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VIRTUAL ELEMENTS FOR THE TRANSMISSION EIGENVALUE PROBLEM ON POLYTOPAL MESHES*

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Abstract. The transmission eigenvalue problem is a challenging model in the inverse scattering theory and has important applications in this topic. The aim of this paper is to analyze a C^1 virtual element method on polytopal meshes in $\mathbb{R}^d$ $(d = 2, 3)$ for solving a quadratic and non-self-adjoint fourth-order eigenvalue problem derived from the transmission eigenvalue problem. Optimal order error estimates for the eigenfunctions and a double order for the eigenvalues are obtained by using the approximation theory for compact non-self-adjoint operators. Finally, a set of numerical tests illustrating the good performance of the virtual scheme are presented.

Key words. transmission eigenvalues, virtual element schemes, spectral problem, polytopal meshes, error estimates

AMS subject classifications. 35P30, 65N15, 65N25, 65N30, 78A46

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1. Introduction.

The transmission eigenvalue problem can be stated as follows (see, for instance, [21, 37]). Find $\kappa \in \mathbb{C}$ and $w_1, w_2 \in L^2(\Omega)$ with $w_1 - w_2 \in H^2(\Omega)$ such that

$$\Delta w_1 + \kappa^2 n w_1 = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$\Delta w_2 + \kappa^2 w_2 = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$w_1 - w_2 = 0 \quad \text{on } \Gamma, \tag{1.1c}$$

$$\partial_\nu w_1 - \partial_\nu w_2 = 0 \quad \text{on } \Gamma. \tag{1.1d}$$

The system (1.1a)–(1.1b) together with the boundary conditions (1.1c)–(1.1d) corresponds to the scattering problem for an isotropic inhomogeneous medium for the Helmholtz equation, where $\Omega \subseteq \mathbb{R}^d$ $(d = 2, 3)$ is a bounded simply connected Lipschitz domain with boundary $\Gamma := \partial\Omega$. Here, ν denotes the outward unit normal vector to Γ, ∂_ν denotes the normal derivative, and n is the index of refraction.

The transmission eigenvalue problem (1.1a)–(1.1d) is a nonlinear and non-self-adjoint eigenvalue problem which plays an important role in inverse scattering theory (see [14, 13]). For instance, the transmission eigenvalues can be determined from the far-field data of the scattered wave and used to obtain estimates for the material properties of the scattering object. The numerical solution of the transmission eigenvalue problem has attracted interests from many researchers in the last years. For instance, several conforming and nonconforming finite element methods, mixed

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formulations, among others, have been proposed. We cite as a minimal sample of them [15, 16, 17, 18, 22, 25, 36, 39, 42].

The transmission eigenvalue problem is often solved by reformulating it as a fourth-order eigenvalue problem. More precisely, by introducing a new unknown $u := w_1 - w_2 \in H_0^2(\Omega)$, the model problem (1.1a)–(1.1d) can be rewritten as follows:

$$(\Delta + \kappa^2 n)\frac{1}{n-1}(\Delta + \kappa^2)u = 0 \quad \text{in } \Omega. \tag{1.2}$$

In [16] it has been introduced and analyzed as a conforming $C^1 - C^0$ variational formulation in two dimensions (2D), using Argyris and Lagrange finite element spaces. A complete analysis of the method including error estimates is proved using the theory for compact non-self-adjoint operators. In [41] it has been written as a weak formulation in $H^2(\Omega) \times L^2(\Omega)$ for the transmission eigenvalue problem which is based on a linearization technique. The authors have proposed a conforming $C^1 - C^1$ finite element discretization in 2D and three dimensions (3D), and error estimates have been obtained. We recall that the construction of C^1-conforming finite elements is difficult in general, since they usually involve a large number of degrees of freedom. They are often viewed as prohibitively expensive due to their high polynomial degree and complexity [20]. For instance, the minimal polynomial degree of the 3DC^1 is 9; thus, a conforming C^1 finite element method (FEM) on a tetrahedral mesh will require 220 degrees of freedom per element.

The aim of the present paper is to introduce and analyze a virtual element method in 2D and 3D to solve the fourth-order transmission eigenvalue problem. The *virtual element method* (VEM) is a new technology introduced in [5] as a generalization of FEM which is characterized by the capability of dealing with very general polygonal/polyhedral meshes, including "hanging nodes" and nonconvex elements (see [12, 23, 24, 29, 30, 34] and refereneces therein). It also permits to easily implement highly regular conforming discrete spaces which make the method very feasible to solve fourth-order problems. For instance, in 2D the method has been applied in a wide range of problems: [2, 3, 7, 8, 11, 31]. In the three-dimensional case, in [6] it has recently been introduced as a C^1 VEM to solve a fourth-order partial differential equation. Regarding the approximation by VEM of the transmission eigenvalue problem, in [32] it has been recently presented as a C^1-conforming VEM to solve the spectral problem on general polygonal meshes (only the two-dimensional case). Optimal order error estimates for the eigenfunctions and a double order for the eigenvalues are derived.

In this paper, we study a new VEM method to solve the transmission eigenvalue problem in 2D and 3D. More precisely, the goal of this work is to introduce and analyze a C^1 virtual element discretization on polytopal meshes to approximate the fourth-order transmission eigenvalue problem. Since (1.2) is a nonlinear equation regarding the parameter κ^2, we introduce a new unknown, which leads to a linear non-self-adjoint variational formulation of the problem written in $H_0^2(\Omega) \times L^2(\Omega)$ as in [40, 41]. Then, a solution operator is introduced whose spectra is related to the solutions of the transmission eigenvalue problem. Next, we use the fact that $H_0^2(\Omega) \subset L^2(\Omega)$ to propose a conforming discrete formulation based on the virtual element spaces introduced in [2] and [6]. Then, we employ the spectral theory for non-self-adjoint compact operators presented in [33] and rather mild assumptions on the polygonal/polyhedral meshes to obtain that the resulting C^1-VEM scheme provides a correct approximation of the spectrum. In addition, optimal order error estimates for the eigenfunctions and a double order for the eigenvalues are also obtained. Finally, we remark that the proposed conforming VEM for three-dimensional transmission eigenvalue problem on tetrahedral meshes employs 16 degrees of freedom per element. This makes the method highly attractive, in terms of computational cost, compared with a conforming FEM.

The paper is organized as follows. In section 2, we introduce the weak formulation associated to the transmission eigenvalue problem and formulate its spectral characterization with a suitable solution operator. In section 3, we present the definitions of the two-dimensional and three-dimensional C^1 virtual element spaces. Then, a virtual element discrete formulation and its spectral characterization are presented in section 4. In addition, we prove that the numerical scheme provides a correct spectral approximation and establish optimal order error estimates for the eigenvalues and eigenfunctions. Finally, in section 5, we report some numerical tests that confirm the theoretical analysis developed.

Throughout the article we will use standard notations for Sobolev spaces, norms, and seminorms. Moreover, we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

2. The continuous spectral formulation.

In this section we introduce a continuous variational formulation associated to the fourth-order transmission eigenvalue problem (cf. (1.2)) and its spectral characterization. With this aim, we multiply the identity (1.2) by $w \in H_0^2(\Omega)$, and we arrive at the following quadratic eigenvalue problem: find $\kappa \in \mathbb{C}$ and $0 \neq u \in H_0^2(\Omega)$ such that

$$\begin{aligned}&\int_\Omega \frac{1}{n-1}\Delta u \Delta \overline{w} + \kappa^2 \int_\Omega \Delta u\Big(\frac{n}{n-1}\overline{w}\Big) + \kappa^2 \int_\Omega \frac{1}{n-1} u\Delta\overline{w} + \kappa^4 \int_\Omega \frac{n}{n-1} u\overline{w} = 0 \\ &\qquad \forall w \in H_0^2(\Omega).\end{aligned} \tag{2.1}$$

One of the main difficulties of the variational formulation (2.1) is the nonlinearity with respect to the parameter κ^2. For the theoretical analysis it is convenient to transform the above variational problem into a linear eigenvalue problem. To do that, in this work we will consider the following auxiliary variable denoted by z and defined as follows (see [40]):

$$z := \kappa^2 u \quad \text{in } \Omega. \tag{2.2}$$

In this work, we suppose that $n(\boldsymbol{x}) =: n \in W^{1,\infty}(\Omega)$, satisfying either one of the following assumptions for all $\boldsymbol{x} \in \Omega$:

$$\begin{aligned} 1 < n_* \leq n(\boldsymbol{x}) \leq n^* < \infty, \\ 0 < n_* \leq n(\boldsymbol{x}) \leq n^* < 1. \end{aligned} \tag{2.3}$$

Now, we denote by $\mathbb{H}$ the product space $\mathbb{H} := H_0^2(\Omega) \times L^2(\Omega)$, endowed with the following product norm

$$||(w,v)||_{\mathbb{H}} := \big(||D^2 w||_{0,\Omega}^2 + ||v||_{0,\Omega}^2\big)^{1/2},$$

where $D^2 w$ denotes the Hessian matrix of w. Moreover, it is clear that the above norm is equivalent with the usual norm in $H_0^2(\Omega) \times L^2(\Omega)$.

Using (2.2) we arrive at the following weak formulation of the transmission eigenvalue problem.

PROBLEM 1. *Find* $(\lambda,(u,z)) \in \mathbb{C}\times\mathbb{H}$ *with* $(u,z)\neq 0$ *such that*

$$A((u,z),(w,v)) := a_1(u,w) + a_2(z,v) = \lambda B((u,z),(w,v)) \qquad \forall (w,v)\in\mathbb{H}, \tag{2.4}$$

where $\lambda := -\kappa^2$ and $a_1(\cdot,\cdot), a_2(\cdot,\cdot), B(\cdot,\cdot)$ are sesquilinear forms defined as follows:

$$a_1 : H_0^2(\Omega)\times H_0^2(\Omega)\to\mathbb{C}, \quad a_1(u,w) := \int_\Omega \frac{1}{n-1}\Delta u\Delta\overline{w}, \tag{2.5}$$

$$a_2 : L^2(\Omega)\times L^2(\Omega)\to\mathbb{C}, \quad a_2(z,v) := \int_\Omega z\overline{v}, \tag{2.6}$$

and

$$\begin{aligned} B : \mathbb{H}\times\mathbb{H}\to\mathbb{C}, \quad B((u,z),(w,v)) := &\int_\Omega \Delta u\Big(\frac{n}{n-1}\overline{w}\Big) + \int_\Omega \frac{1}{n-1}u\Delta\overline{w} \\ &+ \int_\Omega \frac{n}{n-1}z\overline{w} - \int_\Omega u\overline{v}. \end{aligned} \tag{2.7}$$

Our goal is to introduce and analyze a conforming virtual element discretization in 2D and 3D to solve Problem 1. We observe that (2.4) has been considered in [41] where a conforming FEM has been analized and in [40] where a nonconforming C^0 interior penalty Galerkin (IPG) finite element discretization has been presented.

