\begin{cases} a(u,v) + b(v,p) = \displaystyle\int_0^t \left[\widetilde{a}(t,s;(u(s),v)) + \widetilde{b}(t,s;(v,p(s)))\right] ds, \\ b(u,q) - \lambda(p,q)_{\mathcal{Q}} = \langle g, q\rangle_{\mathcal{Q}}, \end{cases}$$

for all $(v,q) \in \mathcal{V}\times\mathcal{Q}$. Notice that in operator form, problem above reads, a.e in $\mathcal{J}$, as follows

$$\begin{cases} \mathbb{A}u + \mathbb{B}^*p = \displaystyle\int_0^t \left[\widetilde{\mathbb{A}}(t,s)u(s) + \widetilde{\mathbb{B}}^*(t,s)p(s)\right] ds, \\ \mathbb{B}u - \lambda\mathfrak{R}_{\mathcal{Q}}p = g, \end{cases} \tag{2.19}$$

Since $f = 0$, we take norms in the first equation of Problem 2 and use the boundedness of the linear operators and the Volterra kernels, in order to obtain

$$\|\mathbb{A}u + \mathbb{B}^*p\|_{\mathcal{V}'} \leq C\int_0^t \|\mathbb{A}u(s) + \mathbb{B}^*p(s)\|_{\mathcal{V}'}ds.$$

Then, from Gronwall's lemma we obtain that $\mathbb{A}u + \mathbb{B}^*p = 0$, or equivalently,

$$a(u,v) + b(v,p) = 0, \qquad \forall v \in \mathcal{V}.$$

Hence, we resort to [6, Theorem 4.3.2] in order to obtain the remaining bounds:

$$\|u\|_{L^\ell(0,t;\mathcal{V})} \leq C\|g\|_{L^\ell(0,t;\mathcal{Q}')} \quad \text{and} \quad \|p\|_{L^\ell(0,t;\mathcal{Q})} \leq C\|g\|_{L^\ell(0,t;\mathcal{Q}')}.$$

Finally, by gathering the bounds for u and p with respect to f and g gives the desired estimate. This concludes the proof. □

Observe that all the constants involved are uniform respect to the perturbation parameter λ as it happens on the non-viscoelastic mixed formulations. This is an important fact, since in real applications, like the analysis of numerical methods for slender structures such as Timoshenko beams, Reissner-Mindlin plates, among others, the thickness parameter is the one that leads to the so called locking phenomenon. Now, when these structures admit viscoelastic properties, these results hold as well. For this reason, if λ represents the thickness parameter of some particular structure, Theorem 2.3 states that all the constants will be uniform with respect to it in our mixed viscoleastic formulation.

3. Semi-discrete problem

In this section we are interested in a discretization by conforming finite element spaces for Problem 1. With this goal in mind, and under suitable assumptions on discrete spaces, we adapt the classic theory for mixed formulations for our viscoelastic approach.

3.1. **Semi-discrete abstract analysis.** The goal of the present section is to analyze the semi-discrete counterpart of the proposed mixed problems and obtain a priori error estimates. Here, we consider the necessary hypotheses for the existence and uniqueness of semi-discrete solutions such as ellipticity in the kernel and the discrete inf-sup condition (see for instance [2, 32] for further details related to the existence of semi-discrete solutions of Volterra equations of the second kind).

Hence, this section will be focused on the derivation of error estimates which are characterized by having constants that do not deteriorate when the parameter λ goes to zero.

Let us introduce the following assumption.

Assumption 3.1. Assume that there exist two finite dimensional spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$ such that $\mathcal{V}_h \subset \mathcal{V}$ and $\mathcal{Q}_h \subset \mathcal{Q}$. Together with the continuous space kernels $\mathcal{K}$ and $\mathcal{H}$, we consider the discrete counterparts

$$\mathcal{K}_h := \left\{ v_h \in \mathcal{V}_h \ : \ b(v_h, q_h) = 0, \ \forall q_h \in \mathcal{Q}_h \right\},$$

such that there exist constants α_d, β_d, both positive and independent of h and λ, such that

$$a(v_h^0, v_h^0) \geq \alpha_d \|v_h^0\|_{\mathcal{V}}^2 \ \ \forall v_h^0 \in \mathcal{K}_h, \qquad \sup_{v \in \mathcal{V}_h} \frac{b(v_h, q_h)}{\|v_h\|_{\mathcal{V}_h}} \geq \beta_d \|q_h\|_{\mathcal{Q}_h}, \ \ \forall q_h \in \mathcal{Q}_h.$$

Let us remark that through this section, we are considering families of conforming finite elements.

For simplicity, we define the corresponding errors as follows

$$\mathsf{e}_u := u_h - u = \xi_u - \eta_u, \qquad \mathsf{e}_p := p_h - p = \xi_p - \eta_p,$$

where $\xi_u := u_h - u_I$, $\xi_p := p_h - p_I$, $\eta_u = u - u_I$, and $\eta_p = p - p_I$. Here, $u_I \in \mathcal{V}_h$ and $p_I \in \mathcal{Q}_h$ represent general interpolations of u and p, respectively (see for instace [6, Chapter 5] or [29, Chapter 4.]).

In what follows we analyze the semi-discretization of the perturbed mixed formulation analyzed in the previous section. The following problem corresponds to the semi-discretized version of Problem 1.

Problem 3. Find $(u_h, p_h) \in L^\ell(\mathcal{J}; \mathcal{V}_h \times \mathcal{Q}_h)$ such that

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle_{\mathcal{V}} + \displaystyle\int_0^t \left[\widetilde{a}(t, s; (u_h(s), v_h)) + \widetilde{b}(t, s; (v_h, p_h(s))) \right] ds, \\ b(u_h, q_h) - \lambda (p_h, q_h)_{\mathcal{Q}} = \langle g, q_h \rangle_{\mathcal{Q}}, \end{cases}$$

for all $(v, q) \in \mathcal{V}_h \times \mathcal{Q}_h$.

The existence and uniqueness, as well as the discrete stability estimate of the semi-discrete solution, follows from Assumption 3.1 and Theorem 2.3. Hence, we obtain the following system, from subtracting Problem 1 and Problem 3:

$$(3.1) \qquad \begin{cases} a(\mathsf{e}_u, v_h) + b(v_h, \mathsf{e}_p) = \displaystyle\int_0^t \left[\widetilde{a}(t, s; (\mathsf{e}_u(s), v_h)) + \widetilde{b}(t, s; (v_h, \mathsf{e}_p(s))) \right] ds, \\ b(\mathsf{e}_u, q_h) - \lambda (\mathsf{e}_p, q_h)_{\mathcal{Q}} = 0, \end{cases}$$

for all $(v,q)\in\mathcal{V}_h\times\mathcal{Q}_h$. Then, from the linearity of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the history bilinear forms, we rewrite the problem above as follows: Find $(\xi_u,\xi_p)\in L^\ell(\mathcal{J};\mathcal{V}_h\times\mathcal{Q}_h)$ such that

(3.2)
$$\begin{cases} a(\xi_u,v_h)+b(v_h,\xi_p)=\langle\mathcal{F},v_h\rangle_{\mathcal{V}}+\displaystyle\int_0^t\left[\widetilde{a}(t,s;(\xi_u(s),v_h))+\widetilde{b}(t,s;(v_h,\xi_p(s)))\right]ds,\\ b(\xi_u,q_h)-\lambda(\xi_p,q_h)_{\mathcal{Q}}=\langle\mathcal{G},q_h\rangle_{\mathcal{Q}},\end{cases}$$

for all $(v,q)\in\mathcal{V}_h\times\mathcal{Q}_h$, where

(3.3)
$$\langle\mathcal{F},v_h\rangle_{\mathcal{V}}=a(\eta_u,v_h)+b(v_h,\eta_p)-\int_0^t\left[\widetilde{a}(t,s;(\eta_u(s),v_h)+\widetilde{b}(t,s;(v_h,\eta_p(s)))\right]ds,$$

(3.4)
$$\langle\mathcal{G},q_h\rangle_{\mathcal{Q}}=b(\eta_u,q_h)-\lambda(\eta_p,q_h)_{\mathcal{Q}}.$$

Then, from Theorem 2.3 we have the following result.

Theorem 3.2. *Together with Assumption 3.1, assume that (u,p) is the unique solution of Problem 1 and let (u_h,p_h) be the unique solution of Problem 3. Then, there exists a constant $C>0$, uniform with respect to λ, such that*

$$\|u_h-u\|_{L^\ell(0,t;\mathcal{V})}+\|p_h-p\|_{L^\ell(0,t;\mathcal{Q})}\leq C\left(\inf_{v_h\in\mathcal{V}_h}\|u-v\|_{L^\ell(0,t;\mathcal{V})}+\inf_{q_h\in\mathcal{Q}_h}\|p-q\|_{L^\ell(0,t;\mathcal{Q})}\right).$$

Proof. Let $u_I\in L^\ell(\mathcal{J};\mathcal{V}_h)$ and $p_I\in L^\ell(\mathcal{J};\mathcal{Q}_h)$. Then, applying Theorem 2.3 in (3.2), we have

$$\|\xi_u\|_{L^\ell(0,t;\mathcal{V})}+\|\xi_p\|_{L^\ell(0,t;\mathcal{Q})}\leq C\big(\|\mathcal{F}\|_{L^\ell(0,t;\mathcal{V}')}+\|\mathcal{G}\|_{L^\ell(0,t;\mathcal{Q}')}\big),$$

On the other hand, estimating (3.3) and (3.4) gives

$$\begin{aligned}\|\mathcal{F}\|_{L^\ell(0,t;\mathcal{V}')}&\leq C\big[\|u-u_I\|_{L^\ell(0,t;\mathcal{V})}+\|p-p_I\|_{L^\ell(0,t;\mathcal{Q})}\big],\\ \|\mathcal{G}\|_{L^\ell(0,t;\mathcal{Q}')}&\leq C\,\|u-u_I\|_{L^\ell(0,t;\mathcal{V})}+\lambda\|p-p_I\|_{L^\ell(0,t;\mathcal{Q})},\end{aligned}$$

hence, from the triangle inequality we obtain

$$\begin{aligned}\|\mathsf{e}_u\|_{L^\ell(0,t;\mathcal{V})}\leq\|\xi_u-\eta_u\|_{L^\ell(0,t;\mathcal{V})}&\leq\|\xi_u\|_{L^\ell(0,t;\mathcal{V})}+\|\eta_u\|_{L^\ell(0,t;\mathcal{V})}\\ &\leq C\big[\|\eta_u\|_{L^\ell(0,t;\mathcal{V})}+\|\eta_p\|_{L^\ell(0,t;\mathcal{Q})}\big],\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\mathsf{e}_p\|_{L^\ell(0,t;\mathcal{Q})}\leq\|\xi_p-\eta_p\|_{L^\ell(0,t;\mathcal{Q})}&\leq\|\xi_p\|_{L^\ell(0,t;\mathcal{Q})}+\|\eta_p\|_{L^\ell(0,t;\mathcal{Q})}\\ &\leq C\big[\|\eta_u\|_{L^\ell(0,t;\mathcal{V})}+\|\eta_p\|_{L^\ell(0,t;\mathcal{Q})}\big],\end{aligned}$$

We conclude the proof by taking the infimum over all u_I and p_I. □

3.2. **Error estimates in weaker norms.** Now we include some additional estimates using a duality argument in the sense of the Volterra theory (See [33] for instance). Let us consider two spaces $\mathcal{V}_-$ and $\mathcal{Q}_-$, where the "-" index indicates a less regular spaces than $\mathcal{V}$ and $\mathcal{Q}$, respectively, satisfying the following dense inclusions

$$\mathcal{V}\hookrightarrow\mathcal{V}_-\quad\text{and}\quad\mathcal{Q}\hookrightarrow\mathcal{Q}_-.\tag{3.5}$$

Our aim is to estimate $\|u-u_h\|_{L^\ell(0,t;\mathcal{V}_-)}$ and $\|p-p_h\|_{L^\ell(0,t;\mathcal{Q}_-)}$. To accomplish this task, we define

$$\mathcal{V}_+' := (\mathcal{V}_-)', \quad \mathcal{Q}_+' := (\mathcal{Q}_-)'.$$

The "+" suggest that we have more regular dual spaces. On the other hand, the inclusions provided in (3.5) imply

$$\mathcal{V}_+' \hookrightarrow \mathcal{V}', \quad \mathcal{Q}_+' \hookrightarrow \mathcal{Q}'.$$

Let $\mathcal{V}_{++}$ and $\mathcal{Q}_{++}$ be two spaces, where the double subindex "++" denotes more regular spaces that $\mathcal{V}$ and $\mathcal{Q}$, respectively, satisfying the inclusions

$$\mathcal{V}_{++} \hookrightarrow \mathcal{V}, \quad \mathcal{Q}_{++} \hookrightarrow \mathcal{Q}.$$

Now we introduce the dual-backward mixed formulation of Problem 1. To do this task, let r denote the Hölder conjugate index of ℓ. Then, for any $\tau \in \mathcal{J}$ and for any $(f_+, g_+) \in L^r(0,\tau;\mathcal{V}_+' \times \mathcal{Q}_+')$, we consider the dual problem: find $(w,m) \in L^r(0,\tau;\mathcal{V}\times\mathcal{Q})$ such that a.e. in $[0,\tau]$,

(3.6)
$$\begin{cases} a(v,w) + b(v,m) = \langle f_+, v\rangle_{\mathcal{V}_+'\times\mathcal{V}} + \displaystyle\int_t^\tau \Big[\widetilde{a}(s,t;v,w(s)) + \widetilde{b}(s,t;v,m(s))\Big]ds, \\ b(w,q) - \lambda(q,m)_{\mathcal{Q}} = \langle g_+, q\rangle_{\mathcal{Q}_+'\times\mathcal{Q}}, \end{cases}$$

for all $(v,q) \in \mathcal{V}\times\mathcal{Q}$.

