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A C^1-virtual element method of high order for the Brinkman equations in stream function formulation with pressure recovery

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In this paper, we propose and analyze a C^1-virtual element method of high order to solve the Brinkman problem formulated in terms of the stream function. The velocity is obtained as a simple post-process from stream function and a novel strategy is written to recover the fluid pressure. We establish optimal *a priori* error estimates for the stream function, velocity and pressure with constants independent of the viscosity. Finally, we report some numerical test illustrating the behavior of the virtual scheme and supporting our theoretical results on different families of polygonal meshes.

Keywords: virtual element method; Brinkman equations; stream function; pressure recovery.

1. Introduction

During the past decades, the numerical solution of incompressible flow problems have acquired great interest due to the variety of applications in different sciences: engineering, biomedicine, oceanography and environmental processes, among others. Different formulation and discretizations of the Stokes, Brinkman, Oseen, Navier–Stokes and Stokes–Darcy equations have been analyzed in the past years; see for instance Gatica *et al.* (2011); Guzmán & Neilan (2012); Vassilevski & Villa (2014); Cai & Chen (2016); John *et al.* (2017); Botti *et al.* (2018, 2019); Camaño *et al.* (2018); di Pietro & Krell (2018); Lederer *et al.* (2018); Anaya *et al.* (2016, 2019); Fu *et al.* (2019) and the references therein.

The aim of the present paper is to introduce and analyze a virtual element method (VEM) to solve the Brinkman problem in polygonal simply connected domains, formulated in terms of the stream function of the velocity field (fourth-order partial differential equation (PDE)), which stands as a suitable framework for the study of Stokes and Darcy flows (cf. Juntunen & Stenberg, 2010; Guzmán & Neilan, 2012; Vassilevski & Villa, 2014; Anaya *et al.*, 2015; Howell *et al.*, 2016), as well as semidiscretizations of transient Stokes equations. The VEM introduced in Beirão da Veiga *et al.* (2013) is a recent generalization of the finite element method that allows to use general polygonal/polyhedral meshes. The method has been applied successfully in a large range of problems in fluid mechanics; see for instance Antonietti *et al.* (2014); Benedetto *et al.* (2016); Cangiani *et al.* (2016); Beirão da Veiga *et al.* (2017, 2018, 2019b); Cáceres & Gatica (2017); Cáceres *et al.* (2017); Vacca (2018); Gatica *et al.* (2018b, 2021); Irisarri & Hauke (2019); Liu *et al.* (2019); Liu & Chen (2019); Zhao *et al.* (2019), where Stokes, Brinkman, Stokes–Darcy and Navier–Stokes equations have been recently developed.

In particular, we are interested in a formulation where the stream function is the principal unknown of the system (Girault & Raviart, 1986). Salient features in formulations of this kind include that there is only one scalar variable, the incompressible condition is satisfied automatically, the stream function is one of the most useful tools in flow visualization and the matrix associated to the linear system turns out to be positive definite. On the other hand, we note that the primary fields velocity and pressure are not present in the formulation. However, if pressure profiles are required, they can be recovered using different manners. For instance, for the Navier–Stokes equation, in Cayco & Nicolaides (1986) has been presented an algorithm for pressure recovery, which is based on a mixed finite element discretization with discrete stream function as a data on the right-hand side (see also Cayco & Nicolaides, 1989). In Beirão da Veiga *et al.* (2019b), it has been recently proposed a VEM least squares method to recover the pressure field from the stream function, and it is based on the discrete inf-sup stable pair proposed in Beirão da Veiga *et al.* (2017) for the Stokes problem. Here, we will also propose a novel strategy to recover the fluid pressure by using the flexibility of the virtual approach.

It is well known that conforming discretizations of a primal formulation to solve fourth-order PDEs require C^1-continuity. The construction of conforming finite elements with C^1-continuity is difficult in general, since they generally involve a large number of degrees of freedom (see for instance Ciarlet, 2002, Section 6.1). However, this can be easily achieved by using VEM. More precisely, we will follow the VEM approach presented in Brezzi & Marini (2013); Chinosi & Marini (2016) (see also Mora *et al.*, 2018; Beirão da Veiga *et al.*, 2019a, 2020; Mora & Velásquez, 2020) to build global discrete spaces of $H^2(\Omega)$ of arbitrary order to solve the fourth-order Brinkman problem on general polygonal meshes. In addition, we will propose a strategy to recover the remaining quantities of interest: velocity and pressure.