It is easy to check that the forms $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ satisfy the following bounds.

LEMMA 2.1. *There exist positive constants* α_0 *and* C *that depend on the index of refraction* n *such that*

$$A((w,v),(w,v)) \geq \alpha_0 ||(w,v)||_{\mathbb{H}}^2, \tag{2.8}$$

$$|A((u,z),(w,v))| \leq C||(u,z)||_{\mathbb{H}}||(w,v)||_{\mathbb{H}}, \tag{2.9}$$

$$|B((u,z),(w,v))| \leq C||(u,z)||_{\mathbb{H}}||(w,v)||_{\mathbb{H}} \tag{2.10}$$

for all $(u,z),(w,v)\in\mathbb{H}$.

According to Lemma 2.1, we are in a position to introduce the solution operator.

$$\begin{aligned} \mathcal{S} : \mathbb{H} &\longrightarrow \mathbb{H} \\ (f,g) &\longmapsto \mathcal{S}(f,g) = (\widetilde{u},\widetilde{z}) \end{aligned}$$

defined as the unique solution (see Lemma 2.1) of the following source problem:

$$A((\widetilde{u},\widetilde{z}),(w,v)) = B((f,g),(w,v)) \qquad \forall (w,v)\in\mathbb{H}. \tag{2.11}$$

Thus, we have that the linear operator $\mathcal{S}$ is well defined and bounded. Moreover, we have that $(\lambda,(u,z))$ solves Problem 1 if and only if $(\mu,(u,z))$ is an eigenpair of $\mathcal{S}$; i.e., $\mathcal{S}(u,z) = \mu(u,z)$, with $\mu := 1/\lambda$.

We observe that no spurious eigenvalues are introduced into the problem. In fact, if $\mu\neq 0$, then $(0,z)$ is not an eigenfunction of the problem.

The following is an additional regularity result associated to the solution of the source problem (2.11).

LEMMA 2.2. *There exist* $s\in(0,1]$ *and a positive constant* C *depending on the index of refraction* n *such that for all* $(f,g)\in\mathbb{H}$, *the solution* $(\widetilde{u},\widetilde{z})$ *of* (2.11) *satisfies* $(\widetilde{u},\widetilde{z})\in H^{2+s}(\Omega)\times H_0^2(\Omega)$ *and*

$$||\widetilde{u}||_{2+s,\Omega} + ||\widetilde{z}||_{2,\Omega} \leq C||(f,g)||_{\mathbb{H}}.$$

Proof. The estimate for $\widetilde{u}$ follows from the classical regularity result for the biharmonic problem with its right-hand side in $H^{-1}(\Omega)$ (see, for instance, [9, 27, 35]). On the other hand, from (2.11) we have that $\widetilde{z} = f \in H_0^2(\Omega)$. We conclude the proof. ∎

Now, from Lemma 2.2 and the fact that the inclusion $H^{2+s}(\Omega) \times H_0^2(\Omega) \hookrightarrow \mathbb{H}$ is compact, we obtain that the operator $\mathcal{S}$ is compact. As consequence, we have the following spectral characterization result.

LEMMA 2.3. *The spectrum of $\mathcal{S}$ satisfies $\operatorname{sp}(\mathcal{S}) = \{0\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, where $\{\mu_k\}_{k\in\mathbb{N}}$ is a sequence of complex eigenvalues which converges to 0 and their corresponding eigenspaces lie in $[H^{2+s}(\Omega)]^2$ and*

$$||u||_{2+s,\Omega} + ||z||_{2+s,\Omega} \leq C||(u,z)||_{\mathbb{H}}.$$

In addition, $\mu = 0$ is an infinite multiplicity eigenvalue of $\mathcal{S}$.

Proof. The proof follows from the compactness of $\mathcal{S}$, Lemma 2.2, and the identity (2.2). ∎

Since Problem 1 is non-self-adjoint, we need to deal with the adjoint operator $\mathcal{S}^*$, which is defined as

$$\begin{aligned} \mathcal{S}^* : \mathbb{H} &\longrightarrow \mathbb{H} \\ (f,g) &\longmapsto \mathcal{S}^*(f,g) = (\widetilde{u}^*, \widetilde{z}^*), \end{aligned}$$

which is defined as the unique solution (see Lemma 2.1) of the following source problem

$$A((w,v),(\widetilde{u}^*,\widetilde{z}^*)) = B((w,v),(f,g)) \qquad \forall (w,v) \in \mathbb{H}. \tag{2.12}$$

It is simple to prove that if μ is an eigenvalue of $\mathcal{S}$ with multiplicity m, $\bar{\mu}$ is an eigenvalue of $\mathcal{S}^*$ with the same multiplicity m. In addition, a result analogous to Lemma 2.2 can be proven in this case.

LEMMA 2.4. *There exist $s \in (0,1]$ and a positive constant C depending on the index of refraction n such that for all $(f,g) \in \mathbb{H}$, the solution $(\widetilde{u}^*,\widetilde{z}^*)$ of (2.12) satisfies $(\widetilde{u}^*,\widetilde{z}^*) \in H^{2+s}(\Omega) \times H_0^2(\Omega)$ and*

$$||\widetilde{u}^*||_{2+s,\Omega} + ||\widetilde{z}^*||_{2,\Omega} \leq C||(f,g)||_{\mathbb{H}}.$$

3. Virtual element spaces.

In this section, we will introduce a virtual element discretization to solve the transmission eigenvalue problem. We start by presenting the virtual element spaces in 2D and 3D to be used in the proposed method.

3.1. The two-dimensional case.

We begin with the mesh construction and the assumptions considered to introduce the discrete virtual element spaces (see, e.g., [1, 5]). Let $\{\Omega_h\}_h$ be a sequence of decompositions of Ω into general polygonal elements P. We will denote by h_P the diameter of the element P and by h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{P\in\Omega_h} h_P$. In addition, we denote by N_P and N_{v}^P the number of polygons in Ω_h and the number of vertices of P, respectively. Moreover, we denote by e a generic edge of $\{\Omega_h\}_h$ and for all $e \in \partial P$, we define a unit normal vector ν_P^e that points outside of P.

For the analysis of the scheme, we will make the following assumptions (see, for instance, [5]): there exists a positive real number C_Ω such that, for every h and every $P \in \Omega_h$,

$\mathbf{A}_1^{2D}$: $P \in \Omega_h$ is star-shaped with respect to every point of a ball of radius $C_\Omega h_P$;
$\mathbf{A}_2^{2D}$: the ratio between the shortest edge and the diameter h_P of P is larger than C_Ω.

Now, for all $m \in \mathbb{N}$, we will denote by $\mathbb{P}_m(\mathcal{O})$ the space of polynomials of degree up to m defined on the subset $\mathcal{O} \subseteq \mathbb{R}^2$.

We introduce on each element $P \in \Omega_h$ the following finite dimensional space $\widetilde{V}_h^{2D}(P)$ introduced in [19]:

$$\widetilde{V}_h^{2D}(P) := \big\{w_h \in H^2(P) : \Delta^2 w_h \in \mathbb{P}_2(P), w_h|_{\partial P} \in C^0(\partial P), w_h|_e \in \mathbb{P}_3(e)\ \forall e \in \partial P,\\ \nabla w_h|_{\partial P} \in [C^0(\partial P)]^2, \partial_{\nu_P^e} w_h|_e \in \mathbb{P}_1(e)\ \forall e \in \partial P\big\}.$$

Moreover, in $\widetilde{V}_h^{2D}(P)$ we define the following sets of linear operators. For all $w_h \in \widetilde{V}_h^{2D}(P)$ we consider

$\mathbf{D_1}^{2D}$: evaluation of w_h at the N_{v}^P vertices of P;
$\mathbf{D_2}^{2D}$: evaluation of ∇w_h at the N_{v}^P vertices of P.

In order to introduce the local virtual space, we define the projector $\Pi_P^{\Delta,2D} : \widetilde{V}_h^{2D}(P) \to \mathbb{P}_2(P)$ as follows:

$$\text{(3.1)}\qquad \begin{cases} \displaystyle\int_P D^2(w - \Pi_P^{\Delta,2D} w) : D^2 q = 0 \quad \forall w \in \widetilde{V}_h^{2D}(P) \quad \forall q \in \mathbb{P}_2(P),\\ \widehat{\Pi_P^{\Delta,2D}} w = \widehat{w}; \qquad \widehat{\nabla \Pi_P^{\Delta,2D}} w = \widehat{\nabla w}, \end{cases}$$

where $\widehat{w}$ is defined as $\widehat{w} := \frac{1}{N_{\mathrm{v}}^P}\sum_{i=1}^{N_{\mathrm{v}}^P} w(\mathrm{v}_i)\,\forall\, w \in C^0(\partial P)$ and $\mathrm{v}_i, 1 \leq i \leq N_{\mathrm{v}}^P$, are the vertices of P. We refer to [11, 19] to prove that the operator $\Pi_P^{\Delta,2D}$ is computable from the output values of the sets $\mathbf{D_1}^{2D}$ and $\mathbf{D_2}^{2D}$.

We introduce on each element $P \in \Omega_h$ the following local virtual space $V_h^{2D}(P)$ (see, for instance, [2]):

$$V_h^{2D}(P) := \left\{w_h \in \widetilde{V}_h^{2D}(P) : \int_P (\Pi_P^{\Delta,2D} w_h) q = \int_P w_h q \qquad \forall q \in \mathbb{P}_2(P)\right\}.$$

Now, since $V_h^{2D}(P) \subseteq \widetilde{V}_h^{2D}(P)$ the projector $\Pi_P^{\Delta,2D}$ is well defined and computable in $V_h^{2D}(P)$. In addition, $\mathbb{P}_2(P) \subseteq V_h^{2D}(P)$, which guarantees the good approximation properties of the space. Moreover, the sets of linear operators $\mathbf{D_1}^{2D}$ and $\mathbf{D_2}^{2D}$ constitutes a set of degrees of freedom for $V_h^{2D}(P)$; we refer to [2, Lemma 2.3] for further details.

Now, we introduce the global virtual space by combining the local spaces $V_h^{2D}(P)$ and incorporating the homogeneous boundary conditions. For every decomposition Ω_h of Ω into simple polygons P, we define

$$V_h^{2D} := \left\{w_h \in H_0^2(\Omega) : w_h|_P \in V_h^{2D}(P)\right\}.$$

A set of degrees of freedom for V_h^{2D} is given by all pointwise values of w_h on all vertices of Ω_h together with all pointwise values of ∇w_h on all vertices of Ω_h, excluding the vertices on the boundary (where the values vanishes). Thus, the dimension of V_h^{2D} is three times the number of interior vertices.