Setting $\xi := \tau - t$, $\chi = \tau - s$, and defining

$$\overline{w}(\cdot) := w(\tau-\cdot), \quad \overline{m}(\cdot) := m(\tau-\cdot), \quad \overline{f}_+(\xi) := f_+(\tau-\xi), \quad \overline{g}_+(\xi) := g_+(\tau-\xi),$$
$$\overline{a}(\xi,\eta,(\cdot,v)) := \widetilde{a}(\tau-\eta,\tau-\xi,(v,\cdot)), \qquad \overline{b}(\xi,\eta,(v,\cdot)) := \widetilde{b}(\tau-\eta,\tau-\xi,(v,\cdot)),$$

it follows that the backward problem can be written in forward form as: Find $(w,m) \in L^r(0,\tau;\mathcal{V}\times\mathcal{Q})$, such that for a.e. $\xi\in[0,\tau]$,

(3.7)
$$\begin{cases} a(\overline{w},v) + b(v,\overline{m}) = \langle \overline{f}_+, v\rangle_{\mathcal{V}_+'\times\mathcal{V}} + \displaystyle\int_0^\xi \Big[\overline{a}(\xi,\eta;(\overline{w}(\eta),v)) + \overline{b}(\xi,\eta;v,\overline{m}(\eta))\Big]d\eta, \\ b(\overline{w},q) - \lambda(\overline{m},q)_{\mathcal{Q}} = \langle \overline{g}_+, q\rangle_{\mathcal{Q}_+'\times\mathcal{Q}}, \end{cases}$$

for all $(v,q) \in \mathcal{V}\times\mathcal{Q}$. Now this is basically the same as Problem 1. Hence, in order to apply all the previous stability and semi-discrete results to this dual problem, we have to guarantee that Assumption 2.1 is satisfied. Assumption 2.1-(i.) is straightforward. Assumption 2.1-(iii.) is satisfied because $\|\overline{f}_+\|_{\mathcal{V}_+'}, \|\overline{q}_+\|_{\mathcal{Q}_+'} \in L^r(0,\tau)$. For Assumption 2.1-(ii.) we observe that

$$|\overline{a}(\xi,\eta;(\overline{w}(\eta),v))| \le C\phi_a\big((\tau-\eta)-(\tau-\xi)\big)\|\overline{w}\|_{\mathcal{V}}\|v\|_{\mathcal{V}} \le C\phi_a(\xi-\eta)\|\overline{w}\|_{\mathcal{V}}\|v\|_{\mathcal{V}}.$$

Similarly, $|\overline{b}(\xi,\eta;v,\overline{\psi}(\eta))| \le C\phi_b(\xi-\eta)\|v\|_{\mathcal{V}}\|\overline{\psi}\|_{\mathcal{Q}}$. Then, Theorem 2.3 guarantees that there exist $C>0$ such that

$$\|w\|_{L^r(0,\tau;\mathcal{V})} + \|m\|_{L^r(0,\tau;\mathcal{Q})} \le C\big(\|f\|_{L^r(0,\tau;\mathcal{V}_+')}\|g\|_{L^r(0,\tau;\mathcal{Q}_+')}\big).$$

In order to simplify the presentation of the material, we denote the errors $w-w_h$ and $m-m_h$ by

$$\mathsf{e}_w := w_h - w, \qquad \mathsf{e}_m := m_h - m,$$

where the dependence on time is omitted if no confusion arises.

Now we are in position to establish our weak norm estimate.

Theorem 3.3. *Under the hypotheses of Theorem 3.2, assume that the solution to the dual problem* (3.6) *belongs to* $L^r(0,\tau;\mathcal{V}_{++}\times\mathcal{Q}_{++})$ *a.e. in* $[0,\tau]$ *and there exists a constant* $C>0$*, independent of* f_+ *and* g_+*, such that*

$$\|w\|_{L^r(0,\tau;\mathcal{V}_{++})}+\|m\|_{L^r(0,\tau;\mathcal{Q}_{++})}\leq C\bigg(\|f_+\|_{L^r(0,\tau;\mathcal{V}_+')}+\|g_+\|_{L^r(0,\tau;\mathcal{Q}_+')}\bigg).$$

Then, there exists a constant C*, independent of* h *and* λ*, such that*

$$\begin{aligned}&\|u-u_h\|_{L^\ell(0,t;\mathcal{V}_-)}+\|p-p_h\|_{L^\ell(0,t;\mathcal{Q}_-)}\\&\qquad\leq C\left(\inf_{v\in\mathcal{V}_h}\|u-v\|_{L^\ell(0,t;\mathcal{V})}+\inf_{q\in\mathcal{Q}_h}\|p-q\|_{L^\ell(0,t;\mathcal{Q})}\right)(l(h)+n(h)),\end{aligned}$$

where

$$\begin{aligned}l(h)&:=\sup_{w\in L^r(0,t;\mathcal{V}_{++})}\inf_{w_h\in L^r(0,\tau;\mathcal{V}_{++})}\frac{\|w-w_h\|_{L^r(0,t;\mathcal{V})}}{\|w\|_{L^r(0,t;\mathcal{V}_{++})}},\\ n(h)&:=\sup_{m\in L^r(0,t;\mathcal{Q}_{++})}\inf_{m_h\in L^r(0,\tau;\mathcal{Q}_{++})}\frac{\|m-m_h\|_{L^r(0,t;\mathcal{Q})}}{\|m\|_{L^r(0,t;\mathcal{Q}_{++})}}.\end{aligned}$$

Moreover, if $l(h)+n(h)\leq Ch$*, then*

$$\begin{aligned}&\|u-u_h\|_{L^\ell(0,t;\mathcal{V}_-)}+\|p-p_h\|_{L^\ell(0,t;\mathcal{Q}_-)}\\&\qquad\leq Ch\left(\inf_{v\in\mathcal{V}_h}\|u-v\|_{L^\ell(0,t;\mathcal{V})}+\inf_{q\in\mathcal{Q}_h}\|p-q\|_{L^\ell(0,t;\mathcal{Q})}\right).\end{aligned}$$

Proof. Taking time dependent test functions $v\in L^\ell(0,\tau;\mathcal{V})$ in the first equation of (3.6), integrating in $[0,\tau]$, and interchanging the order of integration gives,

$$\begin{aligned}\int_0^\tau\langle f_+,v(t)\rangle_{\mathcal{V}_+'\times\mathcal{V}}dt&=\int_0^\tau\bigg\{a\big(v(t),w(t)\big)+b\big(v(t),m(t)\big)\\&\quad-\int_0^t\Big[\widetilde{a}(t,s;(v(s),w(t)))+\widetilde{b}(t,s;(v(s),m(t)))\Big]ds\bigg\}dt.\end{aligned}\tag{3.8}$$

On the other hand, taking $q\in L^\ell(0,\tau;\mathcal{V})$ in the second equation of (3.6) gives

$$\int_0^\tau\langle g_+,q(t)\rangle_{\mathcal{Q}_+'\times\mathcal{Q}}dt=\int_0^\tau\Big[b(w(t),q(t))-\lambda(q(t),m(t))_{\mathcal{Q}}\Big]dt.\tag{3.9}$$

Set $v=u-u_h$ in (3.8) and $q=p-p_h$ in (3.9) in order to obtain

$$\begin{aligned}\int_0^\tau\langle f_+,\mathsf{e}_u(t)\rangle_{\mathcal{V}_+'\times\mathcal{V}}dt&=\int_0^\tau\bigg\{a(\mathsf{e}_u(t),w(t))+b(\mathsf{e}_u(t),m(t))\\&\quad-\int_0^t\Big[\widetilde{a}(t,s;(\mathsf{e}_u(s),w(t))+\widetilde{b}(t,s;(\mathsf{e}_u(s),m(t))\Big]ds\bigg\}dt,\end{aligned}\tag{3.10}$$

and

$$\int_0^\tau\langle g_+,\mathsf{e}_p(t)\rangle_{\mathcal{Q}_+'\times\mathcal{Q}}dt=\int_0^\tau\Big[b(w(t),\mathsf{e}_p(t))-\lambda(\mathsf{e}_p(t),m(t))_{\mathcal{Q}}\Big]dt.\tag{3.11}$$

For $z_1\in L^r(0,\tau)$ we set $f_+(t)=z_1(t)\mathsf{e}_u(t)\|\mathsf{e}_u(t)\|_{\mathcal{V}_-}^{-1}$, then $\|f_+(t)\|_{\mathcal{V}_-}=|z_1(t)|$ and $\|f_+\|_{L^r(0,\tau;\mathcal{V}_+')}=\|z_1\|_{L^r(0,\tau)}$. This implies that $\langle f_+,\mathsf{e}_u\rangle_{\mathcal{V}_+'\times\mathcal{V}_-}=z_1(t)\|\mathsf{e}_u\|_{\mathcal{V}_-}$. This result applied on (3.10) yields to

$$\int_0^\tau z_1(t)\|\mathsf{e}_u(t)\|_{\mathcal{V}_-}\,dt = \int_0^\tau \Big\{a(\mathsf{e}_u(t), w(t)) + b(\mathsf{e}_u(t), m(t)) \\ - \int_0^t \Big[\widetilde{a}(t,s;(\mathsf{e}_u(s), w(t))) + \widetilde{b}(t,s;(\mathsf{e}_u(s), m(t)))\Big]\,ds\Big\}dt.$$

Similarly, for $z_2 \in L^r(0,\tau)$, if $g_+(t) = z_2(t)\mathsf{e}_p(t)\|\mathsf{e}_p(t)\|_{\mathcal{Q}_-}^{-1}$, then we proceed as before to obtain that $\langle g_+, \mathsf{e}_p\rangle_{\mathcal{Q}'_+\times\mathcal{Q}_-} = z_2(t)\|\mathsf{e}_p\|_{\mathcal{Q}_-}$. Replacing this in (3.11), yields to

$$\int_0^\tau z_2(t)\|\mathsf{e}_p(t)\|_{\mathcal{Q}_-}\,dt = \int_0^\tau \Big[b(w(t), \mathsf{e}_p(t)) - \lambda(\mathsf{e}_p(t), m(t))_{\mathcal{Q}}\Big]\,dt.$$

On the other hand from (3.1) and Grönwall's inequality we have that

$$\begin{cases} a(\mathsf{e}_u, w_h) + b(w_h, \mathsf{e}_p) = \displaystyle\int_0^t \Big[\widetilde{a}(t,s;(\mathsf{e}_u(s), w_h)) + \widetilde{b}(t,s;(w_h, \mathsf{e}_p(s)))\Big]\,ds = 0, \\ b(\mathsf{e}_u, m_h) - \lambda(\mathsf{e}_p, m_h)_{\mathcal{Q}} = 0, \end{cases}$$

for all $w_h \in \mathcal{V}_h$ and for all $m_h \in \mathcal{Q}_h$. Hence, using the continuity of the bilinear forms, the viscoelasticity characterization of ϕ_a and ϕ_b, Hölder's inequality, and Lemma 2.2, we obtain

(3.12)

$$\int_0^\tau \Big(z_1(t)\|\mathsf{e}_u\|_{\mathcal{V}_-} + z_2(t)\|\mathsf{e}_p\|_{\mathcal{Q}_-}\Big)\,dt = \int_0^\tau \Big\{a(\mathsf{e}_u, \mathsf{e}_w) + b(\mathsf{e}_u, \mathsf{e}_m) + b(\mathsf{e}_w, \mathsf{e}_p) \\ - \lambda(\mathsf{e}_p, \mathsf{e}_m)_{\mathcal{Q}} - \int_0^t \Big[\widetilde{a}(t,s;(\mathsf{e}_u(s), \mathsf{e}_w)) + \widetilde{b}(t,s;(\mathsf{e}_u(s), \mathsf{e}_m))\Big]\,ds\Big\}dt \\ \leq C\Big(\|\mathsf{e}_u\|_{L^\ell(0,\tau;\mathcal{V})} + \|\mathsf{e}_p\|_{L^\ell(0,\tau;\mathcal{Q})}\Big)\Big(\|\mathsf{e}_w\|_{L^r(0,\tau;\mathcal{V})} + \|\mathsf{e}_m\|_{L^r(0,\tau;\mathcal{Q})}\Big).$$

On the other hand, we have that

$$\|\mathsf{e}_u\|_{L^\ell(0,\tau;\mathcal{V}_-)} = \sup_{z_1}\left\{\left|\int_0^\tau z_1(t)\|\mathsf{e}_u(t)\|_{\mathcal{V}_-}\,dt\right| \;:\; \|z_1\|_{L^r(0,\tau)} = 1\right\},$$

and

$$\|\mathsf{e}_p\|_{L^\ell(0,\tau;\mathcal{Q}_-)} = \sup_{z_2}\left\{\left|\int_0^\tau z_2(t)\|\mathsf{e}_p(t)\|_{\mathcal{Q}_-}\,dt\right| \;:\; \|z_2\|_{L^r(0,\tau)} = 1\right\},$$

for $p \in [1,\infty]$. Therefore, (3.12) becomes

$$\|\mathsf{e}_u\|_{L^\ell(0,\tau;\mathcal{V}_-)} + \|\mathsf{e}_p\|_{L^\ell(0,\tau;\mathcal{Q}_-)} \\ \leq C\Big(\|\mathsf{e}_u\|_{L^\ell(0,\tau;\mathcal{V})} + \|\mathsf{e}_p\|_{L^\ell(0,\tau;\mathcal{Q})}\Big)\Big(\|\mathsf{e}_w\|_{L^r(0,\tau;\mathcal{V})} + \|\mathsf{e}_m\|_{L^r(0,\tau;\mathcal{Q})}\Big). \tag{3.13}$$

Observe that from the definition of $r(h)$ and $n(h)$ we have

$$\inf_{w_h\in\mathcal{V}_h} \|w - w_h\|_{L^r(0,\tau;\mathcal{V})} \leq l(h)\|w\|_{L^r(0,\tau;\mathcal{V}_{++})}, \\ \inf_{m_h\in\mathcal{Q}_h} \|m - m_h\|_{L^r(0,\tau;\mathcal{Q})} \leq n(h)\|m\|_{L^r(0,\tau;\mathcal{Q}_{++})}.$$