According to the above discussion, in the present paper, we are interested in keeping on exploring the flexibility and ability of the VEM to solve fluid flow problems. More precisely, we will propose and analyze a C^1-conforming discretization of arbitrary order $k \geq 2$ using virtual element for the Brinkman equations formulated in terms of the stream function. We will write two well-posed primal discrete formulations (cf. (2.3) and (3.20)) and we will establish optimal order error estimates with constants independent of the viscosity. In addition, the velocity field is then obtained from the discrete stream function by a simple post-process. An error estimate is derived for the velocity in H^1. Moreover, for $k = 3$, we propose a novel strategy to approximate the fluid pressure by means of a second-order variational problem, with a datum coming from the discrete stream function with the help of a proper polynomial projector, which is discretized by employing the enhanced C^0 virtual element space from Ahmad *et al.* (2013). An error estimate in H^1 is derived for the fluid pressure under the assumption that the family of polygonal meshes is quasi-uniform. In summary, the advantages of the proposed VEM are as follows: the possibility to use polygonal meshes, it provides an attractive and competitive alternative to solve the Brinkman problem in terms of the computational cost, the resulting linear system for the stream function is positive definite; it is possible to obtain the velocity and pressure fields in a simple way.

The rest of the paper is organized as follows: in Section 2 we introduce the variational formulation of the Brinkman equations in terms of the stream function of the velocity field. We prove existence and uniqueness of this formulation by using the Lax–Milgram Theorem. In Section 3 we present the virtual element discretization of arbitrary order $k \geq 2$. We also prove the existence and uniqueness of the discrete formulation using the Lax–Milgram Theorem. In Section 4 we prove stability results and obtain error estimates for the stream function. In addition, we recover the velocity field and the fluid pressure. In Section 5 we report a set of numerical examples, which allows us to assess the performance of the proposed method.

Throughout the paper we will follow the usual notation for Sobolev spaces and norms (Adams & Fournier, 2003). We will denote a simply connected polygonal Lipschitz bounded domain of $\mathbb{R}^2$ by Ω, and $\boldsymbol{n} = (n_i)_{1\le i\le 2}$ is the outward unit normal vector to the boundary $\partial\Omega$. The vector $\boldsymbol{t} = (t_i)_{1\le i\le 2}$ is the unit tangent to $\partial\Omega$ oriented such that $t_1 = -n_2$, $t_2 = n_1$. For $\mathcal{D}$ an open bounded domain the $L^2(\mathcal{D})$ inner product will be denoted by $(\cdot,\cdot)_{0,\mathcal{D}}$. Moreover, c and C, with or without subscripts, tildes or hats, will represent a generic constant independent of the mesh parameter h, assuming different values in different occurrences.

2. Model problem

Let $\Omega \subset \mathbb{R}^2$ be a simply connected polygonal domain with boundary $\Gamma := \partial\Omega$. We consider the Brinkman problem (for more details, see for instance, Girault & Raviart, 1986; Quarteroni & Valli, 1994): given a sufficiently smooth force density $\mathbf{f} \in [L^2(\Omega)]^2$, we seek $(\mathbf{u}, p)$ such that:

$$\begin{aligned} \mathbb{K}^{-1}\mathbf{u} - \nu\,\mathbf{div}(\nabla\mathbf{u}) + \nabla p &= \mathbf{f} \qquad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \qquad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \qquad \text{on} \quad \Gamma, \\ (p,1)_{0,\Omega} &= 0, \end{aligned} \tag{2.1}$$

where $\mathbf{u}$ and p are the velocity and the pressure fields, respectively. In the model, ν is the viscosity of the fluid and $\mathbb{K}$ denotes the permeability tensor of the Brinkman region. We assume that the fluid viscosity satisfies $0 < \nu \le C_\nu$, this includes the case where $\nu \to 0$, and the system (2.1) becomes a singular perturbation of the Darcy equations. We assume that $\mathbb{K}^{-1}$ is a sufficiently smooth and uniformly symmetric positive definite tensor, i.e., there exist two positive (uniform) constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1\,\boldsymbol{\eta}^T\boldsymbol{\eta} \le \boldsymbol{\eta}^T\,\mathbb{K}^{-1}\,\boldsymbol{\eta} \le \lambda_2\,\boldsymbol{\eta}^T\,\boldsymbol{\eta} \qquad \forall\,\boldsymbol{\eta} \in \mathbb{R}^2.$$