3.2. The three-dimensional case. In this section, we introduce the C^1 local virtual space, which has been recently introduced in [6]. Let Ω_h be a discretization of Ω composed by polyhedrons P such that

$\mathbf{A}_1^{3D}$: each element P is star shaped with respect to a ball B_P whose radius is uniformly comparable with the polyhedron diameter, h_P,

$\mathbf{A}_2^{3D}$: each face f is star shaped with respect to a disc B_f whose radius is uniformly comparable with the face diameter, h_f,

$\mathbf{A}_3^{3D}$: given a polyhedron P all its edge lengths and face diameters are uniformly comparable with respect to its diameter h_P.

Now, we recall the definitions of the auxiliary local virtual spaces $V_h^{\nabla}(f)$, $V_h^{\Delta}(f)$ and $\widetilde{V}_h^{3D}(P)$ (see [6]), which are needed to define the local virtual space $V_h^{3D}(P)$ (cf. (3.2)) in 3D. For each face f and polyhedron P, we introduce.

$$V_h^{\nabla}(f) := \Bigg\{ w_h \in H^1(f) : \Delta_\tau w_h \in \mathbb{P}_0(f),\ w_h|_{\partial f} \in C^0(\partial f),\ w_h|_e \in \mathbb{P}_1(e)\ \forall e \in \partial f$$
$$\int_f \Pi_f^{\nabla} w_h = \int_f w_h \Bigg\},$$

$$V_h^{\Delta}(f) := \Bigg\{ w_h \in H^2(f) : \Delta_\tau^2 w_h \in \mathbb{P}_1(f), w_h|_{\partial f} \in C^0(\partial f), w_h|_e \in \mathbb{P}_3(e)\ \forall e \in \partial f,$$
$$\nabla_\tau w_h|_{\partial f} \in [C^0(\partial f)]^2,\ \partial_{\nu_f^e} w_h|_e \in \mathbb{P}_1(e)\ \forall e \in \partial f,$$
$$\int_f \Pi_f^{\Delta} w_h p_1 = \int_f w_h p_1\ \forall p_1 \in \mathbb{P}_1(f) \Bigg\},$$

and

$$\widetilde{V}_h^{3D}(P) := \Bigg\{ w_h \in H^2(P) : \Delta^2 w_h \in \mathbb{P}_2(P),\ w_h|_{S_P} \in C^0(S_P), \nabla w_h|_{S_P} \in [C^0(S_P)]^3,$$
$$w_h|_f \in V_h^{\Delta}(f),\ \partial_{\nu_P^f} w_h|_f \in V_h^{\nabla}(f)\, \forall f \in \partial P \Bigg\},$$

where Δ_τ and ∇_τ are the Laplace and gradient operators in the local face coordinates and ∂_ν denotes the normal derivative on each edge or face. In addition, $\Pi_f^{\nabla} : H^1(f) \to \mathbb{P}_1(f)$ is the standard orthogonal projector introduced in [1, 5], in this case defined on each face f of P; $\Pi_f^{\Delta} : V_h^{\Delta}(f) \to \mathbb{P}_2(f)$ is the projection operator defined on each face f of P as the one defined in (3.1) (see [6]) and S_P denotes the skeleton (the union of all edges) of the polyhedron P.

Now, for all $w_h \in \widetilde{V}_h^{3D}(P)$ we consider the following sets of linear operators:

$\mathbf{D_1}^{3D}$: evaluation of w_h at the N_{v}^P vertices of P;

$\mathbf{D_2}^{3D}$: evaluation of ∇w_h at the N_{v}^P vertices of P.

Next, we consider the projection operator $\Pi_P^{\Delta,3D} : \widetilde{V}_h^{3D}(P) \to \mathbb{P}_2(P)$ defined by

$$\begin{cases} \displaystyle\int_P D^2(\Pi_P^{\Delta,3D} w_h - w_h) : D^2 q = 0 & \forall q \in \mathbb{P}_2(P), \\ \displaystyle\int_{\partial P} (\Pi_P^{\Delta,3D} w_h - w_h) q = 0 & \forall q \in \mathbb{P}_1(P). \end{cases}$$

The above projection operator is computable and uniquely determined by the values of the linear operators $\mathbf{D_1}^{3D}$ and $\mathbf{D_2}^{3D}$.

We are in a position to introduce the local virtual space $V_h^{3D}(P)$:

$$V_h^{3D}(P) := \left\{ w_h \in \widetilde{V}_h^{3D}(P) : \int_P \Pi_P^{\Delta,3D} w_h q = \int_P w_h q \quad \forall q \in \mathbb{P}_2(P) \right\}. \tag{3.2}$$

Now, we introduce the global virtual space by combining the local spaces $V_h^{3D}(P)$ and incorporating the homogeneous boundary conditions. For every decomposition Ω_h of Ω into polyhedrons P, we define.

$$V_h^{3D} := \left\{ w_h \in H_0^2(\Omega) : w_h|_P \in V_h^{3D}(P) \right\}. \tag{3.3}$$

A set of degrees of freedom for V_h^{3D} is given by all pointwise values of w_h on all vertices of Ω_h together with all pointwise values of ∇w_h on all vertices of Ω_h, excluding the vertices on the boundary (where the values vanishes). Thus, the dimension of V_h^{3D} is four times the number of interior vertices.

The virtual space (3.3) has been recently considered in [6] to obtain optimal error estimates for fourth-order PDEs in 3D. Here, we will consider the same space to propose a VEM scheme for the transmission eigenvalue problem.

4. Discrete spectral problem. In this section, we will introduce a virtual element discretization to approximate the spectrum of the transmission eigenvalue problem stated in Problem 1. Due to the discrete analysis holds both in the two- and three-dimensional cases, in what follows, we will omit the superscripts $2D$ and $3D$ used in section 3. Moreover, for simplicity, we assume that the index of refraction n is piecewise constant with respect to the decomposition Ω_h; i.e., n is constant on each polygon/polyhedron $P \in \Omega_h$.

Now, for all $m \in \mathbb{N} \cup \{0\}$ and $P \in \Omega_h$, we define the following projectors:

$$\Pi_P^m : L^2(P) \to \mathbb{P}_m(P); \qquad \int_P (v - \Pi_P^m v) q = 0 \qquad \forall q \in \mathbb{P}_m(P), \tag{4.1}$$

$$\Pi_P^0 \Delta : H^2(P) \to \mathbb{P}_0(P); \qquad \int_P (\Delta w - \Pi_P^0 \Delta w) q = 0 \qquad \forall q \in \mathbb{P}_0(P). \tag{4.2}$$

We refer to [2, 6, 8] to check that for all $w_h \in V_h(P)$ the scalar functions $\Pi_P^2 w_h$ and $\Pi_P^0 \Delta w_h$ are computable from the degrees of freedom $\mathbf{D_1}$ and $\mathbf{D_2}$.

Next, we decompose the continuous sesquilinear forms (2.5)–(2.6) in an element by element contribution:

$$a_1(u,w) := \sum_{P \in \Omega_h} a_1^P(u,w) \qquad \forall (u,w) \in H_0^2(\Omega),$$
$$a_2(z,v) := \sum_{P \in \Omega_h} a_2^P(z,v) \qquad \forall (z,v) \in L^2(\Omega).$$

Now, in order to propose the discrete scheme, we need to introduce some definitions. First, we consider $s^{\Delta,P}(\cdot,\cdot)$ and $s^{0,P}(\cdot,\cdot)$ any Hermitian positive definite forms satisfying

$$\alpha_* a_1^P(w_h,w_h) \le s^{\Delta,P}(w_h,w_h) \le \alpha^* a_1^P(w_h,w_h) \quad \forall w_h \in V_h(P) \quad \Pi_P^{\Delta} w_h = 0, \tag{4.3}$$

$$\beta_* a_2(v_h,v_h) \le s^{0,P}(v_h,v_h) \le \beta^* a_2(v_h,v_h) \quad \forall v_h \in V_h(P), \tag{4.4}$$

where, α_*, β_* and α^*, β^* are positive constants independent of the element P.

Next, we define the discrete versions of the sesquilinear forms presented in (2.5)–(2.7) as follows:

$$a_{1h}: V_h \times V_h \to \mathbb{C}; \qquad a_{1h}(u_h, w_h) := \sum_{P\in\Omega_h} a_{1h}^P(u_h, w_h),$$

$$a_{2h}: V_h \times V_h \to \mathbb{C}; \qquad a_{2h}(z_h, v_h) := \sum_{P\in\Omega_h} a_{2h}^P(z_h, v_h),$$

$$B_h: \mathbb{H}_h \times \mathbb{H}_h \to \mathbb{C}; \quad B_h((u_h, z_h), (w_h, v_h)) := \sum_{P\in\Omega_h} B_h^P((u_h, z_h), (w_h, v_h)),$$

where $\mathbb{H}_h := V_h \times V_h$ and

$$a_{1h}^P : V_h(P) \times V_h(P) \to \mathbb{C}, \quad a_{2h}^P : V_h(P) \times V_h(P) \to \mathbb{C}, \quad B_h^P : \mathbb{H}_h^P \times \mathbb{H}_h^P \to \mathbb{C},$$

are local sesquilinear forms given by

$$a_{1h}^P(u_h, w_h) := a_1^P(\Pi_P^{\Delta} u_h, \Pi_P^{\Delta} w_h) + s^{\Delta,P}(u_h - \Pi_P^{\Delta} u_h, w_h - \Pi_P^{\Delta} w_h), \tag{4.5}$$

$$a_{2h}^P(z_h, v_h) := a_2^P(\Pi_P^2 z_h, \Pi_P^2 v_h) + s^{0,P}(z_h - \Pi_P^2 z_h, v_h - \Pi_P^2 v_h), \tag{4.6}$$

$$\begin{aligned} B_h^P((u_h, z_h), (w_h, v_h)) := &\int_P \frac{n}{n-1} \Pi_P^0 \Delta u_h \Pi_P^2 \overline{w}_h + \int_P \frac{1}{n-1} \Pi_P^2 u_h \Pi_P^0 \Delta \overline{w}_h \\ &+ \int_P \frac{n}{n-1} \Pi_P^2 z_h \Pi_P^2 \overline{w}_h - \int_P \Pi_P^2 u_h \Pi_P^2 \overline{v}_h, \end{aligned} \tag{4.7}$$

where $\mathbb{H}_h^P := V_h(P) \times V_h(P)$.