Adding the two inequalities above we have

$$\inf_{w_h\in\mathcal{V}_h} \|w - w_h\|_{L^r(0,\tau;\mathcal{V})} + \inf_{m_h\in\mathcal{Q}_h} \|m - m_h\|_{L^r(0,\tau;\mathcal{Q})} \leq C\big[l(h) + n(h)\big],$$

where we have used that $\|f_+\|_{L^r(0,\tau;\mathcal{V}_+')} + \|g_+\|_{L^r(0,\tau;\mathcal{Q}_+')} = 2$. Taking the infimum in (3.13) for w_h and m_h, in $\mathcal{V}_h$ and $\mathcal{Q}_h$, respectively, gives

$$\|\mathsf{e}_u\|_{L^\ell(0,\tau;\mathcal{V}_-)} + \|\mathsf{e}_p\|_{L^\ell(0,\tau;\mathcal{Q}_-)} \leq C\bigg(\|\mathsf{e}_u\|_{L^\ell(0,\tau;\mathcal{V})} + \|\mathsf{e}_p\|_{L^\ell(0,\tau;\mathcal{Q})}\bigg)\big[l(h) + n(h)\big]. \tag{3.14}$$

Since τ is arbitrary, we conclude the proof taking $\tau = t$ and applying Theorem 3.2 to the semi-discrete error estimates for u and p in the right side of (3.14). □

4. Applications to linear viscoelastic slender structures

This section is devoted to the application of the proposed abstract framework in the formulation and analysis of numerical methods for viscoelastic structures. The study will focus on Timoshenko beams and Reissner-Mindlin plates. These are well-known elastic structures for which mixed formulations have been carried out in order to study the numerical locking. A usual constitutive equation for linear isotropic viscoelastic material is of the form:

$$\sigma_{ij}(t) = Q^{ijkl}(0)\varepsilon_{kl}(t) - \int_0^t \dot{Q}^{ijkl}(t-s)\varepsilon_{kl}(s)ds,$$

where $\dot{Q}^{ijkl}(t-s) = dQ^{ijkl}(t-s)/dt$, and Q^{ijkl} is the general fourth order viscoelastic tensor. The present analysis consider bounded creep materials, which yields to

$$\begin{aligned}
\sigma^{ij}(t) &= E(0)\mathbb{Q}_v^{ijkl}\varepsilon_{kl}(t) - \int_0^t \dot{E}(t-s)\mathbb{Q}_v^{ijkl}\varepsilon_{kl}(s)ds \\
\sigma^{i3} &= G(0)\mathbb{D}_v^{i3k3}\varepsilon_{k3} - \int_0^t \dot{G}(t-s)\mathbb{D}_v^{i3k3}\varepsilon_{k3}(s)ds, \qquad \sigma^{33} = 0,
\end{aligned} \tag{4.1}$$

where $\mathbb{Q}_v^{ijkl}$ and $\mathbb{D}_v^{ijkl}$ are unit elastic tensors that account for the shear, membrane and bending contributions of the structure. The functions $E(t)$ and $G(t)$ correspond to the relaxation and shear modulus, respectively. The action of the unit elastic tensors on ε is given by

$$\mathbb{Q}_v^{ijkl}\varepsilon_{kl} := \frac{1}{(1-\nu^2)}[(1-\nu)\varepsilon^{ij} + v\delta_{ij}\varepsilon_{kk}], \qquad \mathbb{D}_v^{i3k3}\varepsilon_{k3} := \frac{k_s}{1+\nu}\varepsilon_{i3},$$

where k_s is the correction factor, and ν is the Poisson ratio. Also, for bounded creep materials, the relaxation and shear modulus can be expressed in terms of a Prony series of order N as

$$E(t) = E_0 + \sum_{i=1}^N E_i e^{-t/\tau_i^E}, \qquad G(t) = G_0 + \sum_{i=1}^N G_i e^{-t/\tau_i^G}.$$

where $E_0 = E(0), G_0 = G(0)$, and τ_i^E, τ_i^G are relaxation times. If we normalize the Prony series such that $E(0) = G(0) = 1$, then we observe that the constitutive relations (4.1) are reduced to

$$\begin{aligned}
\sigma^{ij}(t) &= \mathbb{Q}_v^{ijkl}\varepsilon_{kl}(t) - \int_0^t \dot{E}(t-s)\mathbb{Q}_v^{ijkl}\varepsilon_{kl}(s)ds \\
\sigma^{i3} &= \mathbb{D}_v^{i3k3}\varepsilon_{k3} - \int_0^t \dot{G}(t-s)\mathbb{D}_v^{i3k3}\varepsilon_{k3}(s)ds, \qquad \sigma^{33} = 0,
\end{aligned} \tag{4.2}$$

Hence, we choose $\phi_a(t) = -\dot{E}(t)$ and $\phi_b(t) = -\dot{G}(t)$. It is important to observe that this selection satisfies

$$\|\phi_a(t)\|_{L^1(0,t)} \leq 1, \qquad \|\phi_b(t)\|_{L^1(0,t)} \leq 1,$$

as required in our analysis.

On the other hand, the natural relation between E and G given by $G(t) = E(t)/(2(1+\nu))$, is assumed (see, for instance [11]). Hence, the corresponding characterization functions satisfy $\phi_a(t) = \phi_b(t)$. For other choices of ϕ, like in isotropic linear materials, we refer to [21, Chapter 8].

In what follows, let $\Omega \subset \mathbb{R}^n$, $n \in \{1,2\}$, be an open and convex domain with boundary $\partial\Omega$. We denote by $L^2(\Omega)$ and $H^l(\Omega)$ the usual Lebesgue and Sobolev spaces, with the convention $H^0(\Omega) = L^2(\Omega)$. The spaces are endowed with standard norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^l(\Omega)}$. Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ consisting of functions that vanish in $\partial\Omega$. For $n=2$, we define by $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^2$ the space of Lebesgue measure space of vector functions. We denote by $\boldsymbol{H}^l(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ the vectorial version of $H^l(\Omega)$ and $H_0^1(\Omega)$, respectively.

We denote by DOF the number of degrees of freedom. To measure the errors for several values of ℓ in $L^\ell(\cdot\,;\,\cdot)$, we define

$$\mathtt{e}_{0,\ell}(f) := \|f - f_h\|_{L^\ell(\mathcal{J};L^2(\Omega))} \quad \text{and} \quad \mathtt{e}_{\mathbf{0},\ell}(\boldsymbol{f}) := \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^\ell(\mathcal{J};\boldsymbol{L}^2(\Omega))},$$

for every scalar function f and every vector function $\boldsymbol{f}$, respectively. Moreover, we define the experimental rates of convergence $\mathtt{r}_i(\cdot)$ and $\mathtt{r}_{\boldsymbol{i}}(\cdot)$ as

$$\mathtt{r}_{0,\ell}(\cdot) := \frac{\log\left(\mathtt{e}_{0,\ell}(\cdot)/\mathtt{e}'_{0,\ell}(\cdot)\right)}{\log(h/h')}, \qquad \mathtt{r}_{\mathbf{0},\ell}(\cdot) := \frac{\log\left(\mathtt{e}_{\mathbf{0},\ell}(\cdot)/\mathtt{e}'_{\mathbf{0},\ell}(\cdot)\right)}{\log(h/h')},$$

where $\mathtt{e}_{0,\ell}$ and $\mathtt{e}'_{0,\ell}$ (resp. $\mathtt{e}_{\mathbf{0},\ell}$ and $\mathtt{e}'_{\mathbf{0},\ell}$) denote two consecutive errors and h and h' their corresponding mesh sizes.

It is well known that the trapezoidal rule error is of order 2. Thus, from the semi-discrete error analysis, we have that given a semi-discrete rate of convergence $\mathcal{O}(h^r)$, we expect that the fully discrete error estimates satisfies

$$\mathtt{e}_0(\cdot) \leq C(h^{r+1} + \Delta t^2), \qquad \mathtt{e}_{\mathbf{0}}(\cdot) \leq C(h^r + \Delta t^2), \tag{4.3}$$

where C is independent of the thickness parameter. We then choose the step size such that $\Delta t^2 \ll h^r$, for $r \geq 1$.

4.1. **Timoshenko beam.** We begin with an application of the developed abstract theory to a clamped linear viscoelastic Timoshenko beam. It is well known that, in the non viscoelastic case, the Timoshenko beam system lead to a parameter dependent problem, where the thickness plays the role of deteriorate the standard numerical methods, which in the viscoelastic setting is expectable as well. Now we will check how our abstract framework helps to avoid the locking effect for the viscoelastic mixed formulation of this beam.

Let $\Omega := [0, L]$, where L represents the length of the beam. We consider the space of square-integrable functions $L^2(\Omega)$ with inner product $(u,v) := \int_\Omega u\,v\,dx$, and its induced norm $\|f\|_{L^2(\Omega)} = \sqrt{(f,f)}$.

We introduce the space

$$\mathrm{H} = \left\{ (v\,,\eta\,) \in H_0^1(\Omega) \times H_0^1(\Omega) \right\},$$

endowed with the product space seminorm

$$\|(\eta,w)\|_{\mathrm{H}}^2 := \|\eta'\|_{L^2(\Omega)}^2 + \|w'\|_{L^2(\Omega)}^2,$$

where $\zeta(x)' := d\zeta/dx$.

Under suitable kinematic assumptions, the constitutive equations given in (4.2) allow to obtain a viscoelastic Timoshenko beam (see for example [30]). In our case, the viscoelastic Timoshenko beam model to be analyzed is the following: Given $\ell\in[1,\infty]$, find $(w,\theta)\in L^\ell(\mathcal{J};\mathrm{H})$ such that

(4.4)

$$\begin{aligned}(I(x)\theta',\eta') + k_s\big(A(x)(\theta-w'),\eta-v'\big) = (\widetilde{f},v) + \int_0^t \dot{E}(t-s)(I(x)\theta'(s),\eta')\,ds\\ + k_s\int_0^t \dot{G}(t-s)\big(A(x)(\theta(s)-w'(s)),\eta-v'\big)\,ds,\end{aligned}$$

for all $(v,\eta)\in\mathrm{H}$, where w represents the displacement of the beam, θ represent the rotations, k_s is the correction factor, $E(t)$ is the relaxation modulus, $G(t) := E(t)/2(1+\nu)$ is the shear modulus, ν is the time-independent Poisson ratio, $I(x)$ is the moment of inertia of the cross-section, $A(x)$ is the area of the cross-section and $\widetilde{f}(x,t)$ is an uniform distributed transverse load.

We rescale the formulation (4.4) to identify a family of viscoelastic problems whose limit is well-posed when the thickness of the beam goes to zero, with the following classic non-dimensional parameter, characteristic of the thickness of the beam

$$\varepsilon^2 = \frac{1}{L}\int_\Omega \frac{I(x)}{A(x)L^2}\,dx,$$

which is assumed to be independent of time and is such that $\varepsilon\in(0,\varepsilon_{\max}]$.

Scaling the load as $\widetilde{f}(x,t)=\varepsilon^3 f(x,t)$, with $f(x,t)$ independent of ε, and defining

$$\hat{I}(x) := \frac{I(x)}{\varepsilon^3},\qquad \hat{A}(x) := k_s\frac{A(x)}{\varepsilon},$$

we have that (4.4) is equivalent to the following problem:

Problem 4. Given $f\in L^\ell\big(\mathcal{J};L^2(\Omega)\big)$, find $(\theta,w)\in L^\ell(\mathcal{J};\mathrm{H})$ such that

$$\begin{aligned}(\hat{I}\theta',\eta') + \frac{\varepsilon^{-2}}{2(1+\nu)}\big(\hat{A}(\theta-w'),\eta-v'\big) = (f,v)\\ +\int_0^t \dot{E}(t-s)\left[(\hat{I}\theta'(s),\eta') + \frac{\varepsilon^{-2}}{2(1+\nu)}\big(\hat{A}(\theta(s)-w'(s)),\eta-v'\big)\right]ds,\end{aligned}$$

for all $(v,\eta)\in\mathrm{H}$.

We introduce the unit shear $\gamma\in L^\ell(\mathcal{J};L^2(\Omega))$ as $\gamma := \frac{\varepsilon^{-2}}{2(1+\nu)}\hat{A}(\theta-w')$. Hence, defining $\lambda := 2(1+\nu)\varepsilon^2$, we rewrite Problem 4 as the following mixed formulation:

Problem 5. Find $(\theta,w,\gamma)\in L^\ell(\mathcal{J};\mathrm{H}\times L^2(\Omega))$ such that

$$\begin{cases}(\hat{I}\theta',\eta') + \big(\gamma,\eta-v'\big) = (f,v) + \displaystyle\int_0^t \dot{E}(t-s)\Big[\big(\hat{I}\theta'(s),\eta'\big) + \big(\gamma(s),\eta-v'\big)\Big]ds,\\ (\theta-w',\psi) - \lambda(\gamma/\hat{A},\psi) = 0,\end{cases}$$

for all $(v,\eta)\in\mathrm{H}$ and for all $\psi\in L^2(\Omega)$.

Now we verify that Problem 5 lies in the framework of the abstract setting provided in the previous section. First, note that according to our abstract framework, $\mathcal{V} := \mathrm{H}$, $\mathcal{Q} := L^2(\Omega)$, $u := (\theta, w)$, $v := (\eta, v), p := \gamma, q := \psi$. Then, the bilinear forms $a : \mathrm{H} \times \mathrm{H} \to \mathbb{R}$ and $b : \mathrm{H} \times L^2(\Omega)$, are given by

$$a\big((\theta, w); (\eta, v)\big) := (\hat{I}\theta', \eta'), \quad b\big((\eta, v); \gamma\big) := (\gamma, \eta - v'),$$

for all $(\theta, w), (\eta, v) \in \mathrm{H}$, $\gamma \in L^2(\Omega)$ a.e. in $\mathcal{J}$. It is straightforward that $a(\cdot, \cdot)$ is $\mathcal{K}$-elliptic and $b(\cdot, \cdot)$ satisfies an inf-sup condition (see for example [4, Section 5]). Then, from Theorem 2.3, there exists $C > 0$, uniform in λ, such that

$$\|(\theta, w)\|_{L^\ell(0,t;\mathrm{H})} + \|\gamma\|_{L^\ell(0,t;L^2(\Omega))} \leq C\|f\|_{L^\ell(0,t;L^2(\Omega))}.$$

Finally, we note that using the differential equations satisfied by the solutions of the mixed formulation in the distributional sense (see for example [8, Proposition 3.] for the case of a rod.), the additional regularity result

$$\|\theta\|_{L^\ell(0,t;H^2(\Omega))} + \|w\|_{L^\ell(0,t;H^2(\Omega))} + \|\gamma\|_{L^1(0,t;H^1(\Omega))} \leq C\|f\|_{L^\ell(0,t;L^2(\Omega))}, \tag{4.5}$$

holds.