We introduce the following spaces

$$\mathbf{H} := [H_0^1(\Omega)]^2 = \left\{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma\right\}$$

and

$$Q := L_0^2(\Omega) = \left\{q \in L^2(\Omega) : (q,1)_{0,\Omega} = 0\right\}.$$

The standard velocity-pressure variational formulation of the Brinkman problem reads as follows: find $(\mathbf{u}, p) \in \mathbf{H} \times Q$, such that

$$\begin{aligned} \int_\Omega \mathbb{K}^{-1}\mathbf{u} \cdot \mathbf{v} + \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \int_\Omega p \,\mathrm{d}iv\, \mathbf{v} &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{H}, \\ \int_\Omega q \,\mathrm{d}iv\, \mathbf{u} &= 0 \qquad \forall\, q \in Q. \end{aligned} \tag{2.2}$$

It is well known that (2.2) admits a unique solution (see Girault & Raviart, 1986). Let us introduce the following space of functions in $\mathbf{H}$ with vanishing divergence

$$\mathbf{Z} := \{\mathbf{v} \in \mathbf{H} : \mathrm{d}iv\, \mathbf{v} = 0\}.$$

Then, (2.2) can be written in the following form: find $\mathbf{u} \in \mathbf{Z}$ such that

$$\int_\Omega \mathbb{K}^{-1}\mathbf{u} \cdot \mathbf{v} + \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{Z}.$$

Now, we reformulate the above problem as follows: since Ω is a simply connected domain, a well-known result states that a vector function $\mathbf{v} \in \mathbf{Z}$ if and only if there exists a scalar function $\varphi \in H^2(\Omega)$ (called *stream function*, see Girault & Raviart, 1986), such that

$$\mathbf{v} = \mathbf{curl}\, \varphi \in \mathbf{H}.$$

The function φ is defined up to a constant. Thus, we consider the following space

$$W := H_0^2(\Omega) = \left\{\varphi \in H^2(\Omega) : \varphi = \partial_{\boldsymbol{n}}\varphi = 0 \quad \text{on} \quad \Gamma\right\},$$

where $\partial_{\boldsymbol{n}}$ denotes the normal derivative. We endow W with the following ν-dependent norm

$$\|\varphi\|_W := \left(|\varphi|_{1,\Omega}^2 + \nu|\varphi|_{2,\Omega}^2\right)^{1/2} \qquad \forall\, \varphi \in W.$$

Then, (2.2) can be formulated as follows: find $\psi \in W$, such that

$$\int_\Omega \mathbb{K}^{-1}\mathbf{curl}\, \psi \cdot \mathbf{curl}\, \phi + \nu \int_\Omega \mathrm{D}^2\psi : \mathrm{D}^2\phi = \int_\Omega \mathbf{f} \cdot \mathbf{curl}\, \phi \qquad \forall\, \phi \in W, \tag{2.3}$$

where $\mathrm{D}^2\phi := (\partial_{ij}\phi)_{1\le i,j\le 2}$ denotes the Hessian matrix of ϕ and ':' denotes the usual scalar product of 2×2-matrices. We have that $\psi \in W$ is the *stream function* of the velocity field $\mathbf{u} \in \mathbf{Z}$ (i.e., $\mathbf{u} = \mathbf{curl}\, \psi$).

Now, in order to rewrite the problem in a compact way, we introduce the following bilinear forms and linear functional, for any $\psi, \phi \in W$:

$$A(\psi, \phi) := A_{\mathbf{curl}}(\psi, \phi) + \nu A_\Delta(\psi, \phi), \tag{2.4}$$

$$A_{\mathbf{curl}}(\psi,\phi) := \int_\Omega \mathbb{K}^{-1}\mathbf{curl}\,\psi \cdot \mathbf{curl}\,\phi, \tag{2.5}$$

$$A_\Delta(\psi,\phi) := \int_\Omega \mathrm{D}^2\psi : \mathrm{D}^2\phi, \tag{2.6}$$

$$F(\phi) := \int_\Omega \mathbf{f}\cdot\mathbf{curl}\,\phi. \tag{2.7}$$

The following lemma will allow us to establish the well posedness of formulation (2.3).

