Mathematical Models and Methods in Applied Sciences
Vol. 28, No. 14 (2018) 2803–2831

DOI: 10.1142/S0218202518500616

A virtual element method for the transmission eigenvalue problem

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Received 1 March 2018
Revised 9 August 2018
Accepted 22 August 2018
Published 1 November 2018
Communicated by L. Beirão da Veiga

In this paper, we analyze a Virtual Element Method (VEM) for solving a non-self-adjoint fourth-order eigenvalue problem derived from the transmission eigenvalue problem. We write a variational formulation and propose a C^1-conforming discretization by means of the VEM. We use the classical approximation theory for compact non-self-adjoint operators to obtain optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. Finally, we present some numerical experiments illustrating the behavior of the virtual scheme on different families of meshes.

Keywords: Virtual Element Method; transmission eigenvalue; spectral problem; error estimates.

AMS Subject Classification: 65N25, 65N30, 65N21, 78A46

1. Introduction

In this work, we study a Virtual Element Method (VEM) for an eigenvalue problem arising in scattering theory. The VEM, introduced in Refs. 5 and 7, is a generalization of the Finite Element Method (FEM) which is characterized by the capability of dealing with very general polygonal/polyhedral meshes, and it also permits to easily implement highly regular discrete spaces. Indeed, by avoiding the

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explicit construction of the local basis functions, the VEM can easily handle general polygons/polyhedrons without complex integrations on the element (see Ref. 7 for details on the coding aspects of the method). The VEM has been developed and analyzed for many problems, see for instance Refs. 2, 3, 6, 9, 11, 13, 16, 18, 20–22, 30, 32, 37, 50 and 55. Regarding VEM for spectral problems, we mention Refs. 14, 38, 39, 46, 47 and 48. We note that there are other methods that can make use of arbitrarily shaped polygonal/polyhedral meshes, we cite as a minimal sample of them.[8,28,36,53]

Due to their important role in many application areas, there has been a growing interest in recent years towards developing numerical schemes for spectral problems (see Ref. 17). In particular, we are going to analyze a virtual element approximation of the transmission eigenvalue problem. The motivation for considering this problem is that it plays an important role in inverse scattering theory.[24,34] This is due to the fact that transmission eigenvalues can be determined from the far-field data of the scattered wave and used to obtain estimates for the material properties of the scattering object.[23,25]

In recent years, various numerical methods have been proposed to solve this eigenvalue problem, see, for example, Refs. 26, 27, 31, 35, 40, 44, 45 and 51. In particular, the transmission eigenvalue problem is often solved by reformulating it as a fourth-order eigenvalue problem. In Ref. 26, a C^1 FEM using Argyris elements has been proposed, a complete analysis of the method including error estimates is proved using the theory for compact non-self-adjoint operators. However, the construction of H^2-conforming finite elements is difficult in general, since they usually involve a large number of degrees of freedom (see Ref. 33). More recently, in Ref. 40, a discontinuous Galerkin method has been proposed and analyzed to solve the fourth-order transmission eigenvalue problem; moreover, in Ref. 31, a C^0 linear FEM has been introduced to solve the spectral problem.

The purpose of the present paper is to introduce and analyze a C^1-VEM for solving a fourth-order spectral problem derived from the transmission eigenvalue problem. We consider a variational formulation of the problem written in $H^2(\Omega)\times H^1(\Omega)$ as in Refs. 26 and 40, where an auxiliary variable is introduced to transform the problem into a linear eigenvalue problem. Here, we exploit the capability of VEM to build highly regular discrete spaces (see Refs. 12 and 20) and propose a conforming $H^2(\Omega) \times H^1(\Omega)$ discrete formulation, which makes use of a very simple set of degrees of freedom, namely 4 degrees of freedom per vertex of the mesh. Then, we use the classical spectral theory for non-self-adjoint compact operators (see Refs. 4 and 49) to deal with the continuous and discrete solution operators, which appear as the solution of the continuous and discrete source problems, and whose spectra are related with the solutions of the transmission eigenvalue problem. Under rather mild assumptions on the polygonal meshes (made by possibly non-convex elements), we establish that the resulting VEM scheme provides a correct approximation of the spectrum and prove optimal-order error estimates for the eigenfunctions and a double order for the eigenvalues. Finally, we note that differently from the FEM where building globally conforming $H^2(\Omega)$ approximation is complicated, here the virtual space can be built with a rather simple construction due to the flexibility of the VEM. In a summary, the advantages of the present virtual element discretization are the possibility to use general polygonal meshes and to build conforming $H^2(\Omega)$ approximations.

The paper is structured as follows: In Sec. 2, we introduce the variational formulation of the transmission eigenvalue problem, define a solution operator and establish its spectral characterization. In Sec. 3, we introduce the virtual element discrete formulation, describe the spectrum of a discrete solution operator and establish some auxiliary results. In Sec. 4, we prove that the numerical scheme provides a correct spectral approximation and establish optimal-order error estimates for the eigenvalues and eigenfunctions using the standard theory for compact and non-self-adjoint operators. Finally, we report some numerical tests that confirm the theoretical analysis developed in Sec. 5.

In this paper, we will employ standard notations for Sobolev spaces, norms and seminorms. In addition, we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences. When the constant depends on the index of refraction, we will write C_n.

2. The Transmission Eigenvalue Problem

Let $\Omega \subset \mathbb{R}^2$ be the polygonal domain. We denote by ν the outward unit normal vector to $\partial\Omega$ and by ∂_ν the normal derivative. Let n be a real value function in $L^\infty(\Omega)$ such that $n-1$ is strictly positive (or strictly negative) almost everywhere in Ω. The transmission eigenvalue problem reads as follows.

Find the so-called transmission eigenvalue $k \in \mathbb{C}$ and a non-trivial pair of functions $(w_1, w_2) \in L^2(\Omega) \times L^2(\Omega)$, such that $(w_1 - w_2) \in H^2(\Omega)$ and

$$\Delta w_1 + k^2 n(x) w_1 = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\Delta w_2 + k^2 w_2 = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$w_1 = w_2 \quad \text{on } \partial\Omega, \tag{2.3}$$

$$\partial_\nu w_1 = \partial_\nu w_2 \quad \text{on } \partial\Omega. \tag{2.4}$$

Now, we rewrite the problem above in the following equivalent form in the new variable $u := (w_1 - w_2) \in H_0^2(\Omega)$ (see Ref. 26).

Find $(k, u) \in \mathbb{C} \times H_0^2(\Omega)$ such that

$$(\Delta + k^2 n)\frac{1}{n-1}(\Delta + k^2)u = 0 \quad \text{in } \Omega. \tag{2.5}$$

The variational formulation of problem (2.5) can be stated as follows: Find $(k, u) \in \mathbb{C} \times H_0^2(\Omega)$, $u \neq 0$ such that

$$\int_\Omega \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v}) = 0 \quad \forall\, v \in H_0^2(\Omega), \tag{2.6}$$

where $\overline{v}$ denotes the complex conjugate of v. Now, expanding the previous expression, we obtain the following quadratic eigenvalue problem:

$$\int_\Omega \frac{1}{n-1}\Delta u \Delta \overline{v} + \tau \int_\Omega \frac{1}{n-1} u \Delta \overline{v} + \tau \int_\Omega \frac{1}{n-1} \Delta u \overline{nv} + \tau^2 \int_\Omega \frac{1}{n-1} u \overline{nv} = 0 \tag{2.7}$$

for all $v \in H_0^2(\Omega)$, where $\tau := k^2$. It is easy to show that $k = 0$ is not an eigenvalue of the problem (see Ref. 26). Moreover, for the sake of simplicity, we will assume that the index of refraction function $n(x)$ is a real constant. Nevertheless, this assumption does not affect the generality of the forthcoming analysis.

For the theoretical analysis, it is convenient to transform problem (2.7) into a linear eigenvalue problem. With this aim, let ϕ be the solution of the problem: Find $\phi \in H_0^1(\Omega)$ such that

$$\Delta \phi = \tau \frac{n}{n-1} u \quad \text{in } \Omega, \tag{2.8}$$

$$\phi = 0 \qquad \text{on } \partial\Omega. \tag{2.9}$$

Therefore, by testing problem (2.8)–(2.9) with functions in $H_0^1(\Omega)$, we arrive at the following weak formulation of the problem.

Problem 1. *Find $(\lambda, u, \phi) \in \mathbb{C} \times H_0^2(\Omega) \times H_0^1(\Omega)$ with $(u, \phi) \neq 0$ such that*

$$a((u,\phi),(v,\psi)) = \lambda b((u,\phi),(v,\psi)) \quad \forall\, (v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega),$$

where $\lambda = -\tau$ and the sesquilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a((u,\phi),(v,\psi)) := \frac{1}{n-1}\int_\Omega D^2 u : D^2 \overline{v} + \int_\Omega \nabla \phi \cdot \nabla \overline{\psi},$$

$$b((u,\phi),(v,\psi)) := \frac{n}{n-1}\int_\Omega \Delta u \overline{v} + \frac{1}{n-1}\int_\Omega u \Delta \overline{v} - \int_\Omega \nabla \phi \cdot \nabla \overline{v} + \frac{n}{n-1}\int_\Omega u \overline{\psi},$$

for all $(u,\phi),(v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega)$. Moreover, ":" denotes the usual scalar product of 2×2 matrices $D^2 u := (\partial_{ij} u)_{1 \le i,j \le 2}$ denotes the Hessian matrix of u.

Remark 2.1. In the definition of $a(\cdot,\cdot)$ (cf. Problem 1), we have considered $a^\Delta(u,v) := \int_\Omega D^2 u : D^2 \overline{v}$ instead of $a^*(u,v) := \int_\Omega \Delta u \Delta \overline{v}$; since, we are considering functions in $H_0^2(\Omega)$, both are equivalent (see Ref. 33). This fact will facilitate the presentation and the analysis of the VEM method. In particular, we will use $a_K^\Delta(u,v)$ to construct the projector Π_2^Δ (cf. (3.1a)–(3.1b)) which will be used to write the discrete scheme. However, once the projector Π_2^Δ is built, it can be used to discretize the $a_K^*(u,v)$ as well (see Appendix of Ref. 15).

We endow $H_0^2(\Omega) \times H_0^1(\Omega)$ with the corresponding product norm, which we will simply denote with $\|(\cdot,\cdot)\|$.

Now, we note that the sesquilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bounded forms (with constants which depend on the index of refraction). Moreover, we have that $a(\cdot,\cdot)$ is elliptic.

Lemma 2.1. *There exists a constant* $C_n > 0$, *such that*

$$a((v,\psi),(v,\psi)) \geq C_n \|(v,\psi)\|^2 \quad \forall (v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega).$$

Proof. The result follows immediately from the fact that $\left\{\|D^2 v\|_{0,\Omega}^2 + \|\nabla\psi\|_{0,\Omega}^2\right\}^{1/2}$ is a norm on $H_0^2(\Omega) \times H_0^1(\Omega)$, equivalent with the norm $\|(\cdot,\cdot)\|$ (see Chap. 6 of Ref. 52). $\square$

We define the solution operator associated with Problem 1:

$$\begin{aligned} T : H_0^2(\Omega) \times H_0^1(\Omega) &\to H_0^2(\Omega) \times H_0^1(\Omega), \\ (f,g) &\mapsto T(f,g) = (\tilde{u},\tilde{\phi}) \end{aligned}$$

as the unique solution (as a consequence of Lemma 2.1) of the corresponding source problem:

$$a((\tilde{u},\tilde{\phi}),(v,\psi)) = b((f,g),(v,\psi)) \quad \forall (v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{2.10}$$

The linear operator T is then well defined and bounded. Note that $(\lambda, u, \phi) \in \mathbb{C} \times H_0^2(\Omega) \times H_0^1(\Omega)$ solves Problem 1 if and only if (μ, u, ϕ), with $\mu := \frac{1}{\lambda}$, is an eigenpair of T, i.e. $T(u,\phi) = \mu(u,\phi)$.