4.1.1. *Finite element analysis.* Now our task is to analyze a conforming finite element semi-discretization for the beam mixed formulation provided previously. The main goal is to derive error estimates, independent of the thickness parameter. As a starting point, consider a finite partition $\mathscr{T}_h = \{\Omega_i\}_{i=1}^n$ of the computational domain Ω such that $\Omega_i =]x_{i-1}, x_i[$, with length $h_i = x_i - x_{i-1}$, and satisfying $\bigcap_{i=1}^n \Omega_i = \emptyset$ and $\Omega = \bigcup_{i=1}^n \Omega_i$, $i = 1, \ldots, n$. The maximum interval length is denoted by $h = \max_{1\leq i\leq n} h_i$.

The approximations will be based in the following finite element spaces:

$$\mathcal{V}_h := \left\{ v \in H_0^1(\Omega) \ : \ v_{|\Omega_i} \in \mathcal{P}_1(\Omega_i), \, \Omega_i \in \mathscr{T}_h \right\},$$

$$\mathcal{Q}_h := \left\{ q \in L^2(\Omega) \ : \ q_{|\Omega_i} \in \mathcal{P}_0(\Omega_i), \, \Omega_i \in \mathscr{T}_h \right\},$$

where $\mathcal{V}_h$ approximates the displacement and rotations, and the shear stress is approximated with the piecewise constants of $\mathcal{Q}_h$.

We also recall the Lagrange interpolant $\mathcal{L}_h : C(\overline{\Omega}) \to V_h$, and the orthogonal projector Π_h, such that the estimates

$$\|u - \mathcal{L}_h u\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}, \quad \text{and } \|v - \Pi(v)\|_{L^2(\Omega)} \leq Ch|v|_{H^1(\Omega)}. \tag{4.6}$$

hold.

From the above estimates, it follows that the more regular spaces required in the abstract setting are given by

$$\mathcal{V}_{++} := H^2(\Omega) \times H^2(\Omega) \qquad \text{and} \qquad \mathcal{Q}_{++} := H^1(\Omega).$$

Define $\mathrm{H}_h := \mathcal{V}_h \times \mathcal{V}_h$ as a finite element subspace of H. Then, the corresponding semi-discrete counterpart of Problem 5 is given as follows:

Problem 6. Find $(\theta_h, w_h, \gamma_h) \in L^\ell(\mathcal{J}; \mathrm{H}_h \times \mathcal{Q}_h)$ such that

$$\begin{cases} (\hat{I}\theta_h', \eta') + \big(\gamma_h, \eta - v'\big) = (f, v) + \displaystyle\int_0^t \dot{E}(t-s)\Big[\big(\hat{I}\theta_h'(s), \eta'\big) + \big(\gamma_h(s), \eta - v'\big)\Big] ds, \\ (\theta_h - w_h', \psi) - \lambda(\gamma_h/\hat{A}, \psi) = 0, \end{cases}$$

for all $(v, \eta) \in \mathrm{H}_h$ and for all $\psi \in \mathcal{Q}_h$.

Following [4, Section 5], we observe that that the restriction of $a(\cdot,\cdot)$ to H_h satisfies the ellipticity condition in $\mathcal{K}_h$, while the restriction $b(\cdot,\cdot)$ to $\mathrm{H}_h\times\mathcal{Q}_h$ satisfies an inf-sup condition. Thus, applying Theorem 3.2 we obtain that there exists a positive constant C, independent of h and λ, such that

$$
\begin{aligned}
\|(\theta,w)-(\theta_h,w_h)\|_{L^\ell(0,t;\mathrm{H})}&+\|\gamma-\gamma_h\|_{L^\ell(0,t;L^2(\Omega))}\\
&\leq C\left(\inf_{(\eta,v)\in\mathrm{H}_h}\|(\theta,w)-(\eta,v)\|_{L^\ell(0,t;\mathrm{H})}+\inf_{\psi\in\mathcal{Q}_h}\|\gamma-\psi\|_{L^\ell(0,t;L^2(\Omega))}\right).
\end{aligned}
\tag{4.7}
$$

Hence, we have the following convergence rate of the semi-discrete mixed Problem 6.

Proposition 4.1. Let $(\theta,w,\gamma)\in L^\ell(\mathcal{J};\mathbb{H}\times L^2(\Omega))$ and $(\theta_h,w_h,\gamma_h)\in L^\ell(\mathcal{J};\mathrm{H}_h\times\mathcal{Q}_h)$ be the solutions of Problem 5 and Problem 6, respectively. Then, if $f\in L^\ell(\mathcal{J};L^2(\Omega))$, there exists a constant $C>0$, independent of h and λ, such that

$$\|(\theta,w)-(\theta_h,w_h)\|_{L^\ell(0,t;\mathrm{H})}+\|\gamma-\gamma_h\|_{L^\ell(0,t;L^2(\Omega))}\leq Ch\|f\|_{L^\ell(0,t;L^2(\Omega))}.$$

Proof. The proof follows from (4.7), the error estimates (4.5), and estimates (4.6). □

In what follows, we will consider the dual-backward version of Problem 6 to obtain an additional error estimate. Note that the estimate for γ can not be improved since the choice of a space less regular that $L^2(\Omega)$ is not available. Also, note that the abstract setting suggests that $(\mathcal{V}_-)'=\mathcal{V}_+'=L^2(\Omega)$.

Proposition 4.2. Under the assumptions of Proposition 4.1, there exists a constant $C>0$, independent of h and λ, such that

$$\|(\theta,w)-(\theta_h,w_h)\|_{L^\ell(0,t;L^2(\Omega))}\leq Ch^2\|f\|_{L^\ell(0,t;L^2(\Omega))}.$$

Proof. Following the construction of the dual-backward problem (3.6) and using the same variable substitution to write it in forward form as in (3.7), we can construct the corresponding forward form of the dual version of Problem 5 and apply all the abstract results from Section 2 in order to have the additional regularity result

$$\|\beta\|_{L^r(0,\tau;H^2(\Omega))}+\|u\|_{L^r(0,\tau;H^2(\Omega))}+\|\psi\|_{L^r(0,\tau;H^1(\Omega))}\leq C\|f_+\|_{L^r(0,\tau;L^2(\Omega))},$$

a.e. in $[0,\tau]$, for any $\tau\in\mathcal{J}$, where (β,u,ψ) are de dual solutions, with $1/\ell+1/r=1$. Then, from Proposition 4.1, estimates (4.6), and Theorem 3.3 we obtain

$$\|(\theta,w)-(\theta_h,w_h)\|_{L^\ell(0,t;L^2(\Omega))}\leq Ch\big[r(h)+n(h)\big].$$

Using (4.6), we readily obtain that $l(h)+n(h)\leq Ch$. □

4.1.2. *Numerical tests.* Now we report a series of numerical tests in order to confirm our theoretical results. The algorithms have been implemented in FEniCS [1].

In the following, the experimental nature of the relaxation modulus is replaced by assumed values of spring constants and viscosity parameters in order to consider the *Standard Linear Solid model* (SLS). The relaxation and shear modulus for this material are given by the truncated Prony series:

$$E(t)=\frac{k_1k_2}{k_1+k_2}+\left(k_1-\frac{k_1k_2}{k_1+k_2}\right)e^{-t/\tau},\qquad G(t)=\frac{E(t)}{2(1+\nu)},$$

where $\tau=\eta/(k_1+k_2)$. We will consider $\nu=0.35$ in all the experiments.

TABLE 1. Error values and experimental rates of convergence for the transverse displacement w in a fully clamped viscoelastic beam.

		$d = 10^{-1}$m		$d = 10^{-2}$m		$d = 10^{-3}$m	
DOF	h	$\mathtt{e}_{0,1}(w)$	$\mathtt{r}_{0,1}(w)$	$\mathtt{e}_{0,1}(w)$	$\mathtt{r}_{0,1}(w)$	$\mathtt{e}_{0,1}(w)$	$\mathtt{r}_{0,1}(w)$
32	0.4	$5.1759e-09$	$--$	$5.1702e-09$	$--$	$5.1701e-09$	$--$
62	0.2	$1.3055e-09$	1.98	$1.3041e-09$	1.98	$1.3041e-09$	1.98
92	0.13	$5.8118e-10$	1.99	$5.8055e-10$	1.99	$5.8054e-10$	1.99
122	0.10	$3.2709e-10$	1.99	$3.2674e-10$	1.99	$3.2673e-10$	1.99
152	0.08	$2.0939e-10$	1.99	$2.0916e-10$	1.99	$2.0916e-10$	1.99
182	0.06	$1.4542e-10$	1.99	$1.4527e-10$	1.99	$1.4526e-10$	1.99
DOF	h	$\mathtt{e}_{0,2}(w)$	$\mathtt{r}_{0,2}(w)$	$\mathtt{e}_{0,2}(w)$	$\mathtt{r}_{0,2}(w)$	$\mathtt{e}_{0,2}(w)$	$\mathtt{r}_{0,2}(w)$
32	0.4	$1.7033e-09$	$--$	$1.7015e-09$	$--$	$1.7014e-09$	$--$
62	0.2	$4.2965e-10$	1.98	$4.2918e-10$	1.98	$4.2917e-10$	1.98
92	0.13	$1.9126e-10$	1.99	$1.9105e-10$	1.99	$1.9105e-10$	1.99
122	0.10	$1.0764e-10$	1.99	$1.0753e-10$	1.99	$1.0752e-10$	1.99
152	0.08	$6.8911e-11$	1.99	$6.8836e-11$	1.99	$6.8835e-11$	1.99
182	0.06	$4.7860e-11$	1.99	$4.7808e-11$	1.99	$4.7808e-11$	1.99
DOF	h	$\mathtt{e}_{0,\infty}(w)$	$\mathtt{r}_{0,\infty}(w)$	$\mathtt{e}_{0,\infty}(w)$	$\mathtt{r}_{0,\infty}(w)$	$\mathtt{e}_{0,\infty}(w)$	$\mathtt{r}_{0,\infty}(w)$
32	0.4	$1.4790e-12$	$--$	$1.4774e-12$	$--$	$1.4774e-12$	$--$
62	0.2	$3.7307e-13$	1.98	$3.7267e-13$	1.98	$3.7266e-13$	1.98
92	0.13	$1.6608e-13$	1.99	$1.6590e-13$	1.99	$1.6589e-13$	1.99
122	0.10	$9.3472e-14$	1.99	$9.3371e-14$	1.99	$9.3370e-14$	1.99
152	0.08	$5.9836e-14$	1.99	$5.9771e-14$	1.99	$5.9771e-14$	1.99
182	0.06	$4.1558e-14$	1.99	$4.1512e-14$	1.99	$4.1512e-14$	1.99

We consider the physical parameters considered in [28]. More precisely, we consider an homogeneous rectangular beam of length $L = 4\,m$, with base $b = 0.08\,m$ and thickness d. The corresponding moment of inertia is $I = 0.08\,d^3/12\text{m}^4$ and the cross section area is $A = 0.08\,d\,\text{m}^2$. We set the thickness parameter as $\varepsilon^2 = I/AL^2$.

On the other hand, the creep load for the beam is $q(t) = 8\,H(t)\,\text{N/m}$. This case considers the SLS parameters $k_1 = 9.8 \times 10^7\,\text{N/m}^2$, $k_2 = 2.44 \times 10^7\,\text{N/m}^2$ and $\eta = 2.74 \cdot 10^8\,\text{N}\cdot\,\text{s/m}^2$. The observation time is $10\,s$ with step size $\Delta t = 0.002$. The quasi-static analytical solution is obtained by means of the corresponding principle [27].

Tables 1 and 2 report the experimental error when $\ell = 1, 2, \infty$, for the transverse displacement w and the rotation θ. Although the measured error for each mesh size is different between the computed norms, we observe that our method recovers the predicted convergence rates for the implemented finite elements. This, together with the fact that the number of DOF's considered is not large, confirms the locking-free nature of the proposed method. We end the test by depicting a comparison between the maximum deflection of w and w_h when $d = 0.001$m in Figure 1. It notes that the method predicts accurately the viscoelastic behavior of the structure, compared with the exact creep compliance.

4.2. **Reissner-Mindlin plate.** We complete the applications section applying our results for a Reissner-Mindlin plate. As it happens in the Timoshenko beam model, the Reissner-Mindlin plate model depends strongly on the thickness of the structure, leading to the locking phenomenon for certain numerical methods. In order to avoid

TABLE 2. Error values and experimental rate of convergence of the rotation θ in a fully clamped viscoelastic beam.