The following result establishes properties of consistency and stability for the local sesquilinear forms $a_{1h}^P(\cdot,\cdot)$ and $a_{2h}^P(\cdot,\cdot)$.

Proposition 4.1. *The local forms $a_{1h}^P(\cdot,\cdot)$ and $a_{2h}^P(\cdot,\cdot)$ satisfy the following properties:*

- *Consistency: For all $h > 0$ and for all $P \in \Omega_h$ we have that*

$$a_{1h}^P(q, w_h) = a_1^P(q, w_h) \qquad \forall q \in \mathbb{P}_2(P) \quad \forall w_h \in V_h(P), \tag{4.8}$$

$$a_{2h}^P(q, v_h) = a_2^P(q, v_h) \qquad \forall q \in \mathbb{P}_2(P) \quad \forall v_h \in V_h(P). \tag{4.9}$$

- *Stability and boundedness: There exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ depending on the index of refraction n and independent of P, such that*

$$\alpha_1 a_1^P(w_h, w_h) \le a_{1h}^P(w_h, w_h) \le \alpha_2 a_1^P(w_h, w_h) \qquad \forall w_h \in V_h(P); \tag{4.10}$$

$$\beta_1 a_2^P(v_h, v_h) \le a_{2h}^P(v_h, v_h) \le \beta_2 a_2^P(v_h, v_h) \qquad \forall v_h \in V_h(P). \tag{4.11}$$

Proof. The proof follows standard arguments in the VEM literature; it is omitted. □

Now, for all $(u_h, z_h), (w_h, v_h) \in \mathbb{H}_h$, we introduce the discrete sesquilinear form

$$A_h: \mathbb{H}_h \times \mathbb{H}_h \to \mathbb{C}; \qquad A_h((u_h, z_h), (w_h, v_h)) := a_{1h}(u_h, w_h) + a_{2h}(z_h, v_h). \tag{4.12}$$

As consequence of Proposition 4.1 we have the following result, which is the discrete version of Lemma 2.1.

Lemma 4.1. *There exist positive constants C and α that depend on the index of refraction n such that for all $(u_h, z_h), (w_h, v_h) \in \mathbb{H}_h$ we have*

$$A_h((w_h, v_h), (w_h, v_h)) \ge \alpha ||(w_h, v_h)||_{\mathbb{H}}^2, \tag{4.13}$$

$$|A_h((u_h, z_h), (w_h, v_h))| \le C ||(u_h, z_h)||_{\mathbb{H}} ||(w_h, v_h)||_{\mathbb{H}}, \tag{4.14}$$

$$|B_h((u_h, z_h), (w_h, v_h))| \le C ||(u_h, z_h)||_{\mathbb{H}} ||(w_h, v_h)||_{\mathbb{H}}. \tag{4.15}$$

Proof. It is straightforward to prove the estimates (4.13)–(4.15) from Proposition 4.1. ∎

Now, we are in a position to write the virtual element discretization of Problem 1.

PROBLEM 2. *Find* $(\lambda_h,(u_h,z_h))\in\mathbb{C}\times\mathbb{H}_h$ *with* $(u_h,z_h)\neq 0$ *such that*

$$A_h((u_h,z_h),(w_h,v_h))=\lambda_h B_h((u_h,z_h),(w_h,v_h))\qquad\forall(w_h,v_h)\in\mathbb{H}_h. \tag{4.16}$$

In order to characterize the spectrum of Problem 2 we introduce the discrete version of the solution operator $\mathcal{S}$:

$$\begin{aligned}\mathcal{S}_h:\mathbb{H}&\longrightarrow\quad\mathbb{H}_h\subseteq\mathbb{H}\\ (f,g)&\longmapsto\quad\mathcal{S}_h(f,g)=(\widetilde{u}_h,\widetilde{z}_h),\end{aligned}$$

defined as the unique solution (as a consequence of Lemma 4.1 and the Lax–Milgram theorem) of the following source problem

$$A_h((\widetilde{u}_h,\widetilde{z}_h),(w_h,v_h))=B_h((f,g),(w_h,v_h))\qquad\forall(w_h,v_h)\in\mathbb{H}_h. \tag{4.17}$$

We have that operator $\mathcal{S}_h$ is well defined and uniformly bounded. Once more, as in the continuous case, we have that $(\lambda,(u_h,z_h))$ solves Problem 2 if and only if $(\mu_h,(u_h,z_h))$ is an eigenpair of $\mathcal{S}_h$, i.e., $\mathcal{S}_h(u_h,z_h)=\mu_h(u_h,z_h)$, with $\mu_h:=1/\lambda_h$.

4.1. Convergence and error estimates. The aim of this section is to prove the convergence properties and to obtain error estimates of the proposed virtual element scheme stated in Problem 2 for the transmission eigenvalue problem. With this aim, we first establish that $\mathcal{S}_h\to\mathcal{S}$ in norm as $h\to 0$. Then, we will establish a similar convergence result for the corresponding adjoint operators $\mathcal{S}_h^*$ and $\mathcal{S}^*$ of $\mathcal{S}_h$ and $\mathcal{S}$, respectively.

First, we recall the following result on star-shaped polygons/polyhedrons, which is derived by interpolation between Sobolev spaces (see, for instance, [26, Theorem I.1.4] from the analogous result for integer values of s). We mention that this result has been stated in [5, Proposition 4.2] for integer values and follows from the classical Scott-Dupont theory (see [10] and [2, Proposition 3.1]).

PROPOSITION 4.2. *There exists a positive constant* C *such that for all* $w\in H^{\delta}(P)$ *there exists* $w_\pi\in\mathbb{P}_k(P)$, $k\geq 0$ *such that*

$$|w-w_\pi|_{\ell,P}\leq Ch_P^{\delta-\ell}|w|_{\delta,P}\quad 0\leq\delta\leq k+1,\ell=0,\dots,[\delta],$$

with $[\delta]$ *denoting largest integer equal or smaller than* $\delta\in\mathbb{R}$.

The following is an interpolation result in the virtual space V_h (see [2, 6]).

PROPOSITION 4.3. *Assume* $\mathbf{A}_1$–$\mathbf{A}_2$ *in the two-dimensional case or* $\mathbf{A}_1$–$\mathbf{A}_3$ *in the three-dimensional case are satisfied; let* $w\in H^{\varepsilon}(\Omega)$ *with* $\varepsilon\in[2,3]$. *Then, there exist* $w_I\in V_h$ *and* $C>0$, *independent of* h, *such that*

$$\|w-w_I\|_{\ell,\Omega}\leq Ch^{\varepsilon-\ell}|w|_{\varepsilon,\Omega},\qquad\ell=0,1,2.$$

Remark 4.1. In the two-dimensional case, the above result can be found in [2, Proposition 3.1]. In the three-dimensional case, the result can be obtained by repeating the arguments in [6, Proposition 5.2 and Corollary 5.3] and using a Clément-type interpolant.

The following lemma shows that $\mathcal{S}_h$ converges in norm to $\mathcal{S}$ as h goes to zero.

LEMMA 4.2. *There exist $s \in (0,1]$ and a positive constant $C > 0$ that depends on the index of refraction n, both independent of the mesh size h such that for all $(f,g) \in \mathbb{H}$, if $(\widetilde{u},\widetilde{z}) = \mathcal{S}(f,g)$ and $(\widetilde{u}_h,\widetilde{z}_h) = \mathcal{S}_h(f,g)$; then*

$$||\left(\mathcal{S}-\mathcal{S}_h\right)(f,g)||_{\mathbb{H}} \leq Ch^s||(f,g)||_{\mathbb{H}}.$$

Proof. Let $(f,g) \in \mathbb{H}$. As a consequence of Lemma 2.2, there exists $s \in (0,1]$ such that $(\widetilde{u},\widetilde{z}) \in H^{2+s}(\Omega)\times H^2(\Omega)$. Let $(\widetilde{u}_I,\widetilde{z}_I) \in \mathbb{H}_h$ be such that Proposition 4.3 holds true. By using the triangular inequality, we have

$$\begin{aligned}||(\mathcal{S}-\mathcal{S}_h)(f,g)||_{\mathbb{H}} &= ||(\widetilde{u},\widetilde{z})-(\widetilde{u}_h,\widetilde{z}_h)||_{\mathbb{H}}\\ &\leq ||(\widetilde{u},\widetilde{z})-(\widetilde{u}_I,\widetilde{z}_I)||_{\mathbb{H}} + ||(\widetilde{u}_I,\widetilde{z}_I)-(\widetilde{u}_h,\widetilde{z}_h)||_{\mathbb{H}}.\end{aligned} \tag{4.18}$$

Now, we define $(w_h,v_h) := (\widetilde{u}_h-\widetilde{u}_I,\widetilde{z}_h-\widetilde{z}_I) \in \mathbb{H}_h$, using the ellipticity of the sesquilinear form $A_h(\cdot,\cdot)$ (cf. (2.8)) and the definition of the operators $\mathcal{S}$ and $\mathcal{S}_h$; for all $\widetilde{u}_\pi,\widetilde{z}_\pi \in \mathbb{P}_2(P)$, we get

$$\begin{aligned}\alpha||(w_h,v_h)||_{\mathbb{H}}^2 &\leq A_h((w_h,v_h),(w_h,v_h)) = A_h((\widetilde{u}_h,\widetilde{z}_h),(w_h,v_h)) - A_h((\widetilde{u}_I,\widetilde{z}_I),(w_h,v_h))\\
&= B_h((f,g),(w_h,v_h)) - \sum_{P\in\Omega_h}\Big\{a_{1h}^P(\widetilde{u}_I,w_h)+a_{2h}^P(\widetilde{z}_I,v_h)\Big\}\\
&= B_h((f,g),(w_h,v_h)) - \sum_{P\in\Omega_h}\Big\{\{a_{1h}^P(\widetilde{u}_I-\widetilde{u}_\pi,w_h)+a_1^P(\widetilde{u}_\pi-\widetilde{u},w_h)\}\\
&\quad + \{a_{2h}^P(\widetilde{z}_I-\widetilde{z}_\pi,v_h)+a_2^P(\widetilde{z}_\pi-\widetilde{z},v_h)\} + \{a_1^P(\widetilde{u},w_h)+a_2^P(\widetilde{z},v_h)\}\Big\}\\
&= \sum_{P\in\Omega_h}\underbrace{\{B_h^P((f,g),(w_h,v_h)) - B^P((f,g),(w_h,v_h))\}}_{G^{1,P}}\\
&\quad - \sum_{P\in\Omega_h}\underbrace{\{a_{1h}^P(\widetilde{u}_I-\widetilde{u}_\pi,w_h)+a_1^P(\widetilde{u}_\pi-\widetilde{u},w_h)\}}_{G^{2,P}}\\
&\quad - \sum_{P\in\Omega_h}\underbrace{\{a_{2h}^P(\widetilde{z}_I-\widetilde{z}_\pi,v_h)+a_2^P(\widetilde{z}_\pi-\widetilde{z},v_h)\}}_{G^{3,P}}\\
&=: \sum_{P\in\Omega_h}G^{1,P} - \sum_{P\in\Omega_h}G^{2,P} - \sum_{P\in\Omega_h}G^{3,P},\end{aligned} \tag{4.19}$$

where we have used the consistency properties (4.8) and (4.9).