We observe that no spurious eigenvalues are introduced into the problem since, if $\mu \neq 0$, $(0,\phi)$ is not an eigenfunction of the problem.

The following is an additional regularity result for the solution of the source problem (2.10) and consequently, for the generalized eigenfunctions of T.

Lemma 2.2. *There exist* $s, t \in (1/2, 1]$ *and* $C_n > 0$ *such that for all* $(f,g) \in H_0^2(\Omega) \times H_0^1(\Omega)$, *the solution* $(\tilde{u},\tilde{\phi})$ *of problem* (2.10) *satisfies* $\tilde{u} \in H^{2+s}(\Omega)$, $\tilde{\phi} \in H^{1+t}(\Omega)$ *and*

$$\|\tilde{u}\|_{2+s,\Omega} + \|\tilde{\phi}\|_{1+t,\Omega} \leq C_n \|(f,g)\|.$$

Proof. The estimate for $\tilde{\phi}$ follows from the classical regularity result for the Laplace problem with its right-hand side in $L^2(\Omega)$. The estimate for $\tilde{u}$ follows from the classical regularity result for the biharmonic problem with its right-hand side in $H^{-1}(\Omega)$ (cf. Ref. 42). $\square$

Remark 2.2. The constant s in the lemma above is the Sobolev regularity for the biharmonic equation with the right-hand side in $H^{-1}(\Omega)$ and homogeneous Dirichlet boundary conditions. The constant t is the Sobolev exponent for the Laplace problem with homogeneous Dirichlet boundary conditions. These constants only depend on the domain Ω. If Ω is convex, then $s = t = 1$. Otherwise, the lemma holds for all $s < s_0$ and $t < t_0$, where $s_0, t_0 \in (1/2, 1]$ depend on the largest reentrant angle of Ω.

Hence, because of the compact inclusions $H^{2+s}(\Omega) \hookrightarrow H_0^2(\Omega)$ and $H^{1+t}(\Omega) \hookrightarrow H_0^1(\Omega)$, we can conclude that T is a compact operator. So, we obtain the following spectral characterization result.

Lemma 2.3. *The spectrum of* T *satisfies* $\mathrm{sp}(T) = \{0\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, *where* $\{\mu_k\}_{k\in\mathbb{N}}$ *is a sequence of complex eigenvalues which converges to* 0 *and their corresponding eigenspaces lie in* $H^{2+s}(\Omega) \times H^{1+t}(\Omega)$. *In addition,* $\mu = 0$ *is an infinite multiplicity eigenvalue of* T.

Proof. The proof is obtained from the compactness of T and Lemma 2.2. □

3. The Virtual Element Discretization

In this section, we will write the C^1-VEM discretization of Problem 1. With this aim, we start with the mesh construction and the assumptions considered to introduce the discrete virtual element spaces.

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K. We will denote by h_K the diameter of the element K and by h the maximum of the diameters of all the elements of the mesh, i.e. $h := \max_{K\in\mathcal{T}_h} h_K$. In what follows, we denote by N_K the number of vertices of K by e a generic edge of $\{\mathcal{T}_h\}_h$ and for all $e \in \partial K$, we denote with ν_K^e the unit normal vector that points outside of K.

In addition, we will make the following assumptions as in Refs. 5 and 14: there exists a positive real number $C_{\mathcal{T}}$ such that, for every h and every $K \in \mathcal{T}_h$,

A1: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$;
A2: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}} h_K$.

In order to introduce the method, we first define two preliminary discrete spaces as follows: For each polygon $K \in \mathcal{T}_h$ (meaning open simply connected set whose boundary is a non-intersecting line made of a finite number of straight line segments), we define the following finite-dimensional spaces:

$$\widetilde{W}_h^K := \big\{v_h \in H^2(K) : \Delta^2 v_h \in \mathbb{P}_2(K), v_h|_{\partial K} \in C^0(\partial K), v_h|_e \in \mathbb{P}_3(e) \forall\, e \in \partial K,$$
$$\nabla v_h|_{\partial K} \in C^0(\partial K)^2, \partial_\nu v_h|_e \in \mathbb{P}_1(e) \forall\, e \in \partial K\big\}$$

and

$$\widetilde{V}_h^K := \big\{\psi_h \in H^1(K) : \Delta\psi_h \in \mathbb{P}_1(K), \psi_h|_{\partial K} \in C^0(\partial K), \psi_h|_e \in \mathbb{P}_1(e) \forall\, e \in \partial K\big\},$$

where Δ^2 represents the biharmonic operator and we have denoted by $\mathbb{P}_k(S)$ the space of polynomials of degree up to k defined on the subset $S \subseteq \mathbb{R}^2$.

The following conditions hold:

- for any $v_h \in \widetilde{W}_h^K$, the trace on the boundary of K is continuous and on each edge is a polynomial of degree 3;
- for any $v_h \in \widetilde{W}_h^K$, the gradient on the boundary is continuous and on each edge its normal (respectively, tangential) component is a polynomial of degree 1 (respectively, 2);

- for any $\psi_h \in \widetilde{V}_h^K$, the trace on the boundary of K is continuous and on each edge is a polynomial of degree 1;
- $\mathbb{P}_2(K) \times \mathbb{P}_1(K) \subseteq \widetilde{W}_h^K \times \widetilde{V}_h^K$.

Next, with the aim to choose the degrees of freedom for both spaces, we will introduce three sets of linear operators $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$. The first two sets $(\mathbf{D_1}, \mathbf{D_2})$ are provided by linear operators from $\widetilde{W}_h^K$ into $\mathbb{R}$ and the set $\mathbf{D_3}$ by linear operators from $\widetilde{V}_h^K$ into $\mathbb{R}$. For all $(v_h, \psi_h) \in \widetilde{W}_h^K \times \widetilde{V}_h^K$, they are defined as follows:

- $\mathbf{D_1}$ contains linear operators evaluating v_h at the N_K vertices of K,
- $\mathbf{D_2}$ contains linear operators evaluating ∇v_h at the N_K vertices of K,
- $\mathbf{D_3}$ contains linear operators evaluating ψ_h at the N_K vertices of K.

Note that, as a consequence of definition of the discrete spaces, the output values of the three sets of operators $\mathbf{D_1}$, $\mathbf{D_2}$ and $\mathbf{D_3}$ are sufficient to uniquely determine v_h and ∇v_h on the boundary of K, and ψ_h on the boundary of K, respectively.

In order to construct the discrete scheme, we need some preliminary definitions. First, we split the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, introduced in the previous section, as follows:

$$a((u,\phi),(v,\psi)) = \sum_{K \in \mathcal{T}_h} \frac{1}{n-1} a_K^{\Delta}(u,v) + a_K^{\nabla}(\phi,\psi)$$

$$\forall\, (u,\phi),(v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega),$$

$$b((u,\phi),(v,\psi)) = \sum_{K \in \mathcal{T}_h} b_K((u,\phi),(v,\psi)) \quad \forall (u,\phi),(v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega)$$

with

$$a_K^{\Delta}(u,v) := \int_K D^2 u : D^2 \overline{v} \quad \forall\, u,v \in H^2(K),$$

$$a_K^{\nabla}(\phi,\psi) := \int_K \nabla\phi \cdot \nabla\overline{\psi} \qquad \forall\, \phi,\psi \in H^1(K)$$

and for all $(u,\phi),(v,\phi) \in H^2(K) \times H^1(K)$,

$$b_K((u,\phi),(v,\psi)) := \frac{n}{n-1}\int_K \Delta u \overline{v} + \frac{1}{n-1}\int_K u \Delta\overline{v} - \int_K \nabla\phi \cdot \nabla\overline{v} + \frac{n}{n-1}\int_K u\overline{\psi}.$$

Now, we define the projector $\Pi_2^{\Delta} : H^2(K) \to \mathbb{P}_2(K) \subseteq \widetilde{W}_h^K$ for each $v \in H^2(K)$ as the solution of

$$a_K^{\Delta}\big(\Pi_2^{\Delta} v, q\big) = a_K^{\Delta}(v,q) \quad \forall\, q \in \mathbb{P}_2(K), \tag{3.1a}$$

$$\big(\big(\Pi_2^{\Delta} v, q\big)\big)_K = ((v,q))_K \quad \forall\, q \in \mathbb{P}_1(K), \tag{3.1b}$$

where $((\cdot,\cdot))_K$ is defined as follows:

$$((u,v))_K = \sum_{i=1}^{N_K} u(P_i)v(P_i) \quad \forall\, u,v \in C^0(\partial K),$$

where $P_i, 1 \leq i \leq N_K$, are the vertices of K. We note that the bilinear form $a_K^{\Delta}(\cdot,\cdot)$ has a non-trivial kernel, given by $\mathbb{P}_1(K)$. Hence, the role of condition (3.1b) is to select an element of the kernel of the operator. We observe that operator Π_2^{Δ} is well defined on $\widetilde{W}_h^K$ and, most important, for all $v \in \widetilde{W}_h^K$, the polynomial $\Pi_2^{\Delta} v$ can be computed using only the values of the operators $\mathbf{D_1}$ and $\mathbf{D_2}$ calculated on v. This follows easily with an integration by parts (see Ref. 3).

In a similar way, we define the projector $\Pi_1^{\nabla} : H^1(K) \to \mathbb{P}_1(K) \subseteq \widetilde{V}_h^K$ for each $\psi \in H^1(K)$ as the solution of

$$a_K^{\nabla}\big(\Pi_1^{\nabla}\psi, q\big) = a_K^{\nabla}(\psi, q) \quad \forall\, q \in \mathbb{P}_1(K), \tag{3.2a}$$

$$\big(\Pi_1^{\nabla}\psi, 1\big)_{\partial K} = (\psi, 1)_{\partial K}. \tag{3.2b}$$

We observe that operator Π_1^{∇} is well defined on $\widetilde{V}_h^K$, and, as before, for all $\psi \in \widetilde{V}_h^K$, the polynomial $\Pi_1^{\nabla}\psi$ can be computed using only the values of the operators $\mathbf{D_3}$ calculated on ψ, which follows by an integration by parts (see Ref. 1).

Now, we introduce our local virtual spaces (see Refs. 1 and 3):

$$W_h^K := \left\{ v_h \in \widetilde{W}_h^K : \int_K \big(\Pi_2^{\Delta} v_h\big) q = \int_K v_h q \ \forall\, q \in \mathbb{P}_2(K) \right\}$$

and

$$V_h^K := \left\{ \psi_h \in \widetilde{V}_h^K : \int_K \big(\Pi_1^{\nabla} \psi_h\big) q = \int_K \psi_h q \ \forall\, q \in \mathbb{P}_1(K) \right\}.$$

It is clear that $W_h^K \times V_h^K \subseteq \widetilde{W}_h^K \times \widetilde{V}_h^K$. Thus, the linear operators Π_2^{Δ} and Π_1^{∇} are well defined on W_h^K and V_h^K, respectively.