		$d = 10^{-1}$m		$d = 10^{-2}$m		$d = 10^{-3}$m	
DOF	h	$\mathtt{e}_{0,1}(\theta)$	$\mathtt{r}_{0,1}(\theta)$	$\mathtt{e}_{0,1}(\theta)$	$\mathtt{r}_{0,1}(\theta)$	$\mathtt{e}_{0,1}(\theta)$	$\mathtt{r}_{0,1}(\theta)$
32	0.4	$3.6416e-09$	$--$	$3.6416e-09$	$--$	$3.6416e-10$	$--$
62	0.2	$9.1367e-10$	1.99	$9.1367e-10$	1.99	$9.1367e-10$	1.99
92	0.13	$4.0634e-10$	1.99	$4.0634e-10$	1.99	$4.0634e-11$	1.99
122	0.10	$2.2861e-10$	1.99	$2.2861e-10$	1.99	$2.2861e-11$	1.99
152	0.08	$1.4632e-10$	1.99	$1.4632e-10$	1.99	$1.4632e-11$	1.99
182	0.06	$1.0161e-10$	1.99	$1.0161e-10$	1.99	$1.0161e-11$	1.99
DOF	h	$\mathtt{e}_{0,2}(\theta)$	$\mathtt{r}_{0,2}(\theta)$	$\mathtt{e}_{0,2}(\theta)$	$\mathtt{r}_{0,2}(\theta)$	$\mathtt{e}_{0,2}(\theta)$	$\mathtt{r}_{0,2}(\theta)$
32	0.4	$1.1984e-09$	$--$	$1.1984e-09$	$--$	$1.1984e-09$	$--$
62	0.2	$3.0068e-10$	1.99	$3.0068e-10$	1.99	$3.0068e-10$	1.99
92	0.13	$1.3372e-10$	1.99	$1.3372e-10$	1.99	$1.3372e-10$	1.99
122	0.10	$7.5237e-11$	1.99	$7.5237e-11$	1.99	$7.5237e-11$	1.99
152	0.08	$4.8155e-11$	1.99	$4.8155e-11$	1.99	$4.8155e-11$	1.99
182	0.06	$3.3442e-11$	1.99	$3.3442e-11$	1.99	$3.3442e-11$	1.99
DOF	h	$\mathtt{e}_{0,\infty}(\theta)$	$\mathtt{r}_{0,\infty}(\theta)$	$\mathtt{e}_{0,\infty}(\theta)$	$\mathtt{r}_{0,\infty}(\theta)$	$\mathtt{e}_{0,\infty}(\theta)$	$\mathtt{r}_{0,\infty}(\theta)$
32	0.4	$1.0406e-12$	$--$	$1.0406e-12$	$--$	$1.0406e-12$	$--$
62	0.2	$2.6109e-13$	1.99	$2.6109e-13$	1.99	$2.6109e-13$	1.99
92	0.13	$1.1611e-13$	1.99	$1.1611e-13$	1.99	$1.1611e-13$	1.99
122	0.10	$6.5329e-14$	1.99	$6.5329e-14$	1.99	$6.5329e-14$	1.99
152	0.08	$4.1814e-14$	1.99	$4.1814e-14$	1.99	$4.1814e-14$	1.99
182	0.06	$2.9038e-14$	1.99	$2.9038e-14$	1.99	$2.9038e-14$	1.99

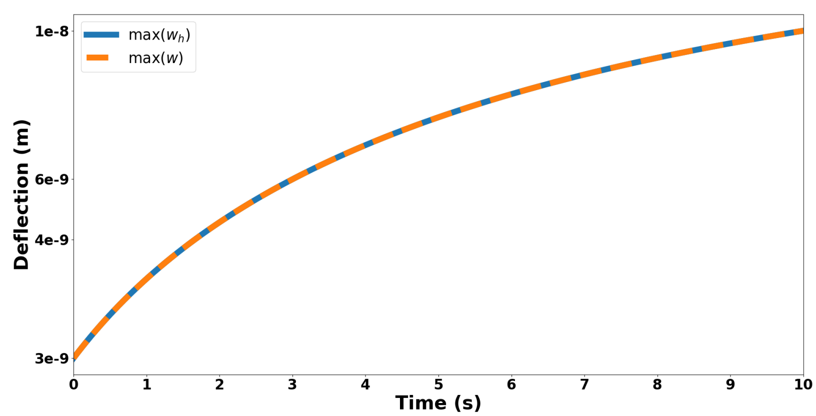

FIGURE 1. Comparison between exact and discrete maximum deflections, w and w_h, in the viscoelastic Timoshenko beam, where $d = 10^{-3}$m and $\Delta t = 0.002$.

this drawback, for instance, the techniques using MITC elements (see for example [17]) are well established.

For our purposes, we consider that the plate is clamped and, under this boundary condition and classic assumptions for the structure, we show that our abstract framework fits for this model, and therefore the theoretical convergence rates provided in the numerical approximation are locking-free.

Let us recall the following standard definitions

$$\operatorname{div}\boldsymbol{\eta} := \partial_1\eta_1 + \partial_2\eta_2, \quad \operatorname{rot}\boldsymbol{\eta} := \partial_1\eta_2 - \partial_2\eta_1, \qquad \nabla v := (\partial_1 v, \partial_2 v)^{\mathrm{t}}$$
$$\operatorname{curl} v := (\partial_2 v, -\partial_1 v)^{\mathrm{t}}, \quad \mathbf{div}\,\boldsymbol{\tau} := \begin{pmatrix} \partial_1\tau_{11} + \partial_2\tau_{12} \\ \partial_1\tau_{21} + \partial_2\tau_{22} \end{pmatrix}, \quad \nabla\boldsymbol{\eta} := \begin{pmatrix} \partial_1\eta_1 & \partial_2\eta_1 \\ \partial_1\eta_2 & \partial_2\eta_2 \end{pmatrix},$$

where t denotes the transpose operator.

For $0 < d \leq 1$, let $\Omega \times (-\frac{d}{2}, \frac{d}{2})$ be the region occupied by the plate, where $\Omega \subset \mathbb{R}^2$ is an open and convex domain with Lipschitz boundary $\partial\Omega$. We denote the inner product in L^2 for tensor, vector and scalar functions by $(\cdot,\cdot)$.

From the constitutive relations (4.2) and inspired by [17], we propose the following viscoelastic Reissner-Mindlin plate system

$$\begin{aligned}
-\mathbf{div}\,\mathbb{C}_v\boldsymbol{\varepsilon}(\boldsymbol{\theta}) - \boldsymbol{\gamma} &= -\boldsymbol{f} - \int_0^t \dot{E}(t-s)\big\{\mathbf{div}\,\mathbb{C}_v\boldsymbol{\varepsilon}(\boldsymbol{\theta}(s)) + \boldsymbol{\gamma}(s)\big\}ds, && \text{in } \Omega,\\
-\operatorname{div}\boldsymbol{\gamma} &= g - \int_0^t \dot{E}(t-s)\operatorname{div}\boldsymbol{\gamma}(s)ds, && \text{in } \Omega,\\
\boldsymbol{\gamma} &= \frac{\kappa_v}{d^2}(\nabla w - \boldsymbol{\theta}), && \text{in } \Omega,\\
w = 0, \quad \boldsymbol{\theta} &= 0, && \text{on } \partial\Omega,
\end{aligned} \tag{4.8}$$

where $\kappa_v = k_s/2(1+\nu)$, g is the scaled distributed transverse load, $\boldsymbol{f}$ is a volume density load, and $\mathbb{C}_v$ is a unit elastic modulus, whose action is given by

$$\mathbb{C}_\nu\boldsymbol{\tau} =:= \frac{1}{12\,(1-\nu^2)}[(1-\nu)\boldsymbol{\tau} + \nu\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}], \quad \boldsymbol{\tau} \in L^2(\Omega)^{2\times 2}.$$

Now we introduce the mixed Volterra formulation for the Reissner-Mindlin plate.

Problem 7. Given $\boldsymbol{f}, g \in L^\ell(\mathcal{J}; \boldsymbol{L}^2(\Omega) \times L^2(\Omega))$, find $(\boldsymbol{\theta}, w, \gamma) \in L^\ell(\mathcal{J}; \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega))$ such that

$$\begin{cases}
a(\boldsymbol{\theta},\boldsymbol{\eta}) + b((\boldsymbol{\eta}, v), \boldsymbol{\gamma}) = L(\boldsymbol{\eta}, v) + \displaystyle\int_0^t \dot{E}(t-s)[a(\boldsymbol{\theta}(s),\boldsymbol{\eta}) + b((\boldsymbol{\eta}, v), \boldsymbol{\gamma}(s))]ds,\\
b((\boldsymbol{\theta}, w), \boldsymbol{q}) - \dfrac{d^2}{\kappa_\nu}(\boldsymbol{\gamma}, \boldsymbol{q}) = 0,
\end{cases}$$

for all $(\boldsymbol{\eta}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ and for all $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)$, where $L(\boldsymbol{\eta}, v) := (g, v) - (\boldsymbol{f}, \boldsymbol{\eta})$.

Here, the bilinear forms $a : \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \to \mathbb{R}$ and $b : \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ are defined by

$$a(\boldsymbol{\theta},\boldsymbol{\eta}) := (\mathbb{C}_v\boldsymbol{\varepsilon}(\boldsymbol{\theta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})), \qquad b((\boldsymbol{\eta}, v), q) := (\boldsymbol{q}, \nabla v - \eta).$$

Observe that the bilinear form $a(\cdot,\cdot)$ is elliptic due to Korn's inequality. Also, our framework suggests that $\mathcal{V} := \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ and $\mathcal{Q} := \boldsymbol{L}^2(\Omega)$. On the other hand, $b(\cdot,\cdot)$ does not satisfy an inf-sup condition in $\mathcal{Q}$ since the associated linear operator $\mathbb{B} : \mathcal{V} \to \mathcal{Q}'$, $\mathbb{B} : (\boldsymbol{\eta}, \psi) \to (\nabla\psi - \boldsymbol{\eta})$ is not surjective. According to [6, Remark 10.4.4], one must consider $\mathcal{Q} = \boldsymbol{H}_0(\operatorname{rot};\Omega)$ in order to satisfy an inf-sup condition, but that implies that the problem becomes into a *singular perturbation problem*, whose analysis is not covered with the present abstract framework. However, an application of a suitable Helmholtz decomposition leads to a system that fits in our theoretical results.

To make matters precise, following [7, Proposition 2.3], we write the unit shear strain tensor as $\boldsymbol{\gamma} = \frac{\kappa_\nu}{d^2}(\nabla w - \beta) = \nabla m + \operatorname{curl} p$, for some $m \in L^\ell(\mathcal{J}; H_0^1(\Omega))$ and $p \in L^\ell(\mathcal{J}; \hat{H}^1(\Omega))$, where the hat represents the subspace of zero mean value, i.e., $\int_\Omega p = 0$. With this definition, we follow [5] and use the strong formulation (4.8) in order to state the following viscoelastic Reissner-Mindlin plate model.

Problem 8. Given $g \in L^\ell(\mathcal{J}; L^2(\Omega))$ and $\boldsymbol{f} \in L^\ell(\mathcal{J}; \boldsymbol{L}^2(\Omega))$, find $(m, p, \boldsymbol{\theta}, w) \in L^\ell(\mathcal{J}; H_0^1(\Omega) \times \hat{H}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)^2 \times H_0^1(\Omega))$ such that

$$(\nabla m, \nabla \mu) = (g, \mu) + \int_0^t \dot{E}(t-s)(\nabla m(s), \nabla \mu)\, ds, \tag{4.9}$$

$$\begin{aligned} a(\boldsymbol{\theta}, \boldsymbol{\eta}) - (\operatorname{curl} p, \boldsymbol{\eta}) &= (\nabla m, \boldsymbol{\eta}) - (\boldsymbol{f}, \boldsymbol{\eta}) \\ &+ \int_0^t \dot{E}(t-s)\Big[a(\boldsymbol{\theta}(s), \boldsymbol{\eta}) - (\operatorname{curl} p(s), \boldsymbol{\eta}) - (\nabla m(s), \boldsymbol{\eta})\Big]\, ds, \end{aligned} \tag{4.10}$$

$$-(\boldsymbol{\theta}, \operatorname{curl} q) - \frac{d^2}{\kappa_\nu}(\operatorname{curl} p, \operatorname{curl} q) = 0, \tag{4.11}$$

$$(\nabla w, \nabla v) = \left(\frac{d^2}{\kappa_\nu}\nabla m + \boldsymbol{\theta}, \nabla v\right), \tag{4.12}$$

for all $(\mu, \boldsymbol{\eta}, q, v) \in (H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \hat{H}^1(\Omega) \times H_0^1(\Omega))$.

Regarding to Problem 8 , we note that (4.9) is decoupled from the rest and corresponds to a Laplacian type Volterra equation [33]. Then, if we solve this problem and obtain m, the solutions β and p are obtained by solving the (4.10)–(4.11). Once this is done, equation (4.12) gives w.

Moreover, for $0 < d \leq 1$, the system formed by the second and third equations corresponds to a mixed system of Volterra of perturbed type, that fits in the abstract framework of our paper. Notice that if $d = 0$, we have a non perturbed system, whose existence and uniqueness follows in a similar fashion as Theorem 2.3.

It is easy to check that Problems 7 and 8 are equivalent. On the other hand, the existence and uniqueness of $(m, p, \boldsymbol{\theta}) \in L^\ell(\mathcal{J}; H_0^1(\Omega) \times \hat{H}^1(\Omega), \boldsymbol{H}_0^1(\Omega))$ for the equations (4.9) and (4.10)-(4.11) is straightforward since we have $\boldsymbol{f}, g \in L^\ell(\mathcal{J}; \boldsymbol{H}^{-1} \times H^{-1}(\Omega))$. The existence of w is is a direct consequence of the Lax-Milgram lemma when is applied to (4.12). Hence, we obtain the following stability result for Problem 8, by which we obtain, respectively, the stability for Problem 7.

Theorem 4.3. *Let Ω be a convex polygonal domain. For any $0 < d \leq 1$, $g \in L^\ell(\mathcal{J}; H^{-1}(\Omega))$ and $\boldsymbol{f} \in L^\ell(\mathcal{J}; \boldsymbol{H}^{-1}(\Omega))$, there exists a unique quadruple $(m, p, \boldsymbol{\theta}, w) \in L^\ell(\mathcal{J}; H_0^1(\Omega) \times \hat{H}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega))$ solving Problem 8. If $\boldsymbol{f} \in L^\ell(\mathcal{J}; \boldsymbol{L}^2(\Omega))$, then there exists C independent of $\boldsymbol{f}$, g, and the thickness parameter d, such that*

$$\begin{aligned} &\|m\|_{L^\ell(0,t;H^1(\Omega))} + \|\boldsymbol{\theta}\|_{L^\ell(0,t;\boldsymbol{H}^2(\Omega))} + \|p\|_{L^\ell(0,t;H^1(\Omega))} + d\|p\|_{L^\ell(0,t;H^2(\Omega))} \\ &+ \|\boldsymbol{\gamma}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + \|w\|_{L^\ell(0,t;H^1(\Omega))} \leq C\bigg(\|\boldsymbol{f}\|_{L^\ell(0,t;L^2(\Omega))} + \|g\|_{L^\ell(0,t;H^{-1}(\Omega))}\bigg). \end{aligned} \tag{4.13}$$

If also $g \in L^\ell(0,t;L^2(\Omega))$, *then*

$$\begin{aligned}(4.14)\quad &\|m\|_{L^\ell(0,t;H^2(\Omega))} + \|w\|_{L^\ell(0,t;H^2(\Omega))} + d\|\boldsymbol{\gamma}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))} \\ &+ \|\operatorname{div}\boldsymbol{\gamma}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} \leq C\Bigg(\|\boldsymbol{f}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + \|g\|_{L^\ell(0,t;L^2(\Omega))}\Bigg).\end{aligned}$$

Proof. The required regularities in are obtained resorting to Theorem 2.3 and [17, Theorem 5.1], together with the results from [26, 25, 16], the regularities of $\boldsymbol{f}$ and g, Gronwall's lemma, interpolation properties and integration on $(0,t)$.