LEMMA 2.1 There exists a constant $\alpha_0 > 0$, independent of ν, such that

$$A(\phi,\phi) \geq \alpha_0 \|\phi\|_W^2 \qquad \forall\,\phi \in W.$$

As a consequence of Lemma 2.1 and the Lax–Milgram Theorem we state the solvability of the continuous problem (2.3).

THEOREM 2.2 There exists a unique $\psi \in W$ solution to problem (2.3), which satisfies the following continuous dependence on the data

$$\|\psi\|_W \leq C\|\mathbf{f}\|_{0,\Omega},$$

where the constant $C > 0$ is independent of ν.

We state the following additional regularity result for the solution of problem (2.3).

THEOREM 2.3 Let $m \geq -1$. Suppose that $\mathbf{f} \in [H^m(\Omega)]^2$, then there exist $s > 1/2$ and a constant $C > 0$, such that the solution ψ of problem (2.3) satisfies $\psi \in H^{2+s}(\Omega)$ and

$$\|\psi\|_{2+s,\Omega} \leq C\|\mathbf{f}\|_{m,\Omega}.$$

Proof. The proof follows from the classical regularity result for the biharmonic problem (see Grisvard, 1985; Bacuta *et al.*, 2002; Brenner & Sung, 2005). □

The constant s in the theorem above is called the index of elliptic regularity of the biharmonic problem with homogeneous Dirichlet boundary conditions. For instance, if Ω is convex, then $s \geq 1$, in this case s will depend on the regularity of $\mathbf{f}$. On the other hand, if Ω is nonconvex, for any $\mathbf{f}$, the theorem holds, but now for all $s < s_0$, where $s_0 \in (1/2, 1)$ depends on the largest reentrant angle of Ω (see Grisvard (1985) for the precise equation determining s_0).

3. Virtual element discretization

In this section we will write a VE discretization for the numerical approximation of problem (2.3) of arbitrary order $k \geq 2$. First, we introduce some basic tools, notations and assumptions to construct a conforming virtual space of W, and to write the corresponding discrete bilinear forms and the discrete linear functional to write the discrete problem. Finally, we prove existence and uniqueness of the discrete formulation.

3.1 *Mesh assumptions*

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into general polygonal elements K. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K\in\mathcal{T}_h} h_K$. In what follows we denote by N_V^K the number of vertices of K, by V_i a generic vertex of K, with $1 \le i \le N_V^K$, by e a generic edge of $\mathcal{T}_h$. For all $e \in \partial K$ we denote by $h_e = |e|$ the length of edge and we define a unit normal vector $\boldsymbol{n}_K^e$ that points outside of K and a unit tangent vector $\boldsymbol{t}_K^e$ to K. Also, we denote by x_e and $\boldsymbol{x}_K$ the midpoint of e and the baricenter of K, respectively.

For the theoretical analysis we will make the following assumptions: there exists a real number $C_{\mathcal{T}} > 0$ such that, for every h and every $K \in \mathcal{T}_h$, we have

A1: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$;

A2: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}} h_K$.

3.2 *Local and global virtual spaces*

For any subset $\mathcal{D} \subseteq \mathbb{R}^2$ and non-negative integer ℓ, we denote by $\mathcal{P}_\ell(\mathcal{D})$ the space of polynomials of degree up to ℓ defined on $\mathcal{D}$, and denote by $\mathcal{M}_\ell^*(K)$ the set of the two-dimensional scaled monomials defined on each polygon K as follows:

$$\mathcal{M}_\ell^*(K) := \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_K}{h_K} \right)^{\boldsymbol{\beta}} : |\boldsymbol{\beta}| = \ell \right\}, \tag{3.1}$$

where for a non-negative multi-index $\boldsymbol{\beta} = (\beta_1, \beta_2)$, we set $|\boldsymbol{\beta}| := \beta_1 + \beta_2$ and $\boldsymbol{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2}$, with $\boldsymbol{x} = (x_1, x_2)$. We define $\mathcal{M}_k(K) := \bigcup_{\ell \le k} \mathcal{M}_\ell^*(K) =: \{m_j\}_{j=1}^{n_k}$ as a basis of $\mathcal{P}_k(K)$, where $n_k = \dim(\mathcal{P}_k(K))$. Analogously, we consider the set of the scaled monomials defined on each edge e:

$$\mathcal{M}_\ell(e) := \left\{ 1, \frac{x - x_e}{h_e}, \left(\frac{x - x_e}{h_e} \right)^2, \ldots, \left(\frac{x - x_e}{h_e} \right)^\ell \right\}.$$