In Lemma 2.1 of Ref. 3, it has been established that the sets of operators $\mathbf{D_1}$ and $\mathbf{D_2}$ constitute a set of degrees of freedom for the space W_h^K. Moreover, the set of operators $\mathbf{D_3}$ constitutes a set of degrees of freedom for the space V_h^K (see Ref. 1).

We also have that $\mathbb{P}_2(K) \times \mathbb{P}_1(K) \subseteq W_h^K \times V_h^K$. This will guarantee the good approximation properties for the spaces.

To continue the construction of the discrete scheme, we will need to consider new projectors. First, we define the projector $\Pi_2^{\nabla} : H^2(K) \to \mathbb{P}_2(K)$ for each $w \in H^2(K)$ as the solution of

$$a_K^{\nabla}\big(\Pi_2^{\nabla} w, q\big) = a_K^{\nabla}(w, q) \quad \forall\, q \in \mathbb{P}_2(K), \tag{3.3a}$$

$$\big(\Pi_2^{\nabla} w, 1\big)_{0,K} = (w, 1)_{0,K}. \tag{3.3b}$$

Moreover, we consider the $L^2(\Omega)$ orthogonal projectors onto $\mathbb{P}_l(K)$, $l = 1, 2$ as follows: we define $\Pi_l^0 : L^2(\Omega) \to \mathbb{P}_l(K)$ for each $p \in L^2(\Omega)$ by

$$\int_K (\Pi_l^0 p) q = \int_K pq \quad \forall\, q \in \mathbb{P}_l(K). \tag{3.4}$$

Now, due to the particular property appearing in definition of the space W_h^K, it can be seen that the right-hand side in (3.4) is computable using $\Pi_2^{\Delta} v_h$, and thus $\Pi_2^0 v_h$ depends only on the values of the degrees of freedom for v_h and ∇v_h. Actually, it is easy to check that on the space W_h^K, the projectors Π_2^0 and Π_2^Δ are the same operators. In fact,

$$\int_K (\Pi_2^0 v_h) q = \int_K v_h q = \int_K (\Pi_2^\Delta v_h) q \quad \forall\, q \in \mathbb{P}_2(K). \tag{3.5}$$

By definition of V_h^K, we have that Π_1^0 and Π_1^∇ are the same operators in V_h^K.

Now, for every decomposition $\mathcal{T}_h$ of Ω into simple polygons K, we introduce the global virtual space denoted by Z_h as follows:

$$Z_h := W_h \times V_h,$$

where

$$W_h := \{v_h \in H_0^2(\Omega) : v_h|_K \in W_h^K\} \quad \text{and} \quad V_h := \{\psi_h \in H_0^1(\Omega) : \psi_h|_K \in V_h^K\}.$$

A set of degrees of freedom for Z_h is given by all pointwise values of v_h and ψ_h on all vertices of $\mathcal{T}_h$ together with all pointwise values of ∇v_h on all vertices of $\mathcal{T}_h$, excluding the vertices on $\partial\Omega$ (where the values vanishes). Thus, the dimension of Z_h is four times the number of interior vertices of $\mathcal{T}_h$.

In what follows, we discuss the construction of the discrete version of the local forms. With this aim, we consider $s_K^\Delta(\cdot,\cdot)$ and $s_K^\nabla(\cdot,\cdot)$ any hermitian positive definite forms satisfying:

$$c_0 a_K^\Delta(v_h, v_h) \le s_K^\Delta(v_h, v_h) \le c_1 a_K^\Delta(v_h, v_h) \qquad \forall\, v_h \in W_h^K \ \text{ with } \Pi_2^\Delta v_h = 0, \tag{3.6}$$

$$c_2 a_K^\nabla(\psi_h, \psi_h) \le s_K^\nabla(\psi_h, \psi_h) \le c_3 a_K^\nabla(\psi_h, \psi_h) \quad \forall\, \psi_h \in V_h^K \ \text{ with } \Pi_1^\nabla \psi_h = 0. \tag{3.7}$$

We define the discrete sesquilinear forms $a_h(\cdot,\cdot) : Z_h \times Z_h \to \mathbb{C}$ and $b_h(\cdot,\cdot) : Z_h \times Z_h \to \mathbb{C}$ by

$$a_h((u_h,\phi_h),(v_h,\psi_h)) := \sum_{K\in\mathcal{T}_h} \frac{1}{n-1} a_{h,K}^\Delta(u_h, v_h) + a_{h,K}^\nabla(\phi_h, \psi_h)$$
$$\forall (u_h,\phi_h),(v_h,\psi_h) \in Z_h,$$

$$b_h((u_h,\phi_h),(v_h,\psi_h)) := \sum_{K\in\mathcal{T}_h} b_{h,K}((u_h,\phi_h),(v_h,\psi_h)) \quad \forall (u_h,\phi_h),(v_h,\psi_h) \in Z_h,$$

where $a_{h,K}^\Delta(\cdot,\cdot)$, $a_{h,K}^\nabla(\cdot,\cdot)$ and $b_{h,K}(\cdot,\cdot)$ are local forms on $W_h^K \times W_h^K$, $V_h^K \times V_h^K$ and $Z_h^K := W_h^K \times V_h^K$, respectively, defined by

$$a_{h,K}^\Delta(u_h, v_h) := a_K^\Delta\big(\Pi_2^\Delta u_h, \Pi_2^\Delta v_h\big) + s_K^\Delta\big(u_h - \Pi_2^\Delta u_h, v_h - \Pi_2^\Delta v_h\big)$$
$$\forall\, u_h, v_h \in W_h^K,$$

$$a_{h,K}^\nabla(\phi_h, \psi_h) := a_K^\nabla\big(\Pi_1^\nabla \phi_h, \Pi_1^\nabla \psi_h\big) + s_K^\nabla\big(\phi_h - \Pi_1^\nabla \phi_h, \psi_h - \Pi_1^\nabla \psi_h\big)$$
$$\forall\, \phi_h, \psi_h \in V_h^K,$$

$$b_{h,K}((u_h,\phi_h),(v_h,\psi_h)) := \frac{n}{n-1}\int_K \Pi_2^0(\Delta u_h)\Pi_2^0\overline{v}_h + \frac{1}{n-1}\int_K \Pi_2^0 u_h \Pi_2^0(\Delta\overline{v}_h) - \int_K \nabla\Pi_1^\nabla\phi_h\cdot\nabla\Pi_2^\nabla\overline{v}_h + \frac{n}{n-1}\int_K \Pi_2^0 u_h\Pi_1^0\overline{\psi_h} \quad \forall (u_h,\phi_h),(v_h,\psi_h)\in Z_h^K.$$

The construction of the local sesquilinear forms guarantees the usual consistency and stability properties, as is stated in the following proposition. Since the proof follows standard arguments in the VEM literature, it is omitted.

Proposition 3.1. *The local forms* $a_{h,K}^{\Delta}(\cdot,\cdot)$ *and* $a_{h,K}^{\nabla}(\cdot,\cdot)$ *on each element* K *satisfy the following*:

- *Consistency*: *for all* $h>0$ *and for all* $K\in\mathcal{T}_h$, *we have that*

$$a_{h,K}^{\Delta}(v_h,q) = a_K^{\Delta}(v_h,q) \quad \forall\, q\in\mathbb{P}_2(K),\ \forall\, v_h\in W_h^K, \tag{3.8}$$

$$a_{h,K}^{\nabla}(\psi_h,q) = a_K^{\nabla}(\psi_h,q) \quad \forall\, q\in\mathbb{P}_1(K),\ \forall\, \psi_h\in V_h^K. \tag{3.9}$$

- *Stability and boundedness*: *There exist positive constants* $\alpha_i, i=1,2,3,4$, *independent of* K, *such that*

$$\alpha_1 a_K^{\Delta}(v_h,v_h) \le a_{h,K}^{\Delta}(v_h,v_h) \le \alpha_2 a_K^{\Delta}(v_h,v_h) \quad \forall\, v_h\in W_h^K, \tag{3.10}$$

$$\alpha_3 a_K^{\nabla}(\psi_h,\psi_h) \le a_{h,K}^{\nabla}(\psi_h,\psi_h) \le \alpha_4 a_K^{\nabla}(\psi_h,\psi_h) \quad \forall\, \psi_h\in V_h^K. \tag{3.11}$$

Now, we are in a position to write the virtual element discretization of Problem 1.

Problem 2. *Find* $(\lambda_h,u_h,\psi_h)\in\mathbb{C}\times Z_h$, $(u_h,\phi_h)\neq 0$ *such that*

$$a_h((u_h,\phi_h),(v_h,\psi_h)) = \lambda_h b_h((u_h,\phi_h),(v_h,\psi_h)). \tag{3.12}$$

It is clear that by virtue of (3.10) *and* (3.11), *the hermitian form* $a_h(\cdot,\cdot)$ *is bounded. Moreover, we will show in the following lemma that* $a_h(\cdot,\cdot)$ *is also uniformly elliptic.*

Lemma 3.1. *There exists constant* $C_n>0$, *independent of* h, *such that*

$$a_h((v_h,\psi_h),(v_h,\psi_h)) \ge C_n\|(v_h,\psi_h)\|^2 \quad \forall\,(v_h,\psi_h)\in Z_h.$$

Proof. The result is deduced from Lemma 2.1, (3.10) and (3.11). □

Now, we introduce the discrete solution operator T_h which is given by

$$T_h : H_0^2(\Omega)\times H_0^1(\Omega) \to H_0^2(\Omega)\times H_0^1(\Omega),$$
$$(f,g)\mapsto T_h(f,g) = (\tilde{u}_h,\tilde{\phi}_h),$$

where $(\tilde{u}_h, \tilde{\phi}_h) \in Z_h$ is the unique solution of the corresponding discrete source problem

$$a_h((\tilde{u}_h, \tilde{\phi}_h), (v_h, \psi_h)) = b_h((f, g), (v_h, \psi_h)) \quad \forall (v_h, \psi_h) \in Z_h. \tag{3.13}$$

Because of Lemma 3.1, the linear operator T_h is well defined and bounded uniformly with respect to h. Once more, as in the continuous case, $(\lambda_h, u_h, \phi_h) \in \mathbb{C} \times Z_h$ solves Problem 2 if and only if (μ_h, u_h, ϕ_h), with $\mu_h := \frac{1}{\lambda_h}$, is an eigenpair of T_h, i.e. $T_h(u_h, \phi_h) = \mu_h(u_h, \phi_h)$.

We end this section with the following remark.

Remark 3.1. In the first and second terms in the definition of $b_{h,K}(\cdot,\cdot)$, we have employed the projector $\Pi_2^0(\Delta\cdot)$ which is a projector of high order. This definition will be useful in the forthcoming analysis of the VEM method. However, this is not the only possibility to discretize $b_K(\cdot,\cdot)$, we can also consider the following alternative definition:

$$\begin{aligned}
&\tilde{b}_{h,K}((u_h, \phi_h), (v_h, \psi_h)) \\
&\quad := \frac{n}{n-1}\int_K \Pi_1^0(\Delta u_h)\Pi_2^0\overline{v}_h + \frac{1}{n-1}\int_K \Pi_2^0 u_h \Pi_1^0(\Delta\overline{v}_h) \\
&\qquad - \int_K \nabla\Pi_1^\nabla \phi_h \cdot \nabla\Pi_2^\nabla \overline{v}_h + \frac{n}{n-1}\int_K \Pi_2^0 u_h \Pi_1^0 \overline{\psi}_h \quad \forall (u_h, \phi_h), (v_h, \psi_h) \in Z_h^K,
\end{aligned}$$

where the projector $\Pi_1^0(\Delta\cdot)$ has been used. The VEM discretization in given this case is as follows.