In what follows, we will bound the terms $G^{1,P}, G^{2,P}$, and $G^{3,P}$. Indeed, for the term $G^{1,P}$ we use the definitions of $B(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$ (cf. (2.7) and (4.7), respectively) to obtain

$$\begin{aligned}G^{1,P} &= \underbrace{\int_P\Big\{\frac{n}{n-1}\Pi_P^0\Delta f\Pi_P^2\overline{w}_h - \frac{n}{n-1}\Delta f\overline{w}_h\Big\}}_{G^{11,P}}\\
&\quad + \underbrace{\int_P\Big\{\frac{1}{n-1}\Pi_P^2 f\Pi_P^0\Delta\overline{w}_h - \frac{1}{n-1}f\Delta\overline{w}_h\Big\}}_{G^{12,P}}\\
&\quad + \underbrace{\int_P\Big\{\frac{n}{n-1}\Pi_P^2 g\Pi_P^2\overline{w}_h - \frac{n}{n-1}g\overline{w}_h\Big\}}_{G^{13,P}} + \underbrace{\int_P\Big\{\Pi_P^2 f\Pi_P^2\overline{v}_h - f\overline{v}_h\Big\}}_{G^{14,P}}\\
&=: G^{11,P}+G^{12,P}+G^{13,P}+G^{14,P}.\end{aligned} \tag{4.20}$$

Now, let us bound each term on the right-hand side of (4.20). We start with the term $G^{11,P}$: We add and subtract the term $\frac{n}{n-1}\Delta f \Pi_P^2 \overline{w}_h$, and we get

$$G^{11,P} = \int_P \frac{n}{n-1}\Big(\Pi_P^0 \Delta f - \Delta f\Big)\Pi_P^2 \overline{w}_h + \int_P \frac{n}{n-1}\Delta f\Big(\Pi_P^2 \overline{w}_h - \overline{w}_h\Big).$$

Next, adding and subtracting the term $\int_P \frac{n}{n-1}(\Pi_P^0 \Delta f - \Delta f)\overline{w}_h$ and using the definition of Π_P^2 in the last equality we obtain

$$\begin{aligned} G^{11,P} &= \int_P \frac{n}{n-1}\Big(\Pi_P^0 \Delta f - \Delta f\Big)\Big(\Pi_P^2 \overline{w}_h - \overline{w}_h\Big) + \int_P \frac{n}{n-1}\Delta f\Big(\Pi_P^2 \overline{w}_h - \overline{w}_h\Big) \\ &\quad + \int_P \Big(\Pi_P^0 \Delta f - \Delta f\Big)\Big(\frac{n}{n-1}\overline{w}_h - \Pi_P^0\Big(\frac{n}{n-1}\overline{w}_h\Big)\Big). \end{aligned} \tag{4.21}$$

Now, using the Cauchy–Schwarz inequality and the fact that $n/(n-1) \in L^\infty(\Omega)$ we have from (4.21) the following estimates:

$$\begin{aligned} G^{11,P} &\leq ||n/(n-1)||_{L^\infty(P)}\Big\{||\Pi_P^0 \Delta f - \Delta f||_{0,P}||\Pi_P^2 \overline{w}_h - \overline{w}_h||_{0,P} \\ &\quad + ||\Delta f||_{0,P}||\Pi_P^0 \overline{w}_h - \overline{w}_h||_{0,P} + ||\Pi_P^0 \Delta f - \Delta f||_{0,P}\|\overline{w}_h - \Pi_P^2 \overline{w}_h\|_{0,P}\Big\} \\ &\leq C||n/(n-1)||_{L^\infty(P)}||\Delta f||_{0,P}\Big\{h_P^2|w_h|_{2,P} + h_P|w_h|_{1,P}\Big\} \\ &\leq Ch_P||n/(n-1)||_{L^\infty(P)}|f|_{2,P}\Big\{|w_h|_{2,P} + |w_h|_{1,P}\Big\}. \end{aligned} \tag{4.22}$$

For the term $G^{12,P}$, we add and subtract $\frac{1}{n-1} f \Pi_P^0 \Delta \overline{w}_h$; then we use the definition of Π_P^0 to obtain

$$\begin{aligned} G^{12,P} &= \int_P \Big\{\frac{1}{n-1}\Big(\Pi_P^2 f - f\Big)\Pi_P^0 \Delta \overline{w}_h + \frac{1}{n-1} f\Big(\Pi_P^0 \Delta \overline{w}_h - \Delta \overline{w}_h\Big)\Big\} \\ &= \int_P \Big\{\frac{1}{n-1}\Big(\Pi_P^2 f - f\Big)\Pi_P^0 \Delta \overline{w}_h\Big\} \\ &\quad + \int_P \Big\{\Big(\frac{1}{n-1} f - \Pi_P^0\big(\frac{1}{n-1} f\big)\Big)\Big(\Pi_P^0 \Delta \overline{w}_h - \Delta \overline{w}_h\Big)\Big\}. \end{aligned}$$

Once again, by the Cauchy–Schwarz inequality and the fact that $1/(n-1) \in L^\infty(\Omega)$, we get

$$\begin{aligned} G^{12,P} &\leq ||1/(n-1)||_{L^\infty(P)}\Big\{||\Pi_P^2 f - f||_{0,P}||\Pi_P^0 \Delta \overline{w}_h||_{0,P} \\ &\quad + \|f - \Pi_P^0 f\|_{0,P}||\Pi_P^0 \Delta \overline{w}_h - \Delta \overline{w}_h||_{0,P}\Big\} \\ &\leq C||1/(n-1)||_{L^\infty(P)}\Big\{h_P^2|f|_{2,P}||\Delta w_h||_{0,P} + Ch_P|f|_{1,P}||\Delta w_h||_{0,P}\Big\} \\ &\leq Ch_P||1/(n-1)||_{L^\infty(P)}\Big\{|f|_{2,P} + |f|_{1,P}\Big\}|w_h|_{2,P}. \end{aligned} \tag{4.23}$$

Now, to bound the term $G^{13,P}$, we use the fact that n is piecewise constant, the definition of Π_P^2, the Cauchy–Schwarz inequality, and $n/(n-1) \in L^\infty(\Omega)$ to have

$$
\begin{aligned}
G^{13,P} &= \int_P \left\{ \frac{n}{n-1}\Pi_P^2 g \Pi_P^2 \overline{w}_h - \frac{n}{n-1} g\overline{w}_h \right\} = \int_P \frac{n}{n-1}\left\{ g(\Pi_P^2\overline{w}_h - \overline{w}_h)\right\} \\
&= \int_P \frac{n}{n-1}\left\{ (g-\Pi_P^2 g)(\Pi_P^2\overline{w}_h - \overline{w}_h)\right\} \\
&\leq C||n/(n-1)||_{L^\infty(P)}||g-\Pi_P^2 g||_{0,P}||\Pi_P^2\overline{w}_h - \overline{w}_h||_{0,P} \\
&\leq Ch_P^2||n/(n-1)||_{L^\infty(P)}||g||_{0,P}|\overline{w}_h|_{2,P}. \qquad (4.24)
\end{aligned}
$$

For the term $G^{14,P}$, we use the definition of Π_P^2 and the Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
G^{14,P} &= \int_P (f-\Pi_P^2 f)(\overline{v}_h - \Pi_P^2\overline{v}_h) \leq ||f-\Pi_P^2 f||_{0,P}||\overline{v}_h - \Pi_P^2\overline{v}_h||_{0,P} \\
&\leq Ch_P^2|f|_{2,P}||\overline{v}_h||_{0,P}. \qquad (4.25)
\end{aligned}
$$

Now, taking sum over P in the terms (4.22), (4.23), (4.24), and (4.25) and applying the Cauchy–Schwarz inequality for sequences we obtain

$$
\sum_{P\in\Omega_h} G^{1,P} \leq Ch\max\{||n/(n-1)||_{L^\infty(\Omega)}, ||1/(n-1)||_{L^\infty(\Omega)}\}||(f,g)||_{\mathbb{H}}||(w_h,v_h)||_{\mathbb{H}}. \qquad (4.26)
$$

On the other hand, to bound the term $\sum_{P\in\Omega_h} G^{2,P}$, we use the Cauchy–Schwarz inequality and the stability and boundedness properties of $a_1^P(\cdot,\cdot)$ (cf. (4.10)) to get

$$
\begin{aligned}
\sum_{P\in\Omega_h} G^{2,P} &= \sum_{P\in\Omega_h}\left\{ a_{1h}^P(\widetilde{u}_I - \widetilde{u}_\pi, w_h) + a_1^P(\widetilde{u}_\pi - \widetilde{u}, w_h)\right\} \\
&\leq \sum_{P\in\Omega_h}\left\{ a_{1h}^P(\widetilde{u}_I - \widetilde{u}_\pi, \widetilde{u}_I - \widetilde{u}_\pi)^{1/2} a_{1h}^P(w_h,w_h) + a_1^P(\widetilde{u}_\pi - \widetilde{u}, \widetilde{u}_\pi - \widetilde{u})^{1/2} a_1^P(w_h,w_h)^{1/2}\right\} \\
&\leq \sum_{P\in\Omega_h}\left\{ |\widetilde{u}_I - \widetilde{u}_\pi|_{2,P}|w_h|_{2,P} + |\widetilde{u}_\pi - \widetilde{u}|_{2,P}|w_h|_{2,P}\right\} \\
&\leq \sum_{P\in\Omega_h}\left\{ |\widetilde{u}_I - \widetilde{u}|_{2,P} + 2|\widetilde{u} - \widetilde{u}_\pi|_{2,P}\right\}|w_h|_{2,P}.
\end{aligned}
$$

Next, from Propositions 4.3 and 4.2 and Lemma 2.2, we have

$$
\sum_{P\in\Omega_h} G^{2,P} \leq Ch^s||(f,g)||_{\mathbb{H}}||(w_h,v_h)||_{\mathbb{H}}. \qquad (4.27)
$$