Now, for any $k \ge 2$ and for every polygon $K \in \mathcal{T}_h$, we introduce the following preliminary local virtual space:

$$\begin{aligned} \widetilde{V}_h^k(K) := \Big\{ \phi_h \in H^2(K) : \Delta^2 \phi_h \in \mathcal{P}_{k-2}(K), \phi_h|_{\partial K} \in C^0(\partial K), \phi_h|_e \in \mathcal{P}_r(e)\ \forall\, e \in \partial K, \\ \nabla \phi_h|_{\partial K} \in [C^0(\partial K)]^2, \partial_{\boldsymbol{n}_K^e} \phi_h|_e \in \mathcal{P}_\alpha(e)\ \forall\, e \in \partial K \Big\}, \end{aligned}$$

where $r := \max\{3, k\}$ and $\alpha := k - 1$.

Next, for a given $\phi_h \in \widetilde{V}_h^k(K)$, we introduce five sets $\mathbf{D}_1 - \mathbf{D}_5$ of linear operators from the local virtual space $\widetilde{V}_h^k(K)$ into $\mathbb{R}$.

- $\mathbf{D}_1$: the values of $\phi_h(V_i)$ for each vertex V_i of K;
- $\mathbf{D}_2$: the values of $h_{V_i} \nabla \phi_h(V_i)$ for each vertex V_i of K;

- $\mathbf{D}_3$: for $\alpha > 1$ the moments $\int_e q(\zeta)\partial_{\boldsymbol{n}_K^e}\phi_h(\zeta)\,\mathrm{d}\zeta \qquad \forall\, q \in \mathcal{M}_{\alpha-2}(e), \quad \forall$ edge e;
- $\mathbf{D}_4$: for $r > 3$ the moments $\frac{1}{|e|}\int_e q(\zeta)\phi_h(\zeta)\,\mathrm{d}\zeta \qquad \forall\, q \in \mathcal{M}_{r-4}(e), \quad \forall$ edge e;
- $\mathbf{D}_5$: for $k \geq 4$ the moments $\frac{1}{|K|}\int_K q(\boldsymbol{x})\phi_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \qquad \forall\, q \in \mathcal{M}_{k-4}(K), \quad \forall$ polygon K,

where h_{V_i} corresponds to the average of the diameters corresponding to the elements with V_i as a vertex.

REMARK 3.1 In the above construction we have used the scaled monomials $\mathcal{M}_k$ as a basis for the space $\mathcal{P}_k$, because they ensure that the linear operators introduced in $\mathbf{D}_3 - \mathbf{D}_5$ scale as 1 with respect to the diameter h_K (see Brezzi & Marini, 2013, and Beirão da Veiga *et al.*, 2014, Remarks 1.1 and 2.5). Additionally, we have that the linear operators introduced in $\mathbf{D}_1$ and $\mathbf{D}_2$ also scale as 1 with respect to the diameter h_K. In turn, this fact allows to build easily bilinear forms satisfying the stability property (cf. (3.7)).

In order to construct the discrete scheme we decompose the bilinear forms (2.4)–(2.6) in the following element by element contribution:

$$A_\Delta(\varphi,\phi) = \sum_{K\in\mathcal{T}_h} A_\Delta^K(\varphi,\phi) := \sum_{K\in\mathcal{T}_h}\int_K \mathrm{D}^2\varphi : \mathrm{D}^2\phi \qquad \forall\,\varphi,\phi \in W,$$

$$A_{\mathbf{curl}}(\varphi,\phi) = \sum_{K\in\mathcal{T}_h} A_{\mathbf{curl}}^K(\varphi,\phi) := \sum_{K\in\mathcal{T}_h}\int_K \mathbb{K}^{-1}\mathbf{curl}\,\varphi\cdot\mathbf{curl}\,\phi \qquad \forall\,\varphi,\phi \in W.$$