Problem 3. *Find $(\lambda_h, u_h, \psi_h) \in \mathbb{C} \times Z_h$, $(u_h, \phi_h) \neq 0$ such that*

$$a_h((u_h, \phi_h), (v_h, \psi_h)) = \lambda_h \tilde{b}_h((u_h, \phi_h), (v_h, \psi_h)). \tag{3.14}$$

We are going to analize in detail the VEM method (3.12) *and summarize the results for the VEM discretization* (3.14) (*see Remark* 4.3).

4. Spectral Approximation and Error Estimates

To prove that T_h provides a correct spectral approximation of T, we will resort to the classical theory for compact operators (see Ref. 4). First, we recall the following approximation result which is derived by interpolation between Sobolev spaces (see for instance Theorem I.1.4 of Ref. 41 from the analogous result for integer values of s). In its turn, the result for integer values is stated in Proposition 4.2 of Ref. 5 and follows from the classical Scott–Dupont theory (see Ref. 19 and Proposition 3.1 of Ref. 3).

Proposition 4.1. *There exists a constant $C > 0$, such that for every $v \in H^\delta(K)$, there exists $v_\pi \in \mathbb{P}_k(K)$, $k \geq 0$ such that*

$$|v - v_\pi|_{\ell,K} \leq C h_K^{\delta-\ell}|v|_{\delta,K}, \quad 0 \leq \delta \leq k+1, \;\; \ell = 0, \ldots, [\delta]$$

with $[\delta]$ denoting the largest integer equal or smaller than $\delta \in \mathbb{R}$.

For the analysis, we will introduce the broken seminorms:

$$|\psi|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} |\psi|_{1,K}^2 \quad \text{and} \quad |v|_{2,h}^2 := \sum_{K\in\mathcal{T}_h} |v|_{2,K}^2,$$

which are well defined for every $(\psi, v) \in [L^2(\Omega)]^2$ such that $(\psi, v)|_K \in H^1(K) \times H^2(K)$ for all polygon $K \in \mathcal{T}_h$.

In what follows, we derive several auxiliary results which will be used in the following to prove convergence and error estimates for the spectral approximation.

Proposition 4.2. *Assume that* **A1**–**A2** *are satisfied, let* $\psi \in H^{1+t}(\Omega)$ *with* $t \in (0, 1]$. *Then, there exist* $\psi_I \in V_h$ *and* $C > 0$ *such that*

$$\|\psi - \psi_I\|_{1,\Omega} \leq Ch^t|\psi|_{1+t,\Omega}.$$

Proof. This result has been proved in Theorem 11 of Ref. 29 (see also Proposition 4.2 of Ref. 47). □

Proposition 4.3. *Assume* **A1**–**A2** *are satisfied, let* $v \in H^{2+s}(\Omega)$ *with* $s \in (0, 1]$. *Then, there exist* $v_I \in W_h$ *and* $C > 0$ *such that*

$$\|v - v_I\|_{2,\Omega} \leq Ch^s|v|_{2+s,\Omega}.$$

Proof. This result has been established in Proposition 3.1 of Ref. 3. □

Now, we establish a result which will be useful to prove the convergence of the operator T_h to T as h goes to zero.

Lemma 4.1. *There exists* C_n *independent of* h *such that for all* $(f, g) \in H_0^2(\Omega) \times H_0^1(\Omega)$, *if* $(\tilde{u}, \tilde{\phi}) := T(f, g)$ *and* $(\tilde{u}_h, \tilde{\phi}_h) := T_h(f, g)$, *then*

$$\begin{aligned}\|(T - T_h)(f, g)\| \leq C_n h\|(f, g)\| + |\tilde{u} - \tilde{u}_I|_{2,\Omega} + |\tilde{u} - \tilde{u}_\pi|_{2,h} \\ + |\tilde{\phi} - \tilde{\phi}_I|_{1,\Omega} + |\tilde{\phi} - \tilde{\phi}_\pi|_{1,h}\end{aligned}$$

for all $(\tilde{u}_I, \tilde{\phi}_I) \in Z_h$ *and for all* $(\tilde{u}_\pi, \tilde{\phi}_\pi) \in [L^2(\Omega)]^2$ *such that* $(\tilde{u}_\pi, \tilde{\phi}_\pi)|_K \in \mathbb{P}_2(K) \times \mathbb{P}_1(K)$.

Proof. Let $(f, g) \in H_0^2(\Omega) \times H_0^1(\Omega)$, for any $(\tilde{u}_I, \tilde{\phi}_I) \in W_h \times V_h$, we have,

$$\|(T - T_h)(f, g)\| \leq \|(\tilde{u}, \tilde{\phi}) - (\tilde{u}_I, \tilde{\phi}_I)\| + \|(\tilde{u}_I, \tilde{\phi}_I) - (\tilde{u}_h, \tilde{\phi}_h)\|. \tag{4.1}$$

Now, we define $(v_h, \psi_h) = (\tilde{u}_h - \tilde{u}_I, \tilde{\phi}_h - \tilde{\phi}_I) \in Z_h$, then from the ellipticity of $a_h(\cdot, \cdot)$ and the definition of T and T_h, we have

$$\begin{aligned}\beta\|(v_h, \psi_h)\|^2 &\leq a_h((v_h, \psi_h), (v_h, \psi_h)) \\ &= a_h((\tilde{u}_h, \tilde{\phi}_h), (v_h, \psi_h)) - a_h((\tilde{u}_I, \tilde{\phi}_I), (v_h, \psi_h))\end{aligned}$$

$$
\begin{aligned}
&= b_h((f,g),(v_h,\psi_h)) - \sum_{K\in\mathcal{T}_h}\left\{\frac{1}{n-1}a_{h,K}^{\Delta}(\tilde{u}_I,v_h) + a_{h,K}^{\nabla}(\tilde{\phi}_I,\psi_h)\right\}\\
&= b_h((f,g),(v_h,\psi_h)) - \sum_{K\in\mathcal{T}_h}\left\{\frac{1}{n-1}\{a_{h,K}^{\Delta}(\tilde{u}_I-\tilde{u}_\pi,v_h)\right.\\
&\quad \left. + a_{h,K}^{\Delta}(\tilde{u}_\pi,v_h)\} + a_{h,K}^{\nabla}(\tilde{\phi}_I-\tilde{\phi}_\pi,\psi_h) + a_{h,K}^{\nabla}(\tilde{\phi}_\pi,\psi_h)\right\}\\
&= b_h((f,g),(v_h,\psi_h)) - \sum_{K\in\mathcal{T}_h}\left\{\frac{1}{n-1}\{a_{h,K}^{\Delta}(\tilde{u}_I-\tilde{u}_\pi,v_h)\right.\\
&\quad + a_{K}^{\Delta}(\tilde{u}_\pi-\tilde{u},v_h) + a_{K}^{\Delta}(\tilde{u},v_h)\} + a_{h,K}^{\nabla}(\tilde{\phi}_I-\tilde{\phi}_\pi,\psi_h)\\
&\quad \left. + a_{K}^{\nabla}(\tilde{\phi}_\pi-\tilde{\phi},\psi_h) + a_{K}^{\nabla}(\tilde{\phi},\psi_h)\right\}\\
&= \underbrace{\sum_{K\in\mathcal{T}_h}\{b_{h,K}((f,g),(v_h,\psi_h)) - b_K((f,g),(v_h,\psi_h))\}}_{E_1}\\
&\quad - \frac{1}{n-1}\underbrace{\sum_{K\in\mathcal{T}_h}\left\{a_{h,K}^{\Delta}(\tilde{u}_I-\tilde{u}_\pi,v_h) + a_{K}^{\Delta}\big(\tilde{u}_\pi-\tilde{u},v_h\big)\right\}}_{E_2}\\
&\quad - \underbrace{\sum_{K\in\mathcal{T}_h}\left\{a_{h,K}^{\nabla}\big(\tilde{\phi}_I-\tilde{\phi}_\pi,\psi_h\big) + a_{K}^{\nabla}\big(\tilde{\phi}_\pi-\tilde{\phi},\psi_h\big)\right\}}_{E_3},
\end{aligned}
\tag{4.2}
$$

where we have used the consistency properties (3.8)–(3.9). We now bound each term $E_i|_K$, $i=1,2,3$.

First, the term $E_1|_K$ can be written as follows:

$$
\begin{aligned}
&b_{h,K}((f,g),(v_h,\psi_h)) - b_K((f,g),(v_h,\psi_h))\\
&\quad = \frac{n}{n-1}\left\{\underbrace{\int_K \Pi_2^0(\Delta f)\Pi_2^0\overline{v}_h - \int_K \Delta f\overline{v_h}}_{E_{11}}\right\}\\
&\qquad + \frac{1}{n-1}\left\{\underbrace{\int_K \Pi_2^0 f\Pi_2^0(\Delta\overline{v}_h) - \int_K f\Delta\overline{v_h}}_{E_{12}}\right\}
\end{aligned}
$$

$$- \Bigg\{ \underbrace{\int_K \nabla \Pi_1^{\nabla} g \cdot \nabla \Pi_2^{\nabla} \overline{v}_h - \int_K \nabla g \cdot \nabla \overline{v_h}}_{E_{13}} \Bigg\}$$

$$+ \frac{n}{n-1} \Bigg\{ \underbrace{\int_K (\Pi_2^0 f)(\Pi_1^0 \overline{\psi}_h) - \int_K f \overline{\psi_h}}_{E_{14}} \Bigg\}. \tag{4.3}$$

Now, we will bound each term $E_{1i}|_K$ $i = 1, 2, 3, 4$. The term E_{11} can be bounded as follows. Using the definition of Π_2^0 and Proposition 4.1, we have

$$E_{11} = \int_K \Delta f \big(\overline{v}_h - \Pi_2^0 \overline{v}_h \big) \leq |f|_{2,K} \big\| v_h - \Pi_2^0 v_h \big\|_{0,K}$$
$$= |f|_{2,K} \inf_{q \in \mathbb{P}_2(K)} \| v_h - q \|_{0,K} \leq C h_K^2 |f|_{2,K} |v_h|_{2,K}.$$

For the term E_{12}, we repeat the same arguments to obtain

$$E_{12} \leq C h_K^2 |f|_{2,K} |v_h|_{2,K}.$$

Now, we bound E_{13}. From the definition of Π_2^{∇}, we have

$$E_{13} = \int_K \nabla \Pi_1^{\nabla} g \cdot \nabla \overline{v_h} - \int_K \nabla g \cdot \nabla \overline{v_h} = \int_K \nabla \big(\Pi_1^{\nabla} g - g \big) \cdot \nabla \overline{v_h}$$
$$= \int_K \nabla \big(\Pi_1^{\nabla} g - g \big) \cdot \nabla (\overline{v_h} - \tilde{v}_\pi) \leq \big| \Pi_1^{\nabla} g - g \big|_{1,K} |v_h - \tilde{v}_\pi|_{1,K}$$
$$\leq C h_K |g|_{1,K} |v_h|_{2,K},$$

where we have used the definition and the stability of Π_1^{∇} with $\tilde{v}_\pi \in \mathbb{P}_1(K)$ such that Proposition 4.1 holds true.