To bound the expression $\sum_{P\in\Omega_h} G^{3,P}$, we use the Cauchy–Schwarz inequality, and we add and subtract the term $\widetilde{z}$ to obtain

$$
\begin{aligned}
\sum_{P\in\Omega_h} G^{3,P} &= \sum_{P\in\Omega_h}\left\{ a_{2h}^P(\widetilde{z}_I - \widetilde{z}_\pi, v_h) + a_2^P(\widetilde{z}_\pi - \widetilde{z}, v_h)\right\} \\
&\leq \sum_{P\in\Omega_h}\left\{ ||\widetilde{z}_I - \widetilde{z}||_{0,P} + 2||\widetilde{z} - \widetilde{z}_\pi||_{0,P}\right\}||v_h||_{0,P}.
\end{aligned}
$$

Hence, applying Proposition 4.2 and Proposition 4.3 (with $\varepsilon = 2$ and $\ell = 0$) and Lemma 2.2 in the above inequality we deduce

$$
\sum_{P\in\Omega_h} G^{3,P} \leq Ch^2||(f,g)||_{\mathbb{H}}||(w_h,v_h)||_{\mathbb{H}}. \qquad (4.28)
$$

Now, by combining (4.19) with (4.26), (4.27), and (4.28), we obtain

$$||(\widetilde{u}_I, \widetilde{z}_I) - (\widetilde{u}_h, \widetilde{z}_h)||_{\mathbb{H}} = ||(w_h, v_h)||_{\mathbb{H}} \leq \frac{C}{\alpha} h^s ||(f,g)||_{\mathbb{H}}. \tag{4.29}$$

Finally, the proof follows from (4.18) and (4.29) and Lemma 2.2. ∎

Now, let $\mathcal{S}_h^* : \mathbb{H} \to \mathbb{H}$ be the adjoint operator of $\mathcal{S}_h$. This operator is defined by $\mathcal{S}_h^*(f,g) := (\tilde{u}_h^*, \tilde{z}_h^*)$, where $(\tilde{u}_h^*, \tilde{z}_h^*)$ is the unique solution of the following source problem:

$$A_h((w_h, v_h), (\tilde{u}_h^*, \tilde{z}_h^*)) = B_h((w_h, v_h), (f,g)) \quad \forall (w_h, v_h) \in \mathbb{H}_h. \tag{4.30}$$

Now, we will show the convergence in norm of the operator $\mathcal{S}_h^*$ (cf. (4.30)) to $\mathcal{S}^*$ (cf. (2.12)) as h goes to zero.

LEMMA 4.3. *There exist a positive constant C that depends on the index of refraction n and $s \in (0,1]$, both independent of the mesh size h, such that for all $(f,g) \in \mathbb{H}$, if $(\widetilde{u}^*, \widetilde{z}^*) = \mathcal{S}^*(f,g)$ and $(\widetilde{u}_h^*, \widetilde{z}_h^*) = \mathcal{S}_h^*(f,g)$, then*

$$||\left(\mathcal{S}^* - \mathcal{S}_h^*\right)(f,g)||_{\mathbb{H}} \leq C h^s ||(f,g)||_{\mathbb{H}}.$$

Proof. The proof is obtained using the same arguments as those used to prove Lemma 4.2. ∎

In what follows, we will establish convergence and obtain error estimates of our discrete scheme. To do that, we will apply the abstract spectral theory from [4, 33] for non-self-adjoint compact operators.

We first recall the definition of the spectral projectors. Let μ be a nonzero eigenvalue of $\mathcal{S}$ with algebraic multiplicity m, and let $\mathcal{D}$ be an open disk in the complex plane centered at μ such that μ is the only eigenvalue of $\mathcal{S}$ lying in $\mathcal{D}$ and $\partial\mathcal{D} \cap \operatorname{sp}(\mathcal{S}) = \emptyset$. The spectral projectors $\mathcal{E}$ and $\mathcal{E}^*$ are defined as follows:

- the spectral projector of $\mathcal{S}$ relative to μ: $\mathcal{E} := (2\pi i)^{-1} \int_{\partial\mathcal{D}} (z - \mathcal{S})^{-1} dz$;
- the spectral projector of $\mathcal{S}^*$ relative to $\overline{\mu}$: $\mathcal{E}^* := (2\pi i)^{-1} \int_{\partial\mathcal{D}} (z - \mathcal{S}^*)^{-1} dz$.

Moreover, $\mathcal{E}$ and $\mathcal{E}^*$ are projections onto the space of generalized eigenvectors $R(\mathcal{E})$ and $R(\mathcal{E}^*)$, respectively. It is easy to check that $R(\mathcal{E}), R(\mathcal{E}^*) \in [H^{2+s}(\Omega)]^2$ (see Lemma 2.3).

As a consequence of the convergence in norm of $\mathcal{S}_h$ to $\mathcal{S}$ (cf. Lemma 4.2), there exist m eigenvalues (which lie in $\mathcal{D}$) $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of $\mathcal{S}_h$ (repeated according to their respective multiplicities) which will converge to μ as h goes to zero.

Analogously, we introduce the following spectral projector $\mathcal{E}_h := (2\pi i)^{-1} \int_{\partial\mathcal{D}} (z - \mathcal{S}_h)^{-1} dz$, which is a projector onto the invariant subspace $R(\mathcal{E}_h)$ of $\mathcal{S}_h$ spanned by the generalized eigenvectors of $\mathcal{S}_h$ corresponding to $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$.

On the other hand, we recall the definition of the *gap* $\widehat{\delta}$ between two closed subspaces $\mathbb{X}$ and $\mathbb{Y}$ of a Hilbert space $\mathbb{H}$:

$$\widehat{\delta}(\mathbb{X}, \mathbb{Y}) := \max\left\{\delta(\mathbb{X}, \mathbb{Y}), \delta(\mathbb{Y}, \mathbb{X})\right\},$$

where

$$\delta(\mathbb{X}, \mathbb{Y}) := \sup_{\mathbf{x} \in \mathbb{X}:\ \|\mathbf{x}\|_{\mathbb{H}} = 1} \delta(\mathbf{x}, \mathbb{Y}), \quad \text{with} \quad \delta(\mathbf{x}, \mathbb{Y}) := \inf_{\mathbf{y} \in \mathbb{Y}} \|\mathbf{x} - \mathbf{y}\|_{\mathbb{H}}.$$

The following theorem establishes the error estimates for the approximation of eigenvalues and eigenfunctions.

THEOREM 4.1. *There exists a strictly positive constant C that depends on the index of refraction such that*

$$\widehat{\delta}(R(\mathcal{E}), R(\mathcal{E}_h)) \leq Ch^s, \tag{4.31}$$

$$|\mu - \hat{\mu}_h| \leq Ch^{2s}, \tag{4.32}$$

where $\hat{\mu}_h := \frac{1}{m}\sum_{k=1}^m \mu_h^{(k)}$ *and* $s \in (0,1]$ *as in Lemma* 2.3.

Proof. The estimate (4.31) follows as a direct consequence of [4, Theorem 7.1] by combining the convergence in norm of $\mathcal{S}_h$ to $\mathcal{S}$ as h goes to zero stated in Lemma 4.2 and the fact that for $(f,g) \in R(\mathcal{E})$, $||(f,g)||_{[H^{2+s}(\Omega)]^2} \leq C||(f,g)||_{\mathbb{H}}$ (cf. Lemma 2.3).

Now, to prove the estimate (4.32), we will use [4, Theorem 7.2]. With this end, we assume that $\mathcal{S}(u_k, z_k) = \mu(u_k, z_k)$, $k = 1, \ldots, m$. Next, since $A(\cdot,\cdot)$ is an inner product in $\mathbb{H}$, we can choose a dual basis for $R(\mathcal{E}^*)$ denoted by $(u_k^*, z_k^*) \in \mathbb{H}$ satisfying

$$A((u_k, z_k), (u_l^*, z_l^*)) = \delta_{k,l}.$$

From [4, Theorem 7.2], we have the following estimate:

$$\begin{aligned}|\mu - \hat{\mu}_h| \leq \frac{1}{m}\sum_{k=1}^m |\langle(\mathcal{S} - \mathcal{S}_h)(u_k, z_k), (u_k^*, z_k^*)\rangle| \\ + C||(\mathcal{S} - \mathcal{S}_h)|_{R(\mathcal{E})}||_{\mathbb{H}}||(\mathcal{S}^* - \mathcal{S}_h^*)|_{R(\mathcal{E}^*)}||_{\mathbb{H}},\end{aligned} \tag{4.33}$$

where $\langle\cdot,\cdot\rangle$ denotes the corresponding duality pairing.

In what follows, we focus on finding upper bounds for the two terms on the right-hand side above. Indeed, the second term can be easily bounded from Lemmas 4.2 and 4.3 as follows:

$$||(\mathcal{S} - \mathcal{S}_h)|_{R(\mathcal{E})}||_{\mathbb{H}}||(\mathcal{S}^* - \mathcal{S}_h^*)|_{R(\mathcal{E}^*)}||_{\mathbb{H}} \leq Ch^{2s}. \tag{4.34}$$

Now, we bound the first term on the right-hand side of (4.33) as follows: Adding and subtracting $(w_h, v_h) \in \mathbb{H}_h$ and using the definition of $\mathcal{S}$ and $\mathcal{S}_h$, we obtain

$$\begin{aligned}\langle(\mathcal{S} - \mathcal{S}_h)(u_k, z_k), (u_k^*, z_k^*)\rangle &= A((\mathcal{S} - \mathcal{S}_h)(u_k, z_k), (u_k^*, z_k^*)) \\ &= \Big\{A((\mathcal{S} - \mathcal{S}_h)(u_k, z_k), (u_k^*, z_k^*) - (w_h, v_h))\Big\} \\ &\quad + \Big\{B((u_k, z_k), (w_h, v_h)) - B_h((u_k, z_k), (w_h, v_h))\Big\} \\ &\quad + \Big\{A_h(\mathcal{S}_h(u_k, z_k), (w_h, v_h)) - A(\mathcal{S}_h(u_k, z_k), (w_h, v_h))\Big\}\end{aligned} \tag{4.35}$$

for all $(w_h, v_h) \in \mathbb{H}_h$. For the first and the third bracket on the right-hand side above, we can repeat the same steps used in the proof of Theorem 4.1 in [32] to obtain that

$$A((\mathcal{S} - \mathcal{S}_h)(u_k, z_k), (u_k^*, z_k^*) - (w_h, v_h)) \leq Ch^{2s}||(u_k^*, z_k^*)||_{\mathbb{H}} \tag{4.36}$$

and

$$A_h(\mathcal{S}_h(u_k, z_k), (w_h, v_h)) - A(\mathcal{S}_h(u_k, z_k), (w_h, v_h)) \leq Ch^{2s}||(u_k, z_k)||_{\mathbb{H}}||(u_k^*, z_k^*)||_{\mathbb{H}}. \tag{4.37}$$

Finally, for the second bracket on the right-hand side of (4.35), we use the additional regularity of $(u_k, z_k) \in R(\mathcal{E}) \subset [H^{2+s}(\Omega)]^2$, and repeating the same steps used to obtain (4.20) (in this case with $(u_k, z_k) \in [H^{2+s}(\Omega)]^2$ instead of $(f,g) \in \mathbb{H}$), we get

$$B_h((u_k, z_k), (w_h, v_h)) - B((u_k, z_k), (w_h, v_h)) \leq Ch^{2s}||(u_k, z_k)||_{\mathbb{H}}||(u_k^*, z_k^*)||_{\mathbb{H}}. \tag{4.38}$$

Next, from (4.35), (4.36), (4.37), and (4.38), we have

$$|\langle(\mathcal{S} - \mathcal{S}_h)(u_k, z_k), (u_k^*, z_k^*)\rangle| \leq Ch^{2s}. \tag{4.39}$$

Therefore, the estimate (4.32) is obtained from (4.34) and (4.39). The proof is complete. □

5. Numerical examples. We report in this section the results of some numerical tests carried out with the discrete scheme presented in Problem 2 in the two-dimensional case, which confirm the theoretical results proved above. The numerical method has been implemented in a MATLAB code.