□

4.2.1. *Finite element analysis.* The following analysis is inspired by [14] and [17, Section 6], together with the semi-discrete techniques that we propose. Let $\mathscr{T}_h$ be a regular family of triangulation of Ω. Let $\boldsymbol{H}_h, W_h, \boldsymbol{\Gamma}_h$ be finite element spaces associated with $\mathscr{T}_h$ such that

$$\boldsymbol{H}_h \subset \boldsymbol{H}_0^1(\Omega), \qquad W_h \subset H_0^1(\Omega), \qquad \boldsymbol{\Gamma}_h \subset \boldsymbol{L}^2(\Omega).$$

For efficient locking-free methods, the relation $\nabla W_h \subset \boldsymbol{\Gamma}_h$ is assumed. Let $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ be an interpolation operator mapping $\boldsymbol{H}_0^1(\Omega)$ to $\boldsymbol{\Gamma}_h$. Then, the corresponding finite element discretization of the viscoelastic Reissner-Mindlin plate is as follows.

Problem 9. Find $(\boldsymbol{\theta}, w, \gamma) \in L^\ell(\mathcal{J};\boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega))$ such that given $\boldsymbol{f}, g \in L^\ell(\mathcal{J};\boldsymbol{L}^2(\Omega) \times L^2(\Omega))$, we have

$$\begin{cases} a(\boldsymbol{\theta}_h,\boldsymbol{\eta}) + b((\boldsymbol{\eta},v),\boldsymbol{\gamma}_h) = L(\boldsymbol{\eta},v) + \displaystyle\int_0^t \dot{E}(t-s)[a(\boldsymbol{\theta}_h(s),\boldsymbol{\eta}) + b((\boldsymbol{\eta},v),\boldsymbol{\gamma}_h(s))]ds, \\ b((\boldsymbol{\theta},w_h),\boldsymbol{q}) - \dfrac{d^2}{\kappa_\nu}(\boldsymbol{\gamma}_h,\boldsymbol{q}) = 0, \end{cases}$$

for all $(\boldsymbol{\eta}, v) \in \boldsymbol{H}_h \times W_h$ and for all $\boldsymbol{q} \in \boldsymbol{\Gamma}_h(\Omega)$.

The bilinear forms for this case are given by:

$$a(\boldsymbol{\theta}_h,\boldsymbol{\eta}) := (\mathbb{C}_v\boldsymbol{\varepsilon}(\boldsymbol{\theta}_h),\boldsymbol{\varepsilon}(\boldsymbol{\eta})), \qquad b((\boldsymbol{\eta},v),q) := (\boldsymbol{q},\nabla v - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta}).$$

The presence of $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ in the formulation leads to a slightly different, but complementary, argument to the one used in Section 3.1. Before we present the following result, we remark several approximation assumptions from [17, Section 7], specific to the finite spaces used and the operator $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$, which will be necessary to obtain convergence results.

$$(4.15)\qquad \|\boldsymbol{\eta} - \boldsymbol{\Pi}^{\gamma}\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \leq Ch\|\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)}, \quad \boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega).$$

Also, for all $\zeta \in \boldsymbol{M}_k$, where $\boldsymbol{M}_k$ denotes the space of discontinuous piecewise polynomials of degree less or equal than k, we also define $k_0 \geq -1$ as the greatest integer k for which

$$(4.16)\qquad (\boldsymbol{\eta} - \boldsymbol{\Pi}^{\gamma}\boldsymbol{\eta}, \zeta) = 0, \qquad \forall \zeta \in \boldsymbol{M}_k.$$

Since the above relation is satisfied with $k = 1$, we let $\boldsymbol{\Pi}^0$ denotes the $\boldsymbol{L}^2$ projection into $\boldsymbol{M}_{k_0}$.

Now we are in position to present the first approximation result, which represents an extension to viscoelastic plate of the results presented in [14, 17]. Let us define

$$\mathsf{e}_{\boldsymbol{\gamma}} := \boldsymbol{\gamma}_h - \boldsymbol{\gamma} = \xi_{\boldsymbol{\gamma}} - \eta_{\boldsymbol{\gamma}}, \qquad \mathsf{e}_{\boldsymbol{\theta}} := \boldsymbol{\theta}_h - \boldsymbol{\theta} = \xi_{\boldsymbol{\theta}} - \eta_{\boldsymbol{\theta}}, \qquad \mathsf{e}_w := w_h - w = \xi_w - \eta_w,$$

where $\xi_{\boldsymbol{\gamma}}=\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I$, $\xi_{\boldsymbol{\theta}}=\boldsymbol{\theta}_h-\boldsymbol{\theta}^I$, $\xi_w=w_h-w^I$, $\eta_{\boldsymbol{\gamma}}=\boldsymbol{\gamma}-\boldsymbol{\gamma}^I$, $\eta_{\boldsymbol{\theta}}=\boldsymbol{\theta}-\boldsymbol{\theta}^I$ and $\eta_w=w-w^I$. Here, $\boldsymbol{\theta}^I\in\boldsymbol{H}_h$ and $\boldsymbol{\gamma}^I\in\boldsymbol{\Gamma}_h$ represent general interpolations of $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$, respectively.

Theorem 4.4. *For $\theta^I\in\boldsymbol{H}_h, w^I\in W_h$ arbitrary, we define $\boldsymbol{\gamma}^I=\kappa_\nu d^{-2}(\nabla w^I-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I)\in\boldsymbol{\Gamma}_h$. Then*

$$\begin{aligned}&\|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))}+d\|\mathsf{e}_{\boldsymbol{\gamma}}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))}\\&\qquad\le C(\|\eta_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))}+d\|\eta_{\boldsymbol{\gamma}}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))}+h\|\boldsymbol{\gamma}-\boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))}),\end{aligned}$$

where the positive constant C is independent of d.

Proof. Adding and subtracting $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta}$ in the first equation of Problem 7 we have

$$\begin{aligned}a(\boldsymbol{\theta},\boldsymbol{\eta})+(\gamma,\nabla v-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta})&=(g,v)-(\boldsymbol{f},\boldsymbol{\eta})-(\boldsymbol{\Gamma},\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta}-\boldsymbol{\eta})\\&\quad+\int_0^t\dot{E}(t-s)\Big[a(\boldsymbol{\theta}(s),\boldsymbol{\eta})+(\boldsymbol{\gamma}(s),\nabla v-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta})+(\boldsymbol{\gamma}(s),\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta}-\boldsymbol{\eta})\Big]ds,\end{aligned}$$

for all $\boldsymbol{\eta}\in\boldsymbol{H}_0^1(\Omega)$. Subtracting the first equation on Problem 9 with this equation, gives the error equation

$$\begin{aligned}a(\mathsf{e}_{\boldsymbol{\theta}},\boldsymbol{\eta})+(\mathsf{e}_{\boldsymbol{\gamma}},\nabla v-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta})=(\boldsymbol{\gamma},\boldsymbol{\eta}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta})+\int_0^t\dot{E}(t-s)\Big[a(\mathsf{e}_{\boldsymbol{\theta}}(s),\boldsymbol{\eta})\\+(\mathsf{e}_{\boldsymbol{\gamma}}(s),\nabla v-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta})+(\boldsymbol{\gamma}(s),\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\eta}-\boldsymbol{\eta})\Big]ds,\end{aligned}$$

for all $\boldsymbol{\eta}\in\boldsymbol{H}_h$ and for all $v\in W_h$.

Following [17, Theorem 7.1], we take $\boldsymbol{\eta}=\xi_{\boldsymbol{\theta}}$, $v=\xi_w$, along with the substitutions $\nabla w^I-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I=\kappa_\nu^{-1}d^2\gamma^I$, $\nabla w_h-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}_h=\kappa_\nu^{-1}d^2\boldsymbol{\gamma}_h$, followed by (4.16), yielding

$$a(\xi_{\boldsymbol{\theta}},\xi_{\boldsymbol{\theta}})+\kappa_\nu^{-1}d^2(\xi_{\boldsymbol{\gamma}},\xi_{\boldsymbol{\gamma}})=\mathcal{F}(t)+\int_0^t\dot{E}(t-s)\chi(s)ds,\tag{4.17}$$

where

$$\begin{aligned}\mathcal{F}(t)&=a(\eta_{\boldsymbol{\theta}},\xi_{\boldsymbol{\theta}})+\kappa_\nu^{-1}d^2(\eta_{\boldsymbol{\gamma}},\xi_{\boldsymbol{\gamma}})+(\boldsymbol{\gamma},\xi_{\boldsymbol{\theta}}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\xi_{\boldsymbol{\theta}})\\&-\int_0^t\dot{E}(t-s)\Big[a(\eta_{\boldsymbol{\theta}}(s),\xi_{\boldsymbol{\theta}})+\kappa_\nu^{-1}d^2(\eta_{\boldsymbol{\gamma}}(s),\xi_{\boldsymbol{\gamma}})-(\boldsymbol{\gamma}(s),\xi_{\boldsymbol{\theta}}+\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\xi_{\boldsymbol{\theta}})\Big],\end{aligned}$$

and

$$\chi(s)=a(\xi_{\boldsymbol{\theta}}(s),\xi_{\boldsymbol{\theta}})+\kappa_\nu^{-1}d^2(\xi_{\boldsymbol{\gamma}}(s),\xi_{\boldsymbol{\gamma}}).$$

Using the bound

$$|(\boldsymbol{\gamma},\xi_{\boldsymbol{\theta}}-\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\xi_{\boldsymbol{\theta}})|\le Ch\|\boldsymbol{\gamma}-\boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_{\boldsymbol{L}^2(\Omega)}\|\xi_{\boldsymbol{\theta}}\|_{\boldsymbol{H}^1(\Omega)},$$

together with the continuity of the bilinear forms, allow to bound $\mathcal{F}$ as

$$\begin{aligned}\mathcal{F}(t)\le C\Big\{&\|\eta_{\boldsymbol{\theta}}\|_{\boldsymbol{H}^1(\Omega)}\|\xi_{\boldsymbol{\theta}}\|_{\boldsymbol{H}^1(\Omega)}+d^2\|\eta_{\boldsymbol{\gamma}}\|_{\boldsymbol{L}^2(\Omega)}\|\xi_{\boldsymbol{\gamma}}\|_{\boldsymbol{L}^2(\Omega)}\\&+h\|\boldsymbol{\gamma}-\boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_{\boldsymbol{L}^2(\Omega)}\|\xi_{\boldsymbol{\theta}}\|_{\boldsymbol{H}^1(\Omega)}+\int_0^t\phi(t-s)\big[\|\eta_{\boldsymbol{\theta}}(s)\|_{\boldsymbol{H}^1(\Omega)}\|\xi_{\boldsymbol{\theta}}\|_{\boldsymbol{H}^1(\Omega)}\\&+d^2\|\eta_{\boldsymbol{\gamma}}(s)\|_{\boldsymbol{L}^2(\Omega)}\|\xi_{\boldsymbol{\gamma}}\|_{\boldsymbol{L}^2(\Omega)}+h\|\boldsymbol{\gamma}(s)-\boldsymbol{\Pi}^0\boldsymbol{\gamma}(s)\|_{\boldsymbol{L}^2(\Omega)}\|\xi_{\boldsymbol{\theta}}\|_{\boldsymbol{H}^1(\Omega)}\big]ds\Big\}.\end{aligned}$$

Note that χ is controlled in a similar way as the first two terms in the bound of $\mathcal{F}$. Then, the Cauchy-Schwartz inequality, Grönwall's lemma, Lemma 2.2, and the triangle inequality yields to the desired estimate. $\square$

The purpose of the above theorem is to give an approximation result that allow to propose an interpolation of $\boldsymbol{\theta}^I, w^I$ and $\boldsymbol{\gamma}^I$ such that there is no dependency on the thickness parameter, i.e.,

$$\boldsymbol{\gamma}^I = \kappa_\nu^{-1} d^2(\nabla w^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}^I) = \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}. \tag{4.18}$$

With this selection of $\boldsymbol{\gamma}^I$, one can obtain

$$\begin{aligned}&\|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))} + d\|\mathsf{e}_{\boldsymbol{\gamma}}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))}\\ &\quad\leq C(\|\eta_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))} + d\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + h\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^0\boldsymbol{\gamma}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))}).\end{aligned}$$

If the geometry and data of the problem allow additional regularities than those described up to this point, we can propose the following convergence theorem, which resembles [17, Theorem 7.3], and generalizes the result from Theorem 4.4 for our viscoelastic plate.