For the term E_{14}, we first use the definition of Π_2^0, the definition and the stability of Π_1^0 with respect to $\hat{f}_\pi \in \mathbb{P}_1(K)$ such that Proposition 4.1 holds true, thus, we have

$$E_{14} = \int_K f \Pi_1^0 \overline{\psi}_h - \int_K f \overline{\psi_h} = \int_K (f - \hat{f}_\pi) \big(\Pi_1^0 \overline{\psi}_h - \overline{\psi_h} \big)$$
$$\leq C h_K^2 |f|_{2,K} \big\| \Pi_1^0 \psi_h - \psi_h \big\|_{0,K}$$
$$\leq C h_K^2 |f|_{2,K} \inf_{q \in \mathbb{P}_1(K)} \| \psi_h - q \|_{0,K} \leq C h_K^3 |f|_{2,K} |\psi_h|_{1,K}.$$

Therefore, using the Cauchy–Schwarz inequality, we can deduce from (4.3) that

$$E_1 \leq C_n h \| (f, g) \| \| (v_h, \psi_h) \|.$$

Now, for the terms E_2 and E_3, we use the Cauchy–Schwarz inequality and the stability of $a_{h,K}^{\Delta}(\cdot, \cdot)$ and $a_{h,K}^{\nabla}(\cdot, \cdot)$ to obtain

$$E_2 + E_3 \leq C_n \big\{ |\tilde{u} - \tilde{u}_I|_{2,\Omega} + |\tilde{u} - \tilde{u}_\pi|_{2,h} \big\} + |\tilde{\phi} - \tilde{\phi}_I|_{1,\Omega} + |\tilde{\phi} - \tilde{\phi}_\pi|_{1,h}.$$

Finally, from (4.2), we have

$$\beta\|(v_h,\psi_h)\| \le C_n\big\{h\|(f,g)\| + |\tilde{u}-\tilde{u}_I|_{2,\Omega} + |\tilde{u}-\tilde{u}_\pi|_{2,h} \\ + |\tilde{\phi}-\tilde{\phi}_I|_{1,\Omega} + |\tilde{\phi}-\tilde{\phi}_\pi|_{1,h}\big\}.$$

Therefore, the proof follows from (4.1) and the previous inequality. □

For the convergence and error analysis of the proposed virtual element scheme for the transmission eigenvalue problem, we first establish that $T_h \to T$ in norm as $h \to 0$. Then, we prove a similar convergence result for the adjoint operators T^* and T_h^* of T and T_h, respectively.

Lemma 4.2. *There exist C_n and $\tilde{s} \in (0,1]$, independent of h, such that*

$$\|T - T_h\| \le C_n h^{\tilde{s}}.$$

Proof. Let $(f,g) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that $\|(f,g)\| = 1$, let $(\tilde{u},\tilde{\phi})$ and $(\tilde{u}_h,\tilde{\phi}_h)$ be the solution of problems (2.10) and (3.13), respectively, so that $(\tilde{u},\tilde{\phi}) := T(f,g)$ and $(\tilde{u}_h,\tilde{\phi}_h) := T_h(f,g)$. From Lemma 4.1 and Poincaré inequality, we have

$$\begin{aligned}\|(T-T_h)(f,g)\| &\le C_n h\|(f,g)\| + \|\tilde{u}-\tilde{u}_I\|_{2,\Omega} + |\tilde{u}-\tilde{u}_\pi|_{2,h} \\ &\quad + \|\tilde{\phi}-\tilde{\phi}_I\|_{1,\Omega} + |\tilde{\phi}-\tilde{\phi}_\pi|_{1,h} \\ &\le C_n(h\|(f,g)\| + h^s\|f\|_{2,\Omega} + h^t\|g\|_{1,\Omega}) \\ &\le C_n h^{\tilde{s}}\|(f,g)\|,\end{aligned}$$

where we have used the Propositions 4.1–4.3, and Lemma 2.2, with $\tilde{s} := \min\{s,t\}$. Thus, we conclude the proof. □

Let T^* and $T_h^* : H_0^2(\Omega) \times H_0^1(\Omega) \to H_0^2(\Omega) \times H_0^1(\Omega)$ the adjoint operators of T and T_h, respectively, defined by $T^*(f,g) := (\tilde{u}^*,\tilde{\phi}^*)$ and $T_h^*(f,g) := (\tilde{u}_h^*,\tilde{\phi}_h^*)$, where $(\tilde{u}^*,\tilde{\phi}^*)$ and $(\tilde{u}_h^*,\tilde{\phi}_h^*)$ are the unique solutions of the following problems:

$$a((v,\psi),(\tilde{u}^*,\tilde{\phi}^*)) = b((v,\psi),(f,g)) \quad \forall (v,\psi) \in H_0^2(\Omega) \times H_0^1(\Omega), \tag{4.4}$$

$$a_h((v_h,\psi_h),(\tilde{u}_h^*,\tilde{\phi}_h^*)) = b_h((v_h,\psi_h),(f,g)) \quad \forall (v_h,\psi_h) \in Z_h. \tag{4.5}$$

It is simple to prove that if μ is an eigenvalue of T with multiplicity m, $\overline{\mu}$ is an eigenvalue of T^* with the same multiplicity m.

Now, we will study the convergence in norm of T_h^* to T^* as h goes to zero. With this aim, first we establish an additional regularity result for the solution $(\tilde{u}^*,\tilde{\phi}^*)$ of problem (4.4).

Lemma 4.3. *There exist $s,t \in (1/2,1]$ and C_n such that for all $(f,g) \in H_0^2(\Omega) \times H_0^1(\Omega)$, the solution $(\tilde{u}^*,\tilde{\phi}^*)$ of problem* (4.4) *satisfies $\tilde{u}^* \in H^{2+s}(\Omega)$, $\tilde{\phi}^* \in H^{1+t}(\Omega)$,* *and*

$$\|\tilde{u}^*\|_{2+s,\Omega} + \|\tilde{\phi}^*\|_{1+t,\Omega} \leq C_n \|(f,g)\|.$$

Proof. The result follows repeating the same arguments used in the proof of Lemma 2.2. □

Remark 4.1. We note that the constants s and t in Lemma 4.3 are the same as in Lemma 2.2.

Now, we are in a position to establish the following result.

Lemma 4.4. *There exist C_n and $\tilde{s} \in (0,1]$, independent of h, such that*

$$\left\|T^* - T_h^*\right\| \leq C_n h^{\tilde{s}}.$$

Proof. It is essentially identical to that of Lemmas 4.1 and 4.2. □

Our final goal is to show convergence and obtain error estimates. With this aim, we will apply to our problem the theory from Refs. 4 and 49 for non-self-adjoint compact operators.

We first recall the definition of spectral projectors. Let μ be a nonzero eigenvalue of T with algebraic multiplicity m and let Γ be an open disk in the complex plane centered at μ, such that μ is the only eigenvalue of T lying in Γ and $\partial\Gamma \cap \mathrm{sp}(T) = \emptyset$. The spectral projectors E and E^* are defined as follows:

- The spectral projector of T relative to μ: $E := (2\pi i)^{-1} \int_{\partial\Gamma} (z - T)^{-1} dz$;
- The spectral projector of T^* relative to $\overline{\mu}$: $E^* := (2\pi i)^{-1} \int_{\partial\Gamma} (z - T^*)^{-1} dz$.

E and E^* are projections onto the space of generalized eigenvectors $R(E)$ and $R(E^*)$, respectively. It is simple to prove that $R(E), R(E^*) \in H^{2+s}(\Omega) \times H^{1+t}(\Omega)$ (see Ref. 26).

Now, since $T_h \to T$ in norm, there exist m eigenvalues (which lie in Γ) $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which will converge to μ as h goes to zero.

In a similar way, we introduce the following spectral projector $E_h := (2\pi i)^{-1} \int_{\partial\Gamma} (z - T_h)^{-1} dz$, which is a projector onto the invariant subspace $R(E_h)$ of T_h spanned by the generalized eigenvectors of T_h corresponding to $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$.

We recall the definition of the gap $\widehat{\delta}$ between two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a Hilbert space $\mathcal{V}$:

$$\widehat{\delta}(\mathcal{X}, \mathcal{Y}) := \max\{\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\},$$

where

$$\delta(\mathcal{X}, \mathcal{Y}) := \sup_{\mathbf{x} \in \mathcal{X}: \|x\|_{\mathcal{V}} = 1} \delta(x, \mathcal{Y}) \quad \text{with } \delta(x, \mathcal{Y}) := \inf_{y \in \mathcal{Y}} \|x - y\|_{\mathcal{V}}.$$

Let $\mathcal{P}_h := \mathcal{P}_h^2 \times \mathcal{P}_h^1 : H_0^2(\Omega) \times H_0^1(\Omega) \to Z_h \subseteq H_0^2(\Omega) \times H_0^1(\Omega)$ be the projector defined by

$$\begin{aligned} a(\mathcal{P}_h(u,\phi) - (u,\phi), (v_h,\psi_h)) &= a^{\Delta}(\mathcal{P}_h^2 u - u, v_h) + a^{\nabla}(\mathcal{P}_h^1 \phi - \phi, \psi_h) \\ &= 0 \quad \forall (v_h,\psi_h) \in Z_h. \end{aligned}$$

We note that the form $a(\cdot,\cdot)$ is the inner product of $H_0^2(\Omega) \times H_0^1(\Omega)$. Therefore, we have

$$|(u,\phi) - \mathcal{P}(u,\phi)|_{H_0^2(\Omega)\times H_0^1(\Omega)} = \inf_{(v_h,\psi_h)\in Z_h} |(u,\phi) - (v_h,\psi_h)|_{H_0^2(\Omega)\times H_0^1(\Omega)} \tag{4.6}$$

and

$$|\mathcal{P}(u,\phi)|_{H_0^2(\Omega)\times H_0^1(\Omega)} \leq |(u,\phi)|_{H_0^2(\Omega)\times H_0^1(\Omega)} \quad \forall (u,\phi) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{4.7}$$

The following error estimates for the approximation of eigenvalues and eigenfunctions hold true.

Theorem 4.1. *There exists a strictly positive constant C_n such that*

$$\widehat{\delta}(R(E), R(E_h)) \leq C_n h^{\min\{s,t\}}, \tag{4.8}$$

$$|\mu - \hat{\mu}_h| \leq C_n h^{2\min\{s,t\}}, \tag{4.9}$$

where $\hat{\mu}_h := \frac{1}{m}\sum_{k=1}^{m} \mu_h^{(k)}$ and with the constants s and t as in Lemmas 2.2 *and* 4.3 (*see also Remark* 2.2).

Proof. As a consequence of Lemma 4.2, T_h converges in norm to T as h goes to zero. Then, the proof of (4.8) follows as a direct consequence of Theorem 7.1 from Ref. 4 and the fact that, for $(f,g) \in R(E)$, $\|(f,g)\|_{H^{2+s}(\Omega)\times H^{1+t}(\Omega)} \leq \|(f,g)\|$, because of Lemma 2.2.