In order to compare our results with those presented in the literature of the transmission eigenvalue problem, we have chosen three configurations for the computational domain Ω:

$$\text{Square domain:} \quad \Omega_{\mathbf{S}} := (0,1) \times (0,1), \tag{5.1}$$

$$\text{L-shaped domain:} \quad \Omega_{\mathbf{L}} := (-1/2, 1/2)^2 \backslash ([0, 1/2] \times [-1/2, 0]), \tag{5.2}$$

$$\text{Circular domain:} \quad \Omega_{\mathbf{C}} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1/4\}. \tag{5.3}$$

On the other hand, we have tested the method by using different families of polygonal meshes (see Figure 5.1):

- Ω_h^s: rectangular meshes;
- Ω_h^t: triangular meshes;
- Ω_h^{dh} non-structured hexagonal meshes made of convex hexagons;
- Ω_h^v: Voronoi meshes which have been partitioned with N_P number of polygons.

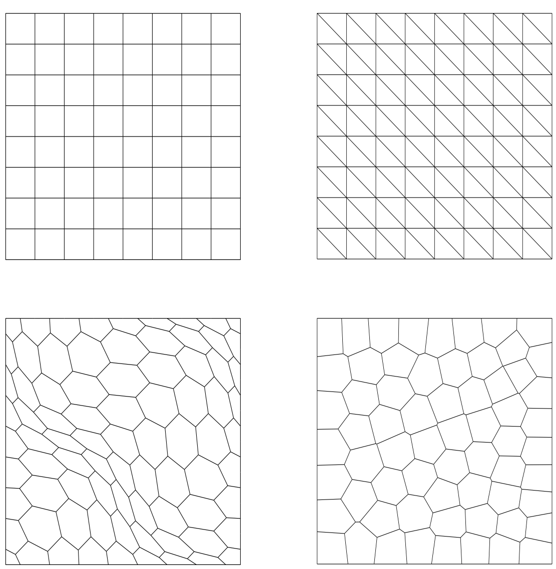

Fig. 5.1. *Sample meshes: Ω_h^s (top left), Ω_h^t (top right), Ω_h^{dh} (bottom left), and Ω_h^v (bottom right).*

We have used successive refinements of an initial mesh (see Figure 5.1). The refinement parameters N and N_P used to label each mesh are the number of elements on each edge of $\Omega_{\mathbf{S}}$ or $\Omega_{\mathbf{L}}$, and the number of polygons inside of the computational domain, respectively.

On the other hand, to complete the choice of the VEM scheme, we had to fix the forms $s^{\Delta,P}(\cdot,\cdot)$ and $s^{0,P}(\cdot,\cdot)$ satisfying (4.3) and (4.4), respectively. In particular, we have considered the form

$$s^P(u_h,w_h) := \sum_{i=1}^{N_{\mathrm{v}}^P}[u_h(\mathrm{v}_i)w_h(\mathrm{v}_i) + h_{\mathrm{v}_i}^2\nabla u_h(\mathrm{v}_i)\cdot\nabla w_h(\mathrm{v}_i)] \qquad \forall u_h,w_h\in V_h(P),$$

where $\mathrm{v}_1,\ldots,\mathrm{v}_{N_{\mathrm{v}}^P}$ are the vertices of P and h_{v_i} corresponds to the maximum diameter of the elements with v_i as a vertex. Thus, we take $s^{\Delta,P}(\cdot,\cdot)$ and $s^{0,P}(\cdot,\cdot)$ in terms of $s^P(\cdot,\cdot)$, properly scaled to satisfy (4.3) and (4.4), respectively (see [2, 19, 31, 32] for further details).

5.1. Test 1: Square domain $\Omega_{\mathbf{S}}$. In this numerical test, we have computed the four lowest transmission eigenvalues κ_{ih}, $i=1,2,3,4$, with three different choice of the index of refraction on the square domain $\Omega_{\mathbf{S}}$ (cf. (5.1)).

We report in Tables 5.1 and 5.2 the lowest transmission eigenvalues κ_{ih}, $i=1,2,3,4$, computed with the discrete virtual scheme (4.16) with indexes of refraction $n=16$ and $n=4$, respectively. We compare the performance of the proposed method with those presented in [22, 28, 32], so we have included in the last row of Tables 5.1 and 5.2 the results reported in these references for the same problem. The table also includes the estimated orders of convergence for each eigenvalue as well as more accurate values of the transmission eigenvalues extrapolated from the computed ones by means of a least-squares fitting of the model

$$\kappa_{ih} \approx \kappa_i + C_i h^{\alpha_i}.$$

Namely, we have computed approximate eigenvalues κ_{ih_j} on several meshes with mesh size h_j, and we find values of κ_i, C_i, and α_i that minimize

$$\sum_j \left(\kappa_{ih_j} - \kappa_i - C_i h_j^{\alpha_i}\right)^2.$$

This minimization problem is nonlinear in α but linear in the other two parameters. Then, for each α, it is easy to compute the minimum in the other parameters. Thus, we obtain a function of one variable, α, that we minimize by a standard procedure. Finally, the fitted parameters κ_i and α_i are the extrapolated transmission eigenvalue (Extrap.) and the estimated order of convergence (Order), respectively.

We can appreciate from Tables 5.1 and 5.2 that the order of convergence of the proposed virtual element scheme (4.16) is quadratic (as predicted by the theory for convex domains). Moreover, we show in Figure 5.2 the eigenfunctions corresponding to the four lowest transmission eigenvalues with an index of refraction $n=16$.

Now, we test the properties of the virtual scheme by considering a nonconstant index of refraction n. More precisely, we consider the following index of refraction $n(x,y) := 8+x-y\,\forall\,(x,y)\in(0,1)^2$.

TABLE 5.1
Test 1: Lowest transmission eigenvalues κ_{ih}, $i = 1,2,3,4$, computed on different families of meshes, on the square domain $\Omega_{\mathbf{S}}$ and with an index of refraction $n = 16$.

$\Omega_{\mathbf{S}}$		κ_{1h}	κ_{2h}	κ_{3h}	κ_{4h}
	$N = 32$	1.8864	2.4547	2.4608	2.8910
	$N = 64$	1.8813	2.4469	2.4484	2.8726
Ω_h^t	$N = 128$	1.8800	2.4449	2.4453	2.8680
	Order	2.00	2.00	2.00	2.00
	Extrap.	1.8796	2.4442	2.4442	2.8664
	$N = 32$	1.8936	2.4667	2.4773	2.9083
	$N = 64$	1.8831	2.4499	2.4528	2.8771
Ω_h^{dh}	$N = 128$	1.8805	2.4457	2.4464	2.8691
	Order	1.98	1.97	1.95	1.97
	Extrap.	1.8796	2.4442	2.4442	2.8664
	$N_P = 1024$	1.8883	2.4611	2.4617	2.8948
	$N_P = 4096$	1.8816	2.4483	2.4484	2.8728
Ω_h^v	$N_P = 16384$	1.8801	2.4452	2.4452	2.8680
	Order	2.15	2.10	2.10	2.18
	Extrap.	1.8797	2.4443	2.4443	2.8666
[22]	[Argyris method]	1.8651	2.4255	2.4271	2.8178
[32]	[VEM]	1.8796	2.4442	2.4442	2.8664

TABLE 5.2
Test 1: Lowest transmission eigenvalues κ_{ih}, $i = 1,2,3,4$, computed on different families of meshes, on the square domain $\Omega_{\mathbf{S}}$ and with an index of refraction $n = 4$.

$\Omega_{\mathbf{S}}$		κ_{1h}	κ_{2h}	κ_{3h}	κ_{4h}
	$N = 32$	4.2558-1.1841i	4.2558+1.1841i	5.6065	5.6065
	$N = 64$	4.2676-1.1567i	4.2676+1.1567i	5.5063	5.5063
Ω_h^s	$N = 128$	4.2707-1.1497i	4.2707+1.1497i	5.4835	5.4835
	Order	1.99	1.99	2.14	2.14
	Extrap.	4.2717-1.1474i	4.2717+1.1474i	5.4768	5.4768
	$N = 32$	4.2516-1.1937i	4.2516+1.1937i	5.6458	5.7298
	$N = 64$	4.2664-1.1595i	4.2664+1.1595	5.5164	5.5343
Ω_h^{dh}	$N = 128$	4.2704-1.1505i	4.2704+1.1505i	5.4861	5.4905
	Order	1.91	1.91	2.09	2.16
	Extrap.	4.2718-1.1473i	4.27181.1473i	5.4767	5.4779
	$N_P = 1024$	4.2573-1.1791i	4.2573+1.1791i	5.6053	5.6063
	$N_P = 4096$	4.2682-1.1554i	4.2682+1.1554i	5.5056	5.5059
Ω_h^v	$N_P = 16384$	4.2708-1.1494i	4.2708+1.1494i	5.4834	5.4834
	Order	1.99	1.99	2.17	2.16
	Extrap.	4.2716-1.1474i	4.2716+1.1474i	5.4771	5.4769
[28]	[Multigrid FEM]	4.2717-1.1474i	4.2717+1.1474i	5.4761	5.4761
[32]	[VEM]	4.2718-1.1475i	4.2718+1.1475i	5.4779	5.4765

With this aim, we report in Table 5.3 the four lowest transmission eigenvalues on a square domain $\Omega_{\mathbf{S}}$ with the family of meshes Ω_h^t and $N = 32, 64, 128$. The table includes orders of convergences as well as accurate values extrapolated by means of a least-squares fitting. We compare the performance of the proposed method with those presented in [18]. Once again, it can be clearly observed from Table 5.3 that the eigenvalue approximation order of our method is quadratic.