Theorem 4.5. *Let $l \geq 1$ and assume for each $\boldsymbol{\theta} \in L^\ell(\mathcal{J}; \boldsymbol{H}^{l+1} \cap \boldsymbol{H}_0^1(\Omega))$ and $w \in L^\ell(\mathcal{J}; H^{l+1} \cap H_0^1(\Omega))$, there exist $\boldsymbol{\theta}^I \in \boldsymbol{H}_h$ and $w^I \in W_h$ such that* (4.18) *holds. If*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_{\boldsymbol{H}^1(\Omega)} \leq Ch^k\|\boldsymbol{\theta}\|_{\boldsymbol{H}^{k+1}(\Omega)}, \tag{4.19}$$

$$\|\boldsymbol{\gamma} - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\gamma}\|_{\boldsymbol{L}^2(\Omega)} \leq Ch^k\|\boldsymbol{\gamma}\|_{\boldsymbol{H}^k(\Omega)}, \tag{4.20}$$

for $1 \leq k \leq l$, then

$$\begin{aligned}&\|\boldsymbol{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))} + d\|\boldsymbol{e}_{\boldsymbol{\gamma}}\|_{L^\ell(\mathcal{J};\boldsymbol{L}^2(\Omega))}\\ &\quad\leq Ch^k\left(\|\boldsymbol{\theta}\|_{L^1(0,t;\boldsymbol{H}^{r+1}(\Omega))} + d\|\boldsymbol{\gamma}\|_{L^1(0,t;\boldsymbol{H}^r(\Omega))} + h^{k_0-k+2}\|\boldsymbol{\gamma}\|_{L^1(0,t;\boldsymbol{H}^{r_0+1}(\Omega))}\right),\end{aligned}$$

where the positive constant C is independent of d.

Once we have this approximation result, we may proceed as in Theorem 3.3 to prove a result in weaker norms for $\boldsymbol{\theta}$ and w. For this purpose, following (3.6), we resort to the dual-backward problem: find $(\boldsymbol{\psi}, u, \boldsymbol{\zeta}) \in L^r(0,\tau; \boldsymbol{H}_0^1 \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega))$ such that a.e. in $[0,\tau]$, for any $\tau \in \mathcal{J}$,

$$\begin{cases} a(\boldsymbol{\eta}, \boldsymbol{\psi}) + b((\boldsymbol{\eta}, v), \boldsymbol{\zeta}) = L_+(\boldsymbol{\eta}, v) + \displaystyle\int_\tau^s \dot{E}(s-t)[a(\boldsymbol{\psi}(s), \boldsymbol{\eta}) + b((\boldsymbol{\eta}, v), \boldsymbol{\zeta}(s))]ds,\\ b((\boldsymbol{\theta}, u), \boldsymbol{q}) - \dfrac{d^2}{\kappa_\nu}(\boldsymbol{\zeta}, \boldsymbol{q}) = 0,\end{cases} \tag{4.21}$$

for all $(\boldsymbol{\eta}, v) \in \boldsymbol{H}_0^1 \times H_0^1(\Omega)$ and for all $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)$, where $L_+(\boldsymbol{\eta}, v) = (g_+, v) - (\boldsymbol{f}_+, \boldsymbol{\eta})$.

Noting that $\mathcal{V}_{++} := \boldsymbol{H}^2(\Omega) \times H^2(\Omega)$ and $\mathcal{Q}_{++} := \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$, we have the following regularity estimate

$$\begin{aligned}&\|u\|_{L^r(0,\tau;H^2(\Omega))} + \|\boldsymbol{\psi}\|_{L^r(0,\tau;\boldsymbol{H}^2(\Omega))} + d\|\boldsymbol{\zeta}\|_{L^r(0,\tau;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{\zeta}\|_{L^r(0,\tau;\boldsymbol{L}^2(\Omega))}\\ &\qquad + d\|\operatorname{div}\boldsymbol{\zeta}\|_{L^r(0,\tau;L^2(\Omega))} \leq C\bigg(\|\boldsymbol{f}_+\|_{L^r(0,\tau;\boldsymbol{L}^2(\Omega))} + \|g_+\|_{L^r(0,\tau;L^2(\Omega))}\bigg).\end{aligned} \tag{4.22}$$

Defining $\xi_{\boldsymbol{\psi}} := \boldsymbol{\psi} - \boldsymbol{\psi}^I$ and $\xi_{\boldsymbol{\zeta}} := \boldsymbol{\zeta} - \boldsymbol{\zeta}^I$, we are able to present an estimate in weaker norms.

Theorem 4.6. *Under the hypotheses of Theorem 4.4, there exists a positive constant C independent of d such that*

$$\|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,\tau;\boldsymbol{L}^2(\Omega))} + \|\mathsf{e}_w\|_{L^\ell(0,\tau;L^2(\Omega))} \leq C\big[\mathcal{E}(\boldsymbol{\theta},\boldsymbol{\gamma})\mathcal{D}(\boldsymbol{\psi},\boldsymbol{\zeta})\big] + \mathcal{N}(\boldsymbol{\gamma},\boldsymbol{\psi}) + \mathcal{M}(\boldsymbol{\theta},\boldsymbol{\zeta}),$$

holds, for all $\tau \in \mathcal{J}$, where

$$E(\boldsymbol{\theta},\boldsymbol{\gamma}) := \|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,\tau;\boldsymbol{H}^1(\Omega))} + d\|\mathsf{e}_{\boldsymbol{\gamma}}\|_{L^\ell(0,\tau;\boldsymbol{L}^2(\Omega))},$$
$$\mathcal{D}(\boldsymbol{\psi},\boldsymbol{\zeta}) := \|\xi_{\boldsymbol{\psi}}\|_{L^r(0,\tau;\boldsymbol{H}^1(\Omega))} + d\|\xi_{\boldsymbol{\zeta}}\|_{L^r(0,\tau;\boldsymbol{L}^2(\Omega))},$$
$$\mathcal{N}(\boldsymbol{\gamma},\boldsymbol{\psi}) := \int_0^\tau \left[(\boldsymbol{\gamma},\boldsymbol{\psi}^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\psi}) - \int_0^t \dot{E}(t-s)(\boldsymbol{\gamma}(s),\boldsymbol{\psi}^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\psi})ds\right]dt,$$
$$\mathcal{M}(\boldsymbol{\theta},\boldsymbol{\zeta}) := \int_0^\tau \left[(\boldsymbol{\theta}_h - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}_h,\boldsymbol{\zeta}) - \int_0^t \dot{E}(t-s)(\boldsymbol{\theta}_h(s) - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}_h(s),\boldsymbol{\zeta})ds\right]dt.$$

Proof. Taking $\boldsymbol{\eta} = \mathsf{e}_{\boldsymbol{\theta}}(t)$ and $v = \mathsf{e}_w(t)$ in the first equation of (4.21), integrating in $[0,\tau]$ and interchanging the order of integration, we observe that

$$\begin{aligned}\int_0^\tau [(\boldsymbol{f}_+,\mathsf{e}_{\boldsymbol{\theta}}) - (g_+,\mathsf{e}_w)]dt = \int_0^\tau \Bigg\{ & a(\mathsf{e}_{\boldsymbol{\theta}},\boldsymbol{\psi}) + (\nabla\mathsf{e}_w - \mathsf{e}_{\boldsymbol{\theta}},\boldsymbol{\zeta}) \\ & - \int_0^t \dot{E}(t-s)\Big[a(\mathsf{e}_{\boldsymbol{\theta}}(s),\boldsymbol{\psi}) + (\nabla\mathsf{e}_w(s) - \mathsf{e}_{\boldsymbol{\theta}}(s),\boldsymbol{\zeta})\Big]ds\Bigg\}dt,\end{aligned} \tag{4.23}$$

holds. Moreover, observing that $\boldsymbol{\zeta}^I = \kappa_\nu d^{-2}(\nabla u^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\psi}^I)$,

$$(\nabla\mathsf{e}_w - \mathsf{e}_{\boldsymbol{\theta}},\boldsymbol{\zeta}) = \kappa_\nu^{-1}d^2(\mathsf{e}_{\boldsymbol{\gamma}},\boldsymbol{\zeta}) + (\boldsymbol{\theta}_h - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\theta}_h,\boldsymbol{\zeta}),$$

and

$$\begin{aligned}a(\mathsf{e}_{\boldsymbol{\theta}},\boldsymbol{\psi}^I) + \kappa_\nu^{-1}d^2(\mathsf{e}_{\boldsymbol{\gamma}},\boldsymbol{\zeta}^I) = (\boldsymbol{\gamma},\boldsymbol{\psi}^I - \boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\psi}^I) + \int_0^t \dot{E}(t-s)\Big[& a(\mathsf{e}_{\boldsymbol{\theta}}(s),\boldsymbol{\psi}^I) \\ & + \kappa_\nu^{-1}d^2(\mathsf{e}_{\boldsymbol{\gamma}}(s),\boldsymbol{\zeta}^I) + (\boldsymbol{\gamma}(s),\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}\boldsymbol{\psi} - \boldsymbol{\psi}^I)\Big]ds,\end{aligned} \tag{4.24}$$

the proof is concluded following the arguments from Theorem 3.3, with $\boldsymbol{f}_+$ and g_+ such that $\|\boldsymbol{f}_+\|_{L^r(0,\tau;\boldsymbol{L}^2(\Omega))} = \|g_+\|_{L^r(0,\tau;L^2(\Omega))} = 1$. ☐

Up to this point we have proved an approximation result in weak norms, from which a bound by a factor h^k depends on the choice of some particular numerical method and the regularity of the solutions. In the following, we explore one of these methods in order to obtain error estimates, independent on the perturbation parameter

4.2.2. *The Durán-Liberman scheme and theoretical rates of convergence.* This section deals with the proposal of a locking-free method. We recall that Ω is assumed to be a convex polygonal domain. Let $\boldsymbol{E}$ and $\boldsymbol{n}$ denote the set of edges in the mesh $\mathscr{T}_h$ and the unit normal vector, respectively. We denote by M_k the space of piecewise polynomials of degree $\leq k$, and the corresponding vector-valued analogue by $\boldsymbol{M}_k := M_k \times M_k$. The Durán-Liberman element [14] corresponds to the choices

$$\boldsymbol{H}_h = \left\{\boldsymbol{\theta} \in \boldsymbol{M}_2 \cap \boldsymbol{H}_0^1 \,:\, \boldsymbol{\theta}\cdot\boldsymbol{n} \in \mathcal{P}_1(e),\, e \in \boldsymbol{E}\right\}, \quad W_h = M_1 \cap H_0^1, \quad \boldsymbol{\Gamma}_h = \boldsymbol{RT}_0^\perp,$$

where $\boldsymbol{RT}_0^\perp$ denotes the Raviart-Thomas discretization of the lowest order to $\boldsymbol{H}(\mathrm{rot})$. From this we take $\boldsymbol{\Pi}^{\boldsymbol{\Gamma}}$ as te usual interpolant into $\boldsymbol{RT}_0^\perp$, defined for

$\boldsymbol{\gamma} \in \boldsymbol{H}^1(\Omega)$ by

$$\int_e \boldsymbol{\Pi}^{\boldsymbol{\Gamma}} \boldsymbol{\gamma} \cdot \boldsymbol{s} = \int_e \boldsymbol{\gamma} \cdot s, \qquad e \in \boldsymbol{E}.$$

Therefore, we have the following error estimate.

Theorem 4.7. *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in L^\ell(\mathcal{J}; \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2)$ and $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in L^\ell(\mathcal{J}; \boldsymbol{H}_h \times W_h \times \boldsymbol{\Gamma}_h)$ be the solutions of the continuous and semi-discrete Problems 7 and 9, respectively. Then,*

$$\|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))} + d\|\mathsf{e}_{\boldsymbol{\gamma}}\|_{L^\ell(0,t;\boldsymbol{H}^1(\Omega))} + \|\mathsf{e}_w\|_{L^\ell(0,t;H^1(\Omega))} \leq Ch(\|\boldsymbol{f}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + \|g\|_{L^\ell(0,t;L^2(\Omega))}),$$

where the positive constant C is independent of d.

Proof. The bounds for $\mathsf{e}_{\boldsymbol{\gamma}}$ and $\mathsf{e}_{\boldsymbol{\theta}}$ follow by the same arguments of [17, Theorem 8.1] and Theorem 4.5. The estimate for e_w follows from [14, Corollary 3.1] together with (4.13)-(4.14). □

The above result gives the necessary bound to exploit Theorem 4.6 and obtain an improved estimate.

Theorem 4.8. *Under the assumptions of Theorem 4.7, there exists a positive constant C independent of d such that*

$$\|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + d\|\mathsf{e}_{\boldsymbol{\gamma}}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + \|\mathsf{e}_w\|_{L^\ell(0,t;H^1(\Omega))} \leq Ch^2(\|\boldsymbol{f}\|_{L^\ell(0,t;\boldsymbol{L}^2(\Omega))} + \|g\|_{L^\ell(0,t;L^2(\Omega))}).$$

Proof. Given $\tau \in \mathcal{J}$, from Theorem 4.6 and Theorem 4.7 we obtain

$$\|\mathsf{e}_{\boldsymbol{\theta}}\|_{L^\ell(0,\tau;\boldsymbol{L}^2(\Omega))} + \|\mathsf{e}_w\|_{L^\ell(0,\tau;L^2(\Omega))} \leq Ch\mathcal{D}(\boldsymbol{\psi}, \boldsymbol{\zeta})(\|\boldsymbol{f}\|_{L^\ell(0,\tau;\boldsymbol{L}^2(\Omega))} + \|g\|_{L^\ell(0,\tau;L^2(\Omega))}) + \mathcal{N}(\boldsymbol{\gamma}, \boldsymbol{\psi}) + \mathcal{M}(\boldsymbol{\theta}, \boldsymbol{\zeta}),$$

The estimate is then obtained from (4.22), [17, Lemma 8.2 and Theorem 8.3] and Lemma 2.2, with $\tau = t$. □

4.2.3. *Numerical tests.* Finally, we report a series of numerical tests in order to test the theoretical results obtained for the viscoelastic plate problem. In order to continue with the uniformity of the experiments, the material selected is the same as the one used in the beam, i.e., the SLS model. For this experiment we consider an observation time $T = 20$s, with 5000 time steps. The plate domain is $\Omega = (a, b)^2$, with $a = 0, b = 1$m, and is assumed to be clamped in its whole boundary. The selected thickness are 10^{-3}m, 10^{-4}m and 10^{-5}m, with Poisson's ratio 0.3. For the numerical implementation, we wrote a FEniCS code. The reduction operator was obtained with the help of FEniCS-shells [22]. The quasi-static analytical solution is derived from [12] and the correspondence principle. For instance, we have

$$w(x, y, t) = J(t)\tilde{w}(x, y), \quad \beta_1(x, y, t) = J(t)\tilde{\beta}_1(x, y), \quad \beta_2(x, y, t) = J(t)\tilde{\beta}_2(x, y),$$

where $J(t)$ is the creep compliance [27], and

$$\tilde{w}(x, y) = \frac{1}{3}x^3(x-1)^3y^3(y-1)^3 - \frac{2t^2}{5(1-v)}\left[y^3(y-1)^3x(x-1)\left(5x^2 - 5x + 1\right) + x^3(x-1)^3y(y-1)\left(5y^2 - 5y + 1\right)\right],$$

$$\tilde{\beta}_1(x, y) = y^3(y-1)^3x^2(x-1)^2(2x-1),$$

$$\tilde{\beta}_2(x, y) = x^3(x-1)^3y^2(y-1)^2(2y-1).$$

Table 3. Error values and experimental rates of convergence of the transverse displacement w in a fully clamped viscoelastic Reissner-Mindlin plate.