In what follows, we will prove (4.9): assume that $T(u_k,\phi_k) = \mu(u_k,\phi_k)$, $k = 1,\ldots,m$. Since $a(\cdot,\cdot)$ is an inner product in $H_0^2(\Omega) \times H_0^1(\Omega)$, we can choose a dual basis for $R(E^*)$ denoted by $(u_k^*,\phi_k^*) \in H_0^2(\Omega) \times H_0^1(\Omega)$ satisfying

$$a\big((u_k,\phi_k),(u_l^*,\phi_l^*)\big) = \delta_{k,l}.$$

Now, from Theorem 7.2 of Ref. 4, we have that

$$\begin{aligned} |\mu - \hat{\mu}_h| \leq \frac{1}{m}\sum_{k=1}^{m} \big|\big\langle (T - T_h)(u_k,\phi_k), \big(u_k^*,\phi_k^*\big)\big\rangle\big| \\ + C_n \|(T - T_h)|_{R(E)}\| \big\|\big(T^* - T_h^*\big)|_{R(E^*)}\big\|, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the corresponding duality pairing.

Thus, in order to obtain (4.9), we need to bound the two terms on the right-hand side above.

The second term can be easily bounded from Lemmas 4.2 and 4.4. In fact, we have

$$\|(T - T_h)|_{R(E)}\| \big\| \big(T^* - T_h^*\big)\big|_{R(E^*)} \| \leq C_n h^{2\min\{s,t\}}. \tag{4.10}$$

Next, we manipulate the first term as follows: adding and subtracting $(v_h, \psi_h) \in Z_h$ and using the definition of T and T_h, we obtain

$$\begin{aligned}
&\big\langle (T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big)\big\rangle \\
&\quad = a\big((T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big)\big) \\
&\quad = a\big((T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big) - (v_h, \psi_h)\big) + a(T(u_k, \phi_k), (v_h, \psi_h)) \\
&\qquad - a(T_h(u_k, \phi_k), (v_h, \psi_h)) \\
&\quad = a\big((T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big) - (v_h, \psi_h)\big) + b((u_k, \phi_k), (v_h, \psi_h)) \\
&\qquad - a(T_h(u_k, \phi_k), (v_h, \psi_h)) + a_h(T_h(u_k, \phi_k), (v_h, \psi_h)) - b_h((u_k, \phi_k), (v_h, \psi_h)) \\
&\quad = \big\{a\big((T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big) - (v_h, \psi_h)\big)\big\} \\
&\qquad + \big\{b((u_k, \phi_k), (v_h, \psi_h)) - b_h((u_k, \phi_k), (v_h, \psi_h))\big\} \\
&\qquad + \big\{a_h(T_h(u_k, \phi_k), (v_h, \psi_h)) - a(T_h(u_k, \phi_k), (v_h, \psi_h))\big\} \quad \forall (v_h, \psi_h) \in Z_h.
\end{aligned} \tag{4.11}$$

Now, we estimate each bracket in (4.11) separately. First, to bound the second bracket, we use the additional regularity of $(u_k, \phi_k) \in R(E) \subset H^{2+s}(\Omega) \times H^{1+t}(\Omega)$ and repeating the same steps used to derive (4.3) (in this case with (u_k, ϕ_k) instead of (f, g)), we have

$$b_{h,K}((u_k, \phi_k), (v_h, \psi_h)) - b_K((u_k, \phi_k), (v_h, \psi_h)) = E_{11} + E_{12} + E_{13} + E_{14}.$$

Now, we will bound each term E_{1i}, $i = 1, 2, 3, 4$, as in the proof of Lemma 4.1, but in this case, exploiting the additional regularity and the estimates in Lemmas 2.2 and 4.3 for $(u_k, \phi_k) \in R(E)$ and $\big(u_k^*, \phi_k^*\big) \in R(E^*)$, respectively.

In particular, the terms E_{11}, E_{12} and E_{14} can be bound exactly as in the proof of Lemma 4.1. However, for the term E_{13}, we proceed as follows:

$$\begin{aligned}
E_{13} &= \int_K \nabla \Pi_1^{\nabla} \phi_k \cdot \nabla \overline{v_h} - \int_K \nabla \phi_k \cdot \nabla \overline{v_h} = \int_K \nabla\big(\Pi_1^{\nabla} \phi_k - \phi_k\big) \cdot \nabla \overline{v_h} \\
&= \int_K \nabla\big(\Pi_1^{\nabla} \phi_k - \phi_k\big) \cdot \nabla\big(\overline{v_h} - \tilde{v}_h^{\pi}\big) \leq \big|\Pi_1^{\nabla} \phi_k - \phi_k\big|_{1,K} \big|v_h - \tilde{v}_h^{\pi}\big|_{1,K} \\
&= \inf_{q_h \in \mathbb{P}_1(K)} |\phi_k - q_h|_{1,K} \big|v_h - \tilde{v}_h^{\pi}\big|_{1,K} \leq C h_K^{1+t} |\phi_k|_{1+t,K} |v_h|_{2,K} \\
&\leq C h_K^{2\min\{s,t\}} |\phi_k|_{1+t,K} |v_h|_{2,K},
\end{aligned}$$

where we have used the definition of Π_1^{∇} with $\tilde{v}_h^{\pi} \in \mathbb{P}_1(K)$ such that Proposition 4.1 holds true and the fact that $\phi_k \in H^{1+t}(\Omega)$ together with Proposition 4.1 again.

Therefore taking sum and using the additional regularity for ϕ_k, together with Lemma 2.2, we obtain for all $(v_h, \psi_h) \in Z_h$ that

$$\{b((u_k, \phi_k), (v_h, \psi_h)) - b_h((u_k, \phi_k), (v_h, \psi_h))\} \leq C_n h^{2\min\{s,t\}} \|(u_k, \phi_k)\| \|(v_h, \psi_h)\|. \tag{4.12}$$

Now, we estimate the third bracket in (4.11). Let $(w_h, \xi_h) := T_h(u_k, \phi_k)$ and Π_h^K be defined by $\big(\Pi_h^K(v, \psi)\big)|_K := (\Pi_2^\Delta v, \Pi_1^\nabla \psi)$ for all $K \in \mathcal{T}_h$ and for all $(v, \psi) \in H_0^2(\Omega) \times H_0^1(\Omega)$, where Π_2^Δ and Π_1^∇ have been defined in (3.1a)–(3.1b) and (3.2a)–(3.2b), respectively. Hence, we have

$$\begin{aligned}
&a_h((w_h, \xi_h), (v_h, \psi_h)) - a((w_h, \xi_h), (v_h, \psi_h)) \\
&\quad = \sum_{K \in \mathcal{T}_h} \{a_{h,K}((w_h, \xi_h), (v_h, \psi_h)) - a_K((w_h, \xi_h), (v_h, \psi_h))\} \\
&\quad = \sum_{K \in \mathcal{T}_h} \big\{a_{h,K}\big((w_h, \xi_h) - \big(\Pi_2^\Delta w_h, \Pi_1^\nabla \xi_h\big), (v_h, \psi_h) - \big(\Pi_2^\Delta v_h, \Pi_1^\nabla \psi_h\big)\big) \\
&\qquad + a_K\big(\big(\Pi_2^\Delta w_h, \Pi_1^\nabla \xi_h\big) - (w_h, \xi_h), (v_h, \psi_h) - \big(\Pi_2^\Delta v_h, \Pi_1^\nabla \psi_h\big)\big)\big\} \\
&\quad \leq C_n \sum_{K \in \mathcal{T}_h} \big\{\big|(w_h, \xi_h) - \big(\Pi_2^\Delta w_h, \Pi_1^\nabla \xi_h\big)\big|_{H^2(K) \times H^1(K)} \\
&\qquad \times \big|(v_h, \psi_h) - \big(\Pi_2^\Delta v_h, \Pi_1^\nabla \psi_h\big)\big|_{H^2(K) \times H^1(K)}\big\} \\
&\quad = C_n \sum_{K \in \mathcal{T}_h} \big\{|T_h(u_k, \phi_k) - \Pi_h^K T_h(u_k, \phi_k)|_{H^2(K) \times H^1(K)} \\
&\qquad \times \big|(v_h, \psi_h) - \Pi_h^K(v_h, \psi_h)\big|_{H^2(K) \times H^1(K)}\big\},
\end{aligned} \tag{4.13}$$

for all $(v_h, \psi_h) \in Z_h$, where we have used (3.8)–(3.9), Cauchy–Schwarz inequality and (3.10)–(3.11). Now, using the triangular inequality, we have that

$$\begin{aligned}
&\big|T_h(u_k, \phi_k) - \Pi_h^K T_h(u_k, \phi_k)\big|_{H^2(K) \times H^1(K)} \\
&\quad \leq |T_h(u_k, \phi_k) - T(u_k, \phi_k)|_{H^2(K) \times H^1(K)} \\
&\qquad + \big|\Pi_h^K T_h(u_k, \phi_k) - \Pi_h^K T(u_k, \phi_k)\big|_{H^2(K) \times H^1(K)} \\
&\qquad + \big|\Pi_h^K T(u_k, \phi_k) - T(u_k, \phi_k)\big|_{H^2(K) \times H^1(K)}.
\end{aligned}$$

Thus, from (4.13), the above estimate, the stability of Π_h^K and the additional regularity for (u_k, ϕ_k) together with Lemma 2.2, we have

$$\begin{aligned}
&a_h(T_h(u_k, \phi_k), (v_h, \psi_h)) - a(T_h(u_k, \phi_k), (v_h, \psi_h)) \\
&\quad \leq C_n h^{\min\{s,t\}} \|(u_k, \phi_k)\| \\
&\qquad \times \sum_{K \in \mathcal{T}_h} \big|(v_h, \psi_h) - \Pi_h^K(v_h, \psi_h)\big|_{H^2(K) \times H^1(K)} \quad \forall (v_h, \psi_h) \in Z_h.
\end{aligned} \tag{4.14}$$

Finally, we take $(v_h, \psi_h) := \mathcal{P}\big(u_k^*, \phi_k^*\big) \in Z_h$ in (4.11). Thus, on the one hand, we bound the first bracket in (4.11) as follows:

$$
\begin{aligned}
& a\big((T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big) - (v_h, \psi_h)\big) \\
&\quad = a\big((T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big) - \mathcal{P}\big(u_k^*, \phi_k^*\big)\big) \\
&\quad \leq |(T - T_h)(u_k, \phi_k)|_{H_0^2(\Omega)\times H_0^1(\Omega)} \big|\big(u_k^*, \phi_k^*\big) - \mathcal{P}\big(u_k^*, \phi_k^*\big)\big|_{H_0^2(\Omega)\times H_0^1(\Omega)} \\
&\quad = |(T - T_h)(u_k, \phi_k)|_{H_0^2(\Omega)\times H_0^1(\Omega)} \inf_{(r_h, s_h)\in Z_h} \big|\big(u_k^*, \phi_k^*\big) - (r_h, s_h)\big|_{H_0^2(\Omega)\times H_0^1(\Omega)} \\
&\quad \leq |(T - T_h)(u_k, \phi_k)|_{H_0^2(\Omega)\times H_0^1(\Omega)} \big|\big(u_k^*, \phi_k^*\big) - \big(\big(u_k^*\big)_I, \big(\phi_k^*\big)_I\big)\big|_{H_0^2(\Omega)\times H_0^1(\Omega)} \\
&\quad \leq C h^{2\min\{s,t\}} \big\|\big(u_k^*, \phi_k^*\big)\big\|,
\end{aligned}
$$

where we have used (4.6), Propositions 4.2 and 4.3, the additional regularity for $\big(u_k^*, \phi_k^*\big)$, Lemmas 4.3 and 4.2.