Also, we split

$$A(\varphi,\phi) = \sum_{K\in\mathcal{T}_h} A^K(\varphi,\phi) := \sum_{K\in\mathcal{T}_h}\Big(A_{\mathbf{curl}}^K(\varphi,\phi) + \nu A_\Delta^K(\varphi,\phi)\Big) \qquad \forall\,\varphi,\phi \in W.$$

In what follows we are going to build discrete version of the local bilinear forms listed above. With this aim, and for $k \geq 2$, we define the following projector operator $\Pi_K^{k,\Delta} : \widetilde{V}_h^k(K) \longrightarrow \mathcal{P}_k(K) \subseteq \widetilde{V}_h^k(K)$ for each $\phi_h \in \widetilde{V}_h^k(K)$, as the solution of the local problems (on each polygon K):

$$A_\Delta^K\big(\Pi_K^{k,\Delta}\phi_h, q\big) = A_\Delta^K(\phi_h, q) \quad \forall\, q \in \mathcal{P}_k(K),$$
$$\widehat{\Pi_K^{k,\Delta}\phi_h} = \widehat{\phi_h}, \quad \widehat{\nabla\Pi_K^{k,\Delta}\phi_h} = \widehat{\nabla\phi_h},$$

where $\widehat{\phi_h}$ is defined as follows:

$$\widehat{\phi_h} := \frac{1}{N_V^K}\sum_{i=1}^{N_V^K}\phi_h(V_i) \qquad \forall\,\phi_h \in C^0(\partial K), \tag{3.2}$$

and V_i, $1 \leq i \leq N_V^K$, are the vertices of K.

The following result establishes that the projector $\Pi_K^{k,\Delta}$ is computable using the output values of the sets $\mathbf{D}_1 - \mathbf{D}_5$ (see Chinosi & Marini, 2016).

LEMMA 3.2 The operator $\Pi_K^{k,\Delta} : \widetilde{V}_h^k(K) \to \mathcal{P}_k(K)$ is explicitly computable for every $\phi_h \in \widetilde{V}_h^k(K)$, using only the information of the linear operators $\mathbf{D}_1 - \mathbf{D}_5$.

For each $k \geq 2$ and for any $K \in \mathcal{T}_h$ our local virtual space is given by:

$$W_h^k(K) := \left\{ \phi_h \in \widetilde{V}_h^k(K) : \int_K q^* \, \Pi_K^{k,\Delta}\phi_h = \int_K q^* \, \phi_h, \quad \forall\, q^* \in \mathcal{M}_{k-3}^*(K) \cup \mathcal{M}_{k-2}^*(K) \right\},$$

where $\mathcal{M}_{k-3}^*(K)$ and $\mathcal{M}_{k-2}^*(K)$ are scaled monomials of degree $k-3$ and $k-2$, respectively (see (3.1)), with the convention that $\mathcal{M}_{-1}^*(K) = \emptyset$ (see for instance Chinosi & Marini, 2016, for further details).

We have that $W_h^k(K) \subseteq \widetilde{V}_h^k(K)$, as a consequence the projector $\Pi_K^{k,\Delta}$ is well defined on $W_h^k(K)$ and computable using the information the linear operators $\mathbf{D}_1 - \mathbf{D}_5$. In addition, we have that $\mathcal{P}_k(K) \subseteq W_h^k(K)$; this will guarantee the good approximation properties for the space. Moreover, it has been established in Chinosi & Marini (2016) that the sets of linear operators $\mathbf{D}_1 - \mathbf{D}_5$ constitutes a set of degrees of freedom for $W_h^k(K)$ (see also Ahmad *et al.*, 2013).

Additionally, we observe that the condition appearing in the definition of the local space $W_h^k(K)$ will be useful to construct an L^2-projection, which will be employed to build the discrete bilinear forms. In particular, we consider the $L^2(K)$-projection onto $\mathcal{P}_{k-2}(K)$. For each $\phi \in L^2(K)$, $\Pi_{k-2}^K\phi \in \mathcal{P}_{k-2}(K)$ satisfies

$$\int_K (\Pi_{k-2}^K\phi)q = \int_K \phi q \qquad \forall\, q \in \mathcal{P}_{k-2}(K). \tag{3.3}$$

The following lemma establishes that Π_{k-2}^K is computable on $W_h^k(K)$. The proof follows from the definition of the local virtual space and the set of degrees of freedom.