		$d=10^{-3}$m		$d=10^{-4}$m		$d=10^{-5}$m	
DOF	h	$\mathtt{e}_{0,1}(w)$	$\mathtt{r}_{0,1}(w)$	$\mathtt{e}_{0,1}(w)$	$\mathtt{r}_{0,1}(w)$	$\mathtt{e}_{0,1}(w)$	$\mathtt{r}_{0,1}(w)$
1323	0.14	$1.7184e-12$	$--$	$1.7183e-12$	$--$	$1.7183e-12$	$--$
2883	0.09	$7.2013e-13$	2.14	$7.1985e-13$	2.14	$7.1978e-13$	2.14
5043	0.07	$3.9015e-13$	2.13	$3.8977e-13$	2.13	$3.8977e-13$	2.13
7803	0.06	$2.4372e-13$	2.10	$2.4327e-13$	2.11	$2.4315e-13$	2.11
11163	0.05	$1.6651e-13$	2.08	$1.6601e-13$	2.09	$1.6584e-13$	2.09
15123	0.04	$1.2095e-13$	2.07	$1.2042e-13$	2.08	$1.2018e-13$	2.08
DOF	h	$\mathtt{e}_{0,2}(w)$	$\mathtt{r}_{0,2}(w)$	$\mathtt{e}_{0,2}(w)$	$\mathtt{r}_{0,2}(w)$	$\mathtt{e}_{0,2}(w)$	$\mathtt{r}_{0,2}(w)$
1323	0.14	$4.0111e-13$	$--$	$4.0109e-13$	$--$	$4.0109e-13$	$--$
2883	0.09	$1.6809e-13$	2.14	$1.6802e-13$	2.14	$1.6800e-13$	2.14
5043	0.07	$9.1069e-14$	2.13	$9.0979e-14$	2.13	$9.0980e-14$	2.13
7803	0.06	$5.6890e-14$	2.10	$5.6785e-14$	2.11	$5.6755e-14$	2.11
11163	0.05	$3.8866e-14$	2.08	$3.8751e-14$	2.09	$3.8709e-14$	2.09
15123	0.04	$2.8232e-14$	2.07	$2.8109e-14$	2.08	$2.8051e-14$	2.08
DOF	h	$\mathtt{e}_{0,\infty}(w)$	$\mathtt{r}_{0,\infty}(w)$	$\mathtt{e}_{0,\infty}(w)$	$\mathtt{r}_{0,\infty}(w)$	$\mathtt{e}_{0,\infty}(w)$	$\mathtt{r}_{0,\infty}(w)$
1323	0.14	$1.7186e-12$	$--$	$1.7185e-12$	$--$	$1.7185e-12$	$--$
2883	0.09	$7.2021e-13$	2.14	$7.1993e-13$	2.14	$7.1986e-13$	2.14
5043	0.07	$3.9019e-13$	2.13	$3.8981e-13$	2.13	$3.8981e-13$	2.13
7803	0.06	$2.4375e-13$	2.10	$2.4330e-13$	2.11	$2.4318e-13$	2.11
11163	0.05	$1.6652e-13$	2.08	$1.6603e-13$	2.09	$1.6586e-13$	2.09
15123	0.04	$1.2096e-13$	2.07	$1.2043e-13$	2.08	$1.2020e-13$	2.08

Tables 3 and 4 show the behavior of the error when is computed with different values of ℓ. As in the beam case, the experimental convergence orders coincide with those predicted by our theory, coinciding by those obtained in elastic plates in [14] and [17]. This, together with the low number of degrees of freedom used, shows that the method is locking-free. In Figures 2 and 3, we present the evolution of the transverse displacement w_h and the components of the rotation vector $\boldsymbol{\theta}_h = (\theta_{1h}, \theta_{2h})$ at different times steps to verify that the method takes into account the presence of the creep compliance, typical of the SLS material.

We end this section reporting the creep compliance for w and w_h in the center of the plate, i.e., the point of maximum deflection. In Figure 4, the bounded creep behavior is clearly visible, and also, is observable how the viscoelastic discrete and exact solution match almost precisely.

5. Conclusions

We have presented an abstract functional framework to deal with mixed formulations for viscoelastic problems, where perturbations parameters arise. We have shown the solvability of mixed viscoelastic formulations, adapting the well known theory for elliptic mixed formulations. The relevance is focused in the independence of the perturbation parameter in every estimate, since in the applications, numerical methods can be affected, deteriorating the stability and convergence. With the well established theory of Volterra equations, we have proved convergence of

TABLE 4. Error values and experimental rates of convergence of the rotation $\boldsymbol{\theta}$ in a fully clamped viscoelastic Reissner-Mindlin plate.

		$d = 10^{-3}$m		$d = 10^{-4}$m		$d = 10^{-5}$m	
DOF	h	$\mathtt{e}_{\mathbf{0},1}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},1}(\boldsymbol{\theta})$	$\mathtt{e}_{\mathbf{0},1}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},1}(\boldsymbol{\theta})$	$\mathtt{e}_{\mathbf{0},1}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},1}(\boldsymbol{\theta})$
1323	0.14	$9.5708e-12$	$--$	$9.5710e-12$	$--$	$9.5709e-12$	$--$
2883	0.09	$4.0839e-12$	2.10	$4.0833e-12$	2.10	$4.0830e-12$	2.10
5043	0.07	$2.2382e-12$	2.09	$2.2372e-12$	2.09	$2.2373e-12$	2.09
7803	0.06	$1.4091e-12$	2.07	$1.4080e-12$	2.07	$1.4074e-12$	2.07
11163	0.05	$9.6805e-13$	2.05	$9.6675e-13$	2.06	$9.6600e-13$	2.06
15123	0.04	$7.0599e-13$	2.04	$7.0459e-13$	2.05	$7.0354e-13$	2.05
DOF	h	$\mathtt{e}_{\mathbf{0},2}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},2}(\boldsymbol{\theta})$	$\mathtt{e}_{\mathbf{0},2}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},2}(\boldsymbol{\theta})$	$\mathtt{e}_{\mathbf{0},2}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},2}(\boldsymbol{\theta})$
1323	0.14	$2.2339e-12$	$--$	$2.2340e-12$	$--$	$2.23405e-12$	$--$
2883	0.09	$9.5325e-13$	2.10	$9.5313e-13$	2.10	$9.5305e-13$	2.10
5043	0.07	$5.2243e-13$	2.09	$5.2222e-13$	2.09	$5.2222e-13$	2.09
7803	0.06	$3.2892e-13$	2.07	$3.2865e-13$	2.07	$3.2852e-13$	2.07
11163	0.05	$2.2596e-13$	2.05	$2.2565e-13$	2.06	$2.2547e-13$	2.06
15123	0.04	$1.6479e-13$	2.04	$1.6446e-13$	2.05	$1.6420e-13$	2.05
DOF	h	$\mathtt{e}_{\mathbf{0},\infty}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},\infty}(\boldsymbol{\theta})$	$\mathtt{e}_{\mathbf{0},\infty}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},\infty}(\boldsymbol{\theta})$	$\mathtt{e}_{\mathbf{0},\infty}(\boldsymbol{\theta})$	$\mathtt{r}_{\mathbf{0},\infty}(\boldsymbol{\theta})$
1323	0.14	$9.5718e-12$	$--$	$9.5720e-12$	$--$	$9.5719e-12$	$--$
2883	0.09	$4.0843e-12$	2.10	$4.0838e-12$	2.10	$4.0835e-12$	2.10
5043	0.07	$2.2384e-12$	2.09	$2.2375e-12$	2.09	$2.2375e-12$	2.09
7803	0.06	$1.4093e-12$	2.07	$1.4081e-12$	2.07	$1.4076e-12$	2.07
11163	0.05	$9.6815e-13$	2.05	$9.6685e-13$	2.06	$9.6610e-13$	2.06
15123	0.04	$7.0607e-13$	2.04	$7.0466e-13$	2.05	$7.0361e-13$	2.05

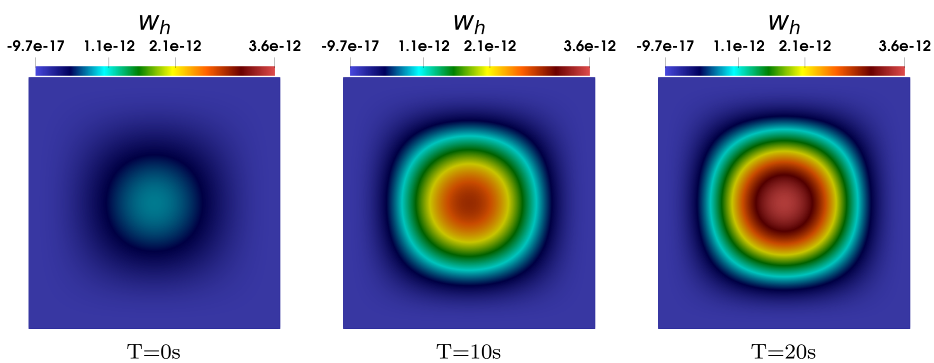

FIGURE 2. Evolution of the viscoelastic displacement w_h in the fully clamped plate, where $d = 10^{-5}$m and $\Delta t = 0.004$.

mixed conforming numerical methods for the mixed viscoelastic problem, where the convergence is independent of the perturbation parameter.

The applications that we performed, for Timoshenko beams and Reissner-Mindlin plates, confirm that the proposed abstract framework is suitable for slender structures, where the thickness parameter do not produces difficulties for numerical methods, when viscoelastic materials are considered.

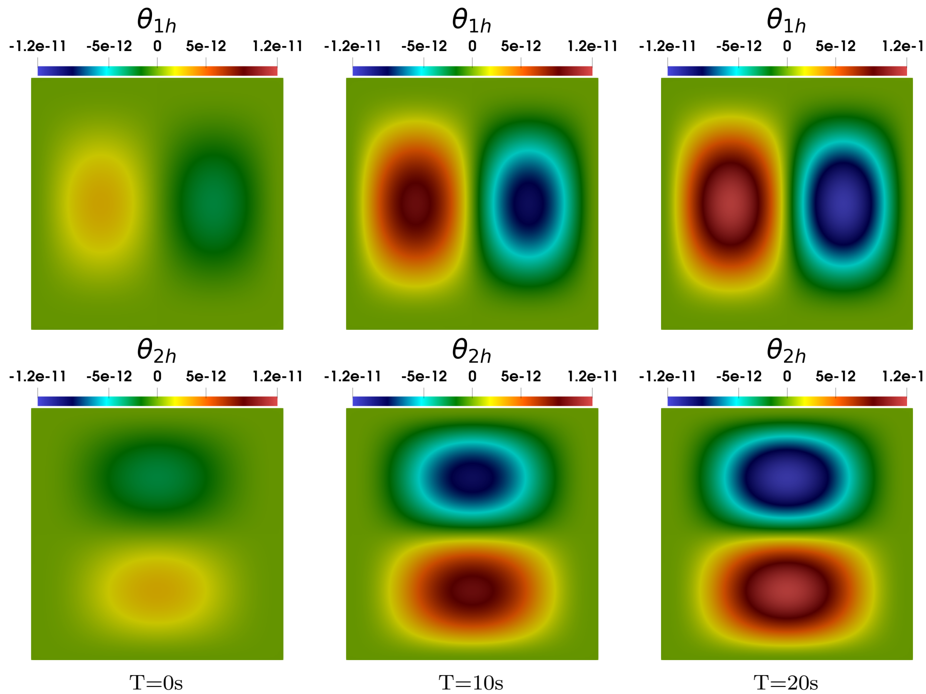

FIGURE 3. Evolution of the viscoelastic rotation components β_{ih} in the fully clamped, where $d = 10^{-5}$m and $\Delta t = 0.04$.

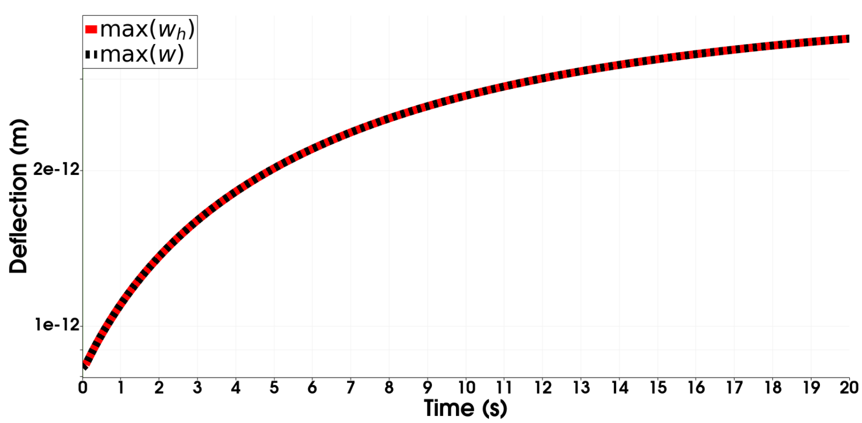

FIGURE 4. Comparison between the exact and discrete maximum transverse displacement in the viscoelastic Reissner-Mindlin plate, where $d = 10^{-5}$m and $\Delta t = 0.04$.

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