5.2. Test 2: L-shaped domain $\Omega_{\mathbf{L}}$. In this numerical test we consider an L-shaped domain $\Omega_{\mathbf{L}}$ (cf. (5.2)). We take the index of refraction $n = 16$, and we compute the four lowest transmission eigenvalues κ_{ih}, $i = 1,2,3,4$.

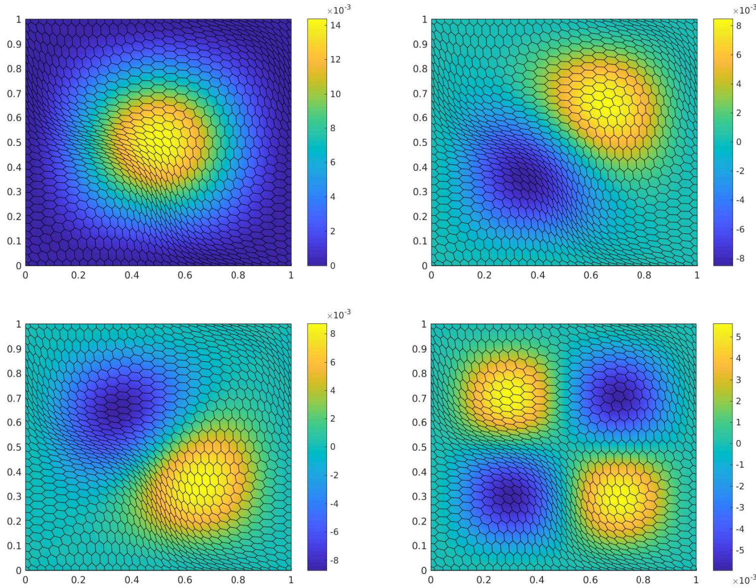

FIG. 5.2. *Test 1: Eigenfunctions u_{1h} (top left), u_{2h} (top right), u_{3h} (bottom left), and u_{4h} (bottom right) associated to the eigenvalues $\kappa_{1h}, \kappa_{2h}, \kappa_{3h}$, and κ_{4h}, respectively.*

TABLE 5.3
Test 1: Lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4, 5$ computed on the mesh Ω_h^t and with an index of refraction $n(x, y) := 8 + x - y \,\forall\, (x, y) \in (0, 1)^2$.

$\Omega_{\mathbf{S}}$		κ_{1h}	κ_{2h}	κ_{3h}	κ_{4h}	κ_{5h}
	$N = 32$	2.8329	3.5512	3.5571	4.1374	4.5322
	$N = 64$	2.8248	3.5418	3.5434	4.1225	4.5093
Ω_h^t	$N = 128$	2.8228	3.5395	3.5401	4.1189	4.5036
	Order	2.03	2.03	2.03	2.05	2.02
	Extrap.	2.8222	3.5387	3.5390	4.1178	4.5017
[18]		2.822052	3.538328	3.538691	4.117093	4.501074

We show in Table 5.4 the lowest transmission eigenvalues κ_{ih} computed by the discrete scheme (4.16). In this case we have employed a family of uniform triangular meshes Ω_h^t (see bottom left picture in Figure 5.1). We compare our results with those reported in [16, 32]. The table includes orders of convergence, as well as accurate values extrapolated by means of a least-squares fitting.

It can be seen from Table 5.4 that for the first eigenvalue, where the associated eigenfunction presents a singularity, the method converges with order close to 1.54, which corresponds to the Sobolev regularity for the biharmonic equation (see [27]). Instead, the method presents an optimal order of convergence for the second, third, and fourth transmission eigenvalues where the associated eigenfunctions are smoother. Moreover, the results obtained by our virtual scheme agree perfectly well with those reported in [16, 32].

Finally, Figure 5.3 illustrates the eigenfunctions corresponding to the four lowest transmission eigenvalues computed in this test.

TABLE 5.4
Test 2: Lowest transmission eigenvalues κ_{ih}, $i = 1, 2, 3, 4$ computed on meshes Ω_h^t and with an index of refraction $n = 16$.

$\Omega_{\mathbf{L}}$		κ_{1h}	κ_{2h}	κ_{3h}	κ_{4h}
	$N = 32$	2.9706	3.1472	3.4237	3.5779
	$N = 64$	2.9589	3.1414	3.4141	3.5691
Ω_h^t	$N = 128$	2.9549	3.1400	3.4114	3.5670
	Order	1.53	1.96	1.82	2.00
	Extrap.	2.9528	3.1394	3.4103	3.5662
[16]	[Argyris method]	2.9553	-	-	
[32]	[VEM]	2.9527	3.1395	3.4103	3.5662

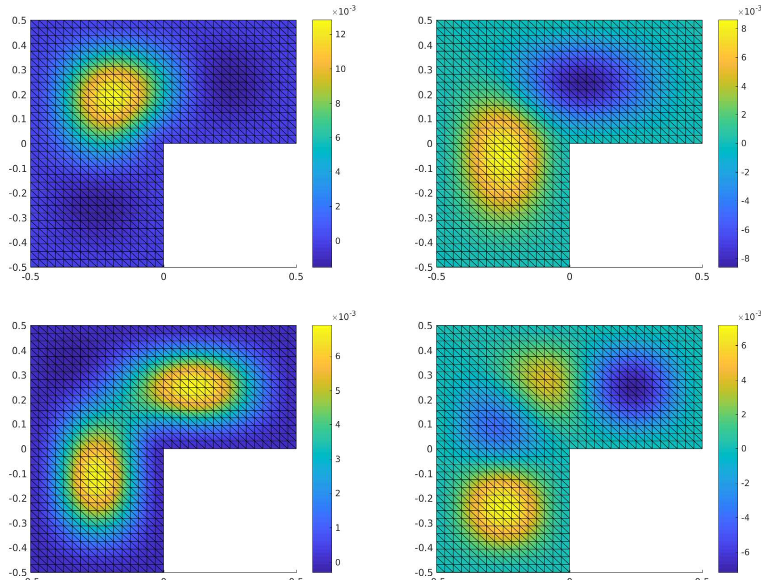

FIG. 5.3. *Test 2: Eigenfunctions u_{1h} (top left), u_{2h} (top right), u_{3h} (bottom left), and u_{4h} (bottom right) associated to the eigenvalues $\kappa_{1h}, \kappa_{2h}, \kappa_{3h}$ and κ_{4h}, respectively.*

5.3. Test 3: Circular domain $\Omega_{\mathbf{C}}$. We end this section by computing the four lowest transmission eigenvalues κ_{ih}, $i = 1, 2, 3, 4$ on the circular domain $\Omega_{\mathbf{C}}$ (cf. (5.3)). We considered a constant index of refraction $n = 16$ in order to compare our resutls with those showed in [16, 18, 22, 32]. We have employed a family of polygonal meshes (see Figure 5.1) created with PolyMesher [38].

Table 5.5 reports the four lowest transmission eigenvalues κ_{ih}, $i = 1, 2, 3, 4$ computed with the virtual method (4.16). The table also includes computed orders of convergence as well as more accurate values extrapolated by means of a least-squares fitting.

Once again, it can be seen from Table 5.5 that the computed transmission eigenvalues converge with an optimal quadratic order as predicted by the theory. Finally, in Figure 5.4, we present the eigenfunctions corresponding to the four lowest transmission eigenvalues computed in this numerical test.

TABLE 5.5
Test 3: Lowest transmission eigenvalues κ_{ih}, $i = 1,2,3,4$ computed on the circular domain $\Omega_{\mathbf{C}}$ and with an index of refraction $n = 16$.

$\Omega_{\mathbf{C}}$		κ_{1h}	κ_{2h}	κ_{3h}	κ_{4h}
Ω_h^v	$N_P = 1024$	1.9961	2.6301	2.6308	3.2611
	$N_P = 4096$	1.9900	2.6173	2.6173	3.2349
	$N_P = 16384$	1.9885	2.6140	2.6140	3.2287
	Order	2.03	1.97	2.03	2.08
	Extrap.	1.9880	2.6129	2.6129	3.2268
[16]	[Argyris method]	1.9881	-	-	-
[18]	[C^0-FEM]	1.9879	2.6124	2.6124	3.2255
[22]	[Continuous method]	2.0301	2.6937	2.6974	3.3744
[32]	[VEM]	1.9880	2.6129	2.6129	3.2267

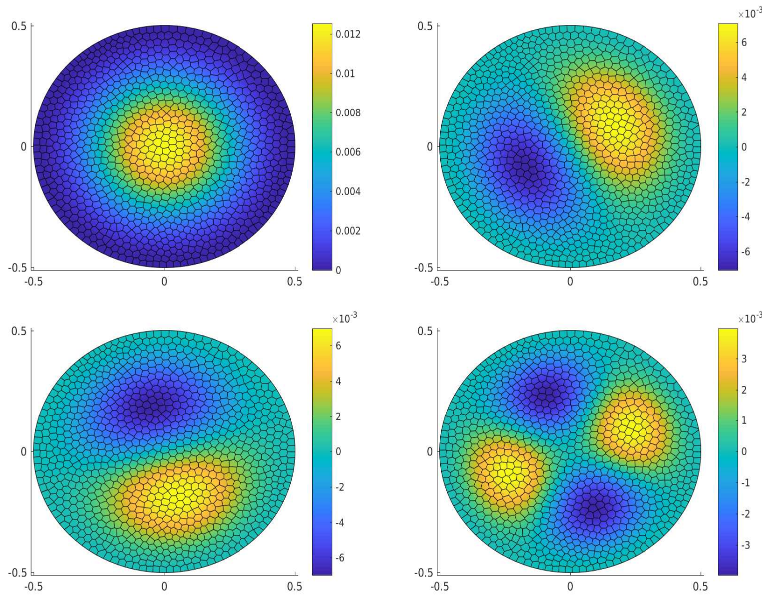

FIG. 5.4. *Test 3: Eigenfunctions u_{1h} (top left), u_{2h} (top right), u_{3h} (bottom left), and u_{4h} (bottom right) associated to the eigenvalues $\kappa_{1h}, \kappa_{2h}, \kappa_{3h}$, and κ_{4h}, respectively.*

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