LEMMA 3.3 The operator $\Pi_{k-2}^K : W_h^k(K) \to \mathcal{P}_{k-2}(K)$ is explicitly computable for each $\phi_h \in W_h^k(K)$, using only the information of the set of degrees freedom $\mathbf{D}_1 - \mathbf{D}_5$.

In what follows, for each polygon $K \in \mathcal{T}_h$, we denote by $N_K^{\mathrm{d}of}$ the number of degrees freedom of $W_h^k(K)$ and by $\mathrm{d}of_i$, with $1 \leq i \leq \dim(W_h^k(K))$, the operator that to each smooth enough function ϕ associates the ith local degree of freedom $\mathrm{d}of_i(\phi)$.

Now, we will consider the following projection onto the polynomial space $[\mathcal{P}_{k-1}(K)]^2$, which will be used to construct a local approximation of $A_{\mathbf{curl}}^K(\cdot,\cdot)$: we define $\boldsymbol{\Pi}_{k-1}^K : [L^2(K)]^2 \to [\mathcal{P}_{k-1}(K)]^2$, for each $\mathbf{v} \in [L^2(K)]^2$ by

$$\int_K \boldsymbol{\Pi}_{k-1}^K\mathbf{v} \cdot \mathbf{q} = \int_K \mathbf{v} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in [\mathcal{P}_{k-1}(K)]^2. \tag{3.4}$$

We observe that for any $\phi_h \in W_h^k(K)$, the vector function $\boldsymbol{\Pi}_{k-1}^K \mathbf{curl}\, \phi_h \in [\mathcal{P}_{k-1}(K)]^2$ can be explicitly computed from the degrees of freedom $\mathbf{D}_1 - \mathbf{D}_5$. In fact, for all $K \in \mathcal{T}_h$ and for all $\phi_h \in W_h^k(K)$, using integration by parts in the right-hand side of (3.4) (with $\mathbf{curl}\,\phi_h$ instead of $\mathbf{v}$), we have

$$\int_K \mathbf{curl}\,\phi_h \cdot \mathbf{q} = \int_K \phi_h \operatorname{rot} \mathbf{q} - \int_{\partial K} \phi_h (\mathbf{q} \cdot \boldsymbol{t}_K^e) \qquad \forall\, \mathbf{q} \in [\mathcal{P}_{k-1}(K)]^2$$

$$= \int_K \left(\Pi_{k-2}^K \phi_h\right) \operatorname{rot} \mathbf{q} - \int_{\partial K} \phi_h (\mathbf{q} \cdot \boldsymbol{t}_K^e) \qquad \forall\, \mathbf{q} \in [\mathcal{P}_{k-1}(K)]^2,$$

where we have used (3.3). The first term on the right-hand side above depends only on $\Pi_{k-2}^K \phi_h$, and this depends only on the values of the degrees of freedom (see Lemma 3.3). The second term is an integral on the boundary of the element K, which is fully computable.

Now, for $k \geq 2$, we introduce an additional projector, which will be used to write the virtual scheme; we define $\Pi_K^{k,\nabla^\perp} : W_h^k(K) \longrightarrow \mathcal{P}_k(K) \subseteq W_h^k(K)$ for each $\phi_h \in W_h^k(K)$ as the solution of the following local problem.

$$\int_K \mathbf{curl}\, \Pi_K^{k,\nabla^\perp} \phi_h \cdot \mathbf{curl}\, q = \int_K \mathbf{curl}\,\phi_h \cdot \mathbf{curl}\, q \qquad \forall\, q \in \mathcal{P}_k(K), \tag{3.5a}$$

$$\widehat{\Pi_K^{k,\nabla^\perp} \phi_h} = \widehat{\phi_h}, \tag{3.5b}$$

where $\widehat{\phi_h}$ has been defined in (3.2). The following result states that this operator is fully computable.

LEMMA 3.4 The operator $\Pi_K^{k,\nabla^\perp} : W_h^k(K) \to \mathcal{P}_k(K) \subseteq W_h^k(K)$ is explicitly computable for each $\phi_h \in W_h^k(K)$, using only the information of the set of degrees freedom $\mathbf{D}_1 - \mathbf{D}_5$.