On the other hand, from (4.14), we have that

$$
\begin{aligned}
& \big|(v_h, \psi_h) - \Pi_h^K(v_h, \psi_h)\big|_{H^2(K)\times H^1(K)} \\
&\quad = \big|\mathcal{P}\big(u_k^*, \phi_k^*\big) - \Pi_h^K \mathcal{P}\big(u_k^*, \phi_k^*\big)\big|_{H^2(K)\times H^1(K)} \\
&\quad \leq \big|\mathcal{P}\big(u_k^*, \phi_k^*\big) - \big(u_k^*, \phi_k^*\big)\big|_{H^2(K)\times H^1(K)} + \big|\big(u_k^*, \phi_k^*\big) - \Pi_h^K\big(u_k^*, \phi_k^*\big)\big|_{H^2(K)\times H^1(K)} \\
&\qquad + \big|\Pi_h^K\big(\big(u_k^*, \phi_k^*\big) - \mathcal{P}\big(u_k^*, \phi_k^*\big)\big)\big|_{H^2(K)\times H^1(K)}.
\end{aligned}
$$

Then, using again (4.6), Propositions 4.2 and 4.3, the additional regularity for $\big(u_k^*, \phi_k^*\big)$, Lemmas 4.3 and 4.2, we obtain from (4.14) that

$$
\begin{aligned}
& a_h(T_h(u_k, \phi_k), (v_h, \psi_h)) - a(T_h(u_k, \phi_k), (v_h, \psi_h)) \\
&\quad \leq C_n h^{2\min\{s,t\}} \|(u_k, \phi_k)\| \big\|\big(u_k^*, \phi_k^*\big)\big\|.
\end{aligned} \tag{4.15}
$$

Thus, from (4.11), (4.12) and (4.15), we obtain

$$
\big|\big\langle (T - T_h)(u_k, \phi_k), \big(u_k^*, \phi_k^*\big)\big\rangle\big| \leq C_n h^{2\min\{s,t\}}. \tag{4.16}
$$

Therefore, the proof follows from estimates (4.10) and (4.16). □

Remark 4.2. The error estimate for the eigenvalue μ of T yields analogous estimate for the approximation of the eigenvalue $\lambda = 1/\mu$ of Problem 1 by means of $\hat{\lambda}_h := \frac{1}{m}\sum_{k=1}^{m} \lambda_h^{(k)}$, where $\lambda_h^{(k)} = 1/\mu_h^{(k)}$.

Now, we state in the following remark the approximation properties of Problem 3.

Remark 4.3. A result analogous to Theorem 4.1 can be proven for the alternative discretization of Problem 1 proposed in Remark 3.1. We do not include proofs to avoid repeating step-by-step those of Sec. 4. However, we will present a numerical test to confirm the error estimates in this case.

5. Numerical Results

In this section, we present a series of numerical experiments to solve the transmission eigenvalue problem with the Virtual Element schemes (3.12) and (3.14). However, to complete the choice of the VEM, we had to fix the forms $s_K^{\Delta}(\cdot,\cdot)$ and $s_K^{\nabla}(\cdot,\cdot)$ satisfying (3.6) and (3.7), respectively. For $s_K^{\Delta}(\cdot,\cdot)$, we consider the same definition as in Ref. 48:

$$s_K^{\Delta}(u_h, v_h) := \sigma_K \sum_{i=1}^{N_K} \left[u_h(P_i) v_h(P_i) + h_{P_i}^2 \nabla u_h(P_i) \cdot \nabla v_h(P_i) \right] \quad \forall\, u_h, v_h \in W_h^K,$$

where $P_1, \ldots, P_{N_K}$ are the vertices of K, h_{P_i} corresponds to the maximum diameter of the elements with P_i as a vertex and $\sigma_K > 0$ is a multiplicative factor to take into account the magnitude of the parameter and the h-scaling, for instance, in the numerical tests, we have picked $\sigma_K > 0$ as the mean value of the eigenvalues of the local matrix $a_K^{\Delta}\left(\Pi_2^{\Delta}\varphi_i, \Pi_2^{\Delta}\varphi_j\right)$, $i, j = 1, \ldots, \dim W_h^K$ and $\{\varphi_i\}_{i=1}^{\dim W_h^K}$ is a basis of W_h^K. This ensures that the stabilizing term scales as $a_K^{\Delta}(v_h, v_h)$. Now, a choice for $s_K^{\nabla}(\cdot,\cdot)$ is given by

$$s_K^{\nabla}(\phi_h, \psi_h) := \sum_{i=1}^{N_K} \phi_h(P_i)\psi_h(P_i) \quad \forall\, \phi_h, \psi_h \in V_h^K,$$

which corresponds to the identity matrix of dimension N_K. A proof of (3.6) and (3.7) for the above choices could be derived following the arguments in Ref. 10. Finally, we mention that the previous definitions are in accordance with the analysis presented in Refs. 47 and 48 in order to avoid spectral pollution.

We have implemented in a MATLAB code the proposed VEM on arbitrary polygonal meshes, by following the ideas presented in Ref. 7. Moreover, we compare our results with those existing in the literature, for example, Refs. 26, 31, 35 and 45. We have considered three different domains, namely, a square domain, a circular domain centered at the origin and an L-shaped domain.

5.1. *Test 1: Square domain*

The goal of this test is to assess and compare the performance of the VEM discretizations (3.12) and (3.14). With this aim, we have taken $\Omega := (0,1)^2$ and index of refraction $n = 4$ and $n = 16$. We have tested the methods by using different families of meshes (see Fig. 1):

- $\mathcal{T}_h^1$: triangular meshes;
- $\mathcal{T}_h^2$: rectangular meshes;
- $\mathcal{T}_h^3$: hexagonal meshes;
- $\mathcal{T}_h^4$: non-structured hexagonal meshes made of convex hexagons.

The refinement parameter N used to label each mesh is the number of elements on each edge of the domain.

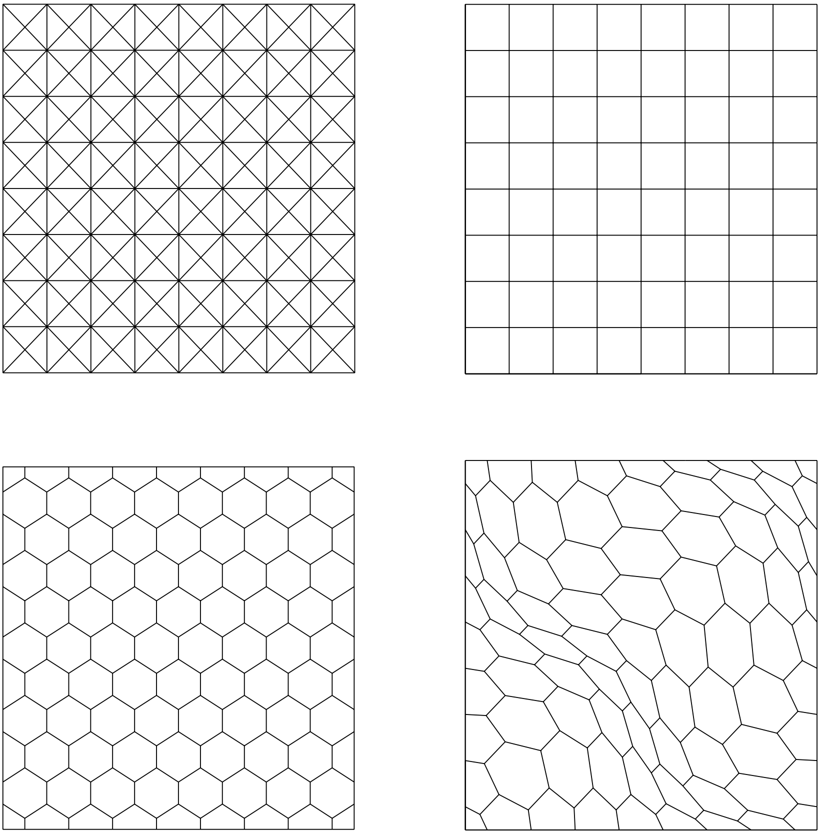

Fig. 1. Sample meshes: $\mathcal{T}_h^1$ (top left), $\mathcal{T}_h^2$ (top right), $\mathcal{T}_h^3$ (bottom left) and $\mathcal{T}_h^4$ (bottom right) for $N = 8$.

We report in Tables 1 and 2 the lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4$ computed by the scheme (3.12) with two different families of meshes and $N = 32, 64, 128$, and with the index of refraction $n = 4$ and $n = 16$, respectively. The tables include computed orders of convergence as well as more accurate values extrapolated by means of a least-squares fitting. Moreover, we compare the performance of the proposed method with those presented in Refs. 35 and 45. With this aim, we include in the last row of Tables 1 and 2 the results reported in that references, for the same problem.

It can be seen from Tables 1 and 2 and that the eigenvalue approximation order of method (3.12) is quadratic (as predicted by the theory for convex domains).

On the other hand, in Table 3, we report the five lowest transmission eigenvalues computed with the VEM (3.14). The table includes orders of convergence as well as accurate values extrapolated by means of a least-squares fitting.

It can be seen from Table 3 that the computed lowest transmission eigenvalues converge with an optimal quadratic order as predicted by the theory (see

Table 1. Test 1: Lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4$ computed on different meshes and with index of refraction $n = 4$.

Meshes	k_{ih}	k_{1h}	k_{2h}	k_{3h}	k_{4h}
$\mathcal{T}_h^3$	$N = 32$	$4.2835 - 1.1367i$	$4.2835 + 1.1367i$	5.3373	5.4172
	$N = 64$	$4.2745 - 1.1446i$	$4.2745 + 1.1446i$	5.4375	5.4599
	$N = 128$	$4.2724 - 1.1467i$	$4.2724 + 1.1467i$	5.4661	5.4719
	Order	2.10 & 1.89	2.10 & 1.89	1.81	1.84
	Extrapolated	$4.2717 - 1.1475i$	$4.2717 + 1.1475i$	5.4775	5.4765
$\mathcal{T}_h^4$	$N = 32$	$4.2870 - 1.1341i$	$4.2870 + 1.1341i$	5.3245	5.4178
	$N = 64$	$4.2753 - 1.1438i$	$4.2753 + 1.1438i$	5.4329	5.4602
	$N = 128$	$4.2726 - 1.1465i$	$4.2726 + 1.1465i$	5.4647	5.4719
	Order	2.12 & 1.86	2.12 & 1.86	1.77	1.85
	Extrapolated	$4.2718 - 1.1475i$	$4.2718 + 1.1475i$	5.4779	5.4765
	Ref. 45	$4.2717 - 1.1474i$	$4.2717 + 1.1474i$	5.4761	5.4761

Table 2. Test 1: Lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4$ computed on different meshes and with index of refraction $n = 16$.

Meshes	k_{ih}	k_{1h}	k_{2h}	k_{3h}	k_{4h}
$\mathcal{T}_h^1$	$N = 32$	1.8805	2.4467	2.4467	2.8691
	$N = 64$	1.8798	2.4449	2.4449	2.8671
	$N = 128$	1.8796	2.4444	2.4444	2.8666
	Order	2.01	2.00	2.00	2.01
	Extrapolated	1.8796	2.4442	2.4442	2.8664
$\mathcal{T}_h^2$	$N = 32$	1.8764	2.4318	2.4318	2.8645
	$N = 64$	1.8788	2.4410	2.4410	2.8658
	$N = 128$	1.8794	2.4434	2.4434	2.8663
	Order	1.95	1.95	1.95	1.61
	Extrapolated	1.8796	2.4443	2.4443	2.8665
	Ref. 35 (Argyris method)	1.8651	2.4255	2.4271	2.8178
	Ref. 35 (Continuous method)	1.9094	2.5032	2.5032	2.9679
	Ref. 35 (Mixed method)	1.8954	2.4644	2.4658	2.8918
	Ref. 45	1.8796	2.4442	2.4442	2.8664

Table 3. Test 1: Lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4$ computed on different meshes and with index of refraction $n = 16$.

Meshes	k_{ih}	k_{1h}	k_{2h}	k_{3h}	k_{4h}
$\mathcal{T}_h^1$	$N = 32$	1.8823	2.4504	2.4504	2.8749
	$N = 64$	1.8803	2.4458	2.4458	2.8686
	$N = 128$	1.8798	2.4446	2.4446	2.8670
	Order	1.99	1.99	1.99	1.99
	Extrapolated	1.8796	2.4442	2.4442	2.8664
$\mathcal{T}_h^2$	$N = 32$	1.8815	2.4421	2.4421	2.8811
	$N = 64$	1.8801	2.4438	2.4438	2.8702
	$N = 128$	1.8797	2.4441	2.4441	2.8674
	Order	1.91	2.15	2.15	1.96
	Extrapolated	1.8796	2.4442	2.4442	2.8664

Remark 4.3). It can be observed that the two methods agree perfectly well (see Tables 2 and 3).

5.2. *Test 2: Circular domain*

In this test, we have taken as domain the circle $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1/2\}$. We have used polygonal meshes created with PolyMesher[54] (see Fig. 2). The refinement parameter N is the number of elements intersecting the boundary.

We report in Table 4 the five lowest transmission eigenvalues computed with the VEM (3.12). The table includes orders of convergence as well as accurate values extrapolated by means of a least-squares fitting. Once again, the last rows show the values obtained by extrapolating those computed with different methods presented in Refs. 26, 31 and 35.

Once more, a quadratic order of convergence can be clearly appreciated from Table 4.

We show in Fig. 2 the eigenfunctions corresponding to the four lowest transmission eigenvalues.

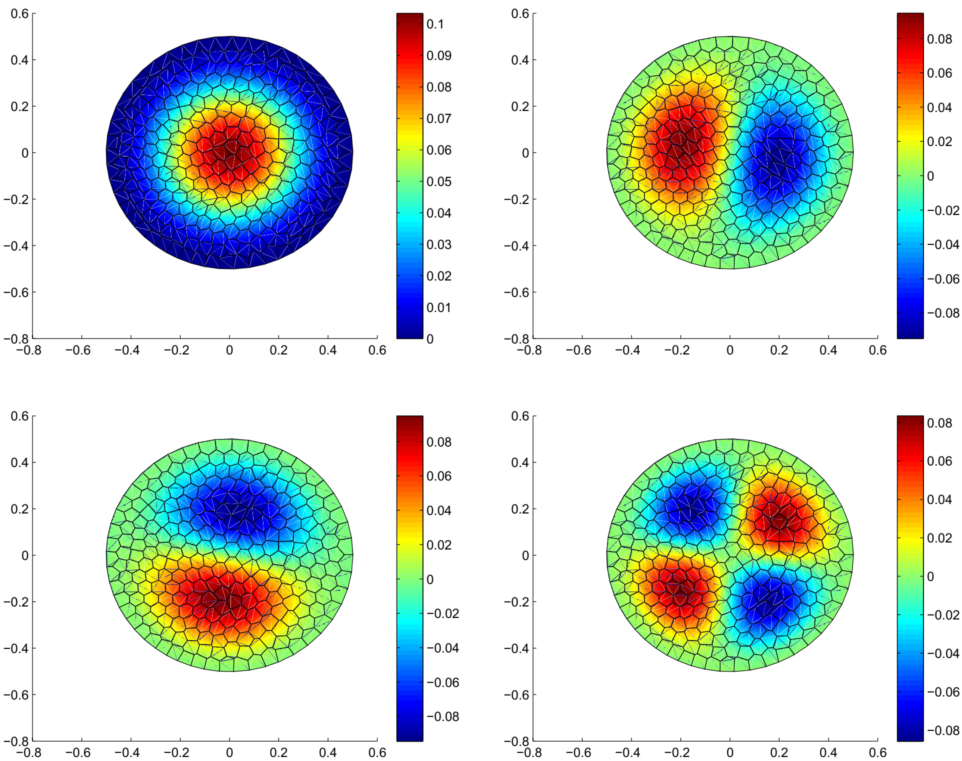

Fig. 2. Test 2: Eigenfunctions u_{1h} (top left), u_{2h} (top right), u_{3h} (bottom left) and u_{4h} (bottom right).

Table 4. Test 2: Computed lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4, 5$ with index of refraction $n = 16$.

	k_{1h}	k_{2h}	k_{3h}	k_{4h}	k_{5h}
$N = 32$	1.9835	2.6032	2.6037	3.2115	3.2117
$N = 64$	1.9869	2.6105	2.6106	3.2225	3.2227
$N = 128$	1.9877	2.6123	2.6123	3.2255	3.2256
Order	1.98	1.97	2.01	1.86	1.90
Extrapolated	1.9880	2.6129	2.6129	3.2267	3.2267
Ref. 26	1.9881	—	—	—	—
Ref. 31	1.9879	2.6124	2.6124	3.2255	3.2255
Ref. 35 (Argyris method)	2.0076	2.6382	2.6396	3.2580	3.2598
Ref. 35 (Continuous method)	2.0301	2.6937	2.6974	3.3744	3.3777
Ref. 35 (Mixed method)	1.9912	2.6218	2.6234	3.2308	3.2397

5.3. *Test 3: L-shaped domain*

Finally, we have considered an L-shaped domain: $\Omega := (-1/2, 1/2)^2 \backslash ([0, 1/2] \times [-1/2, 0])$. We have used uniform triangular meshes as those shown in Fig. 3. The meaning of the refinement parameter N is the number of elements on each edge.

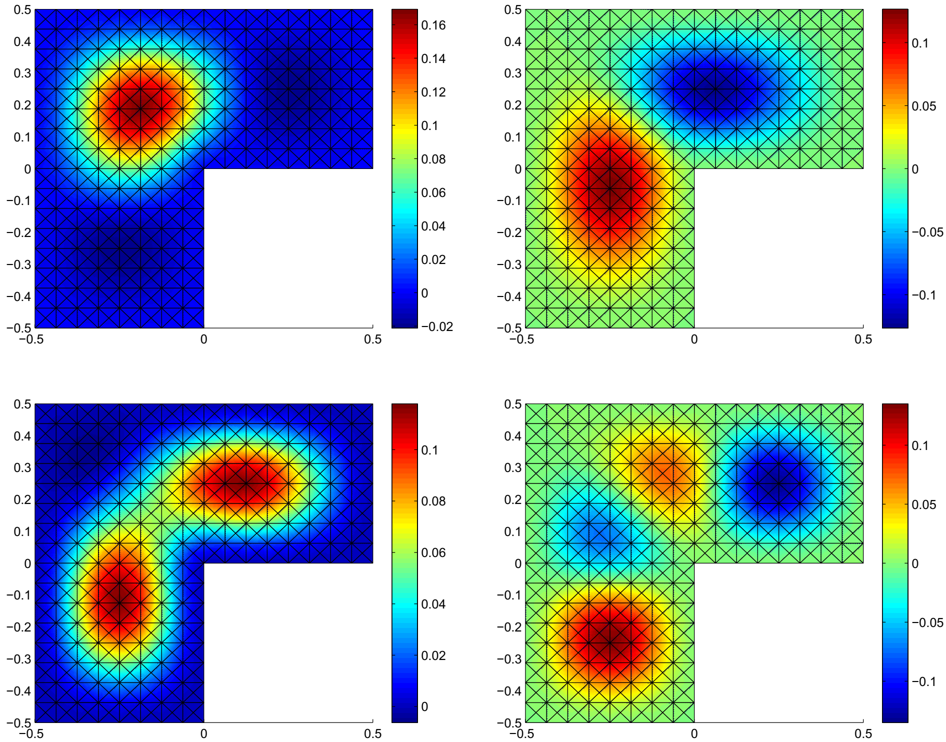

Fig. 3. Test 3: Eigenfunctions u_{1h} (top left), u_{2h} (top right), u_{3h} (bottom left) and u_{4h} (bottom right).

Table 5. Test 3: Computed lowest transmission eigenvalues k_{ih}, $i = 1, 2, 3, 4$ with index of refraction $n = 16$.

k_{ih}	k_{1h}	k_{2h}	k_{3h}	k_{4h}
$N = 32$	2.9690	3.1480	3.4216	3.5744
$N = 64$	2.9590	3.1417	3.4136	3.5683
$N = 128$	2.9551	3.1400	3.4113	3.5667
Order	1.37	1.94	1.76	2.00
Extrapolated	2.9527	3.1395	3.4103	3.5662
Ref. 26	2.9553	—	—	—

We report in Table 5 the four lowest transmission eigenvalues computed with the virtual scheme (3.12). The table includes orders of convergence as well as accurate values extrapolated by means of a least-squares fitting. Once again, we compare the performance of the proposed virtual scheme with the one presented in Ref. 26 for the same problem, using triangular meshes.

In this numerical test, according to Refs. 42 and 43, we have that for the Laplace problem, the Lemma 2.2 holds for all $t < t_0$, where $t_0 := \pi/\omega$ with ω being the largest interior angle of Ω (in this test, $\omega = 3\pi/2$). On the other hand, for the biharmonic equation, the Lemma 2.2 holds for all $s < s_0 := \alpha - 1$, where $\alpha > 1$ is the smallest positive root of the following characteristic equation:

$$\sin^2(\alpha - 1)\omega = (\alpha - 1)^2 \sin^2 \omega.$$

As a consequence, for the first transmission eigenvalue, since the singularity of the solution in the L-shaped domain, the method converges with order close to $\min\{1.089, 1.333\}$, which corresponds to the Sobolev regularity for the biharmonic equation and Laplace equation, respectively. Moreover, the method converges with larger orders for the rest of the transmission eigenvalues.

Finally, Fig. 3 shows the eigenfunctions corresponding to the four lowest transmission eigenvalues with index of refraction $n = 16$.

Acknowledgments

The authors are deeply grateful to Prof. Carlo Lovadina (Università degli Studi di Milano, Italy) and Prof. Rodolfo Rodríguez (Universidad de Concepción, Chile) for the fruitful discussions. The first author was partially supported by CONICYT (Chile) through FONDECYT Project 1180913 and by DIUBB through Project 171508 GI/VC Universidad del Bío-Bío, Chile. The second author was partially supported by a CONICYT (Chile) fellowship.

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