Proof. First we note that (3.5b) is computable using the information of the set $\mathbf{D}_1$. On the other hand, we integrate by parts on the right-hand side of (3.5a) to obtain:

$$\int_K \mathbf{curl}\,\phi_h \cdot \mathbf{curl}\, q = -\int_K \phi_h \Delta q - \int_{\partial K} \phi_h \partial_{\boldsymbol{n}_K^e} q \qquad \forall\, q \in \mathcal{P}_k(K)$$

$$= -\int_K \Pi_{k-2}^K \phi_h \Delta q - \int_{\partial K} \phi_h \partial_{\boldsymbol{n}_K^e} q \qquad \forall\, q \in \mathcal{P}_k(K),$$

where once again we have used the fact that $\Delta q \in \mathcal{P}_{k-2}(K)$ and the definition of the projection Π_{k-2}^K (cf. (3.3)). The previous equality allows us to conclude that the polynomial $\Pi_K^{k,\nabla^\perp} \phi_h$ can be explicitly computed from the degrees of freedom $\mathbf{D}_1 - \mathbf{D}_5$. □

Now, we introduce the global virtual space by combining the local spaces $W_h^k(K)$ and incorporating the homogeneous boundary conditions. For every decomposition $\mathcal{T}_h$ of Ω into polygons K, we define

$$W_h := \left\{ \phi_h \in W : \ \phi_h|_K \in W_h^k(K) \right\}.$$

We have that the dimension of W_h, $R_1 := \dim(W_h)$, is given by:

$$R_1 = 3N_V + N_E(\alpha - 1 + r - 3) + N_K \dim(\mathcal{P}_{k-4}(K)) \\ = 3N_V + N_E(\alpha + r - 4) + N_K \frac{(k-3)(k-2)}{2}, \tag{3.6}$$

where N_E is the number of internal edges, N_V is the number of internal vertices and N_K the number of elements of $\mathcal{T}_h$ (see Brezzi & Marini, 2013; Chinosi & Marini, 2016).

3.3 *Construction of discrete forms*

In this section we will construct the discrete version of the continuous local bilinear forms and the right-hand side, using the projection operators introduced in Section 3.2.

First, let $s_\Delta^K(\cdot,\cdot)$ and $s_{\mathbf{curl}}^K(\cdot,\cdot)$ be any symmetric positive definite bilinear forms to be chosen as to satisfy:

$$\begin{aligned} c_0 A_\Delta^K(\phi_h,\phi_h) \le s_\Delta^K(\phi_h,\phi_h) \le c_1 A_\Delta^K(\phi_h,\phi_h) \qquad & \forall\ \phi_h \in W_h^k(K),\ \text{with}\ \Pi_K^{k,\Delta}\phi_h = 0, \\ c_2 A_{\mathbf{curl}}^K(\phi_h,\phi_h) \le s_{\mathbf{curl}}^K(\phi_h,\phi_h) \le c_3 A_{\mathbf{curl}}^K(\phi_h,\phi_h) \qquad & \forall\ \phi_h \in W_h^k(K),\ \text{with}\ \Pi_K^{k,\nabla^\perp}\phi_h = 0, \end{aligned} \tag{3.7}$$

with c_0, c_1, c_2 and c_3 positive constants independent of h and K. A classical choice for the bilinear forms $s_\Delta^K(\cdot,\cdot)$ and $s_{\mathbf{curl}}^K(\cdot,\cdot)$ satisfying (3.7) is given by the Euclidean scalar product associated to the degrees of freedom (see Beirão da Veiga *et al.*, 2013; Brezzi & Marini, 2013; Cangiani *et al.*, 2017b). We will choose the following representation:

$$s_\Delta^K(\psi_h,\phi_h) := h_K^{-2} \sum_{i=1}^{N_K^{\mathrm{d}of}} \mathrm{d}of_i(\psi_h)\mathrm{d}of_i(\phi_h) \quad \text{and} \quad s_{\mathbf{curl}}^K(\psi_h,\phi_h) := \sigma_{\mathbb{K}}^K \sum_{i=1}^{N_K^{\mathrm{d}of}} \mathrm{d}of_i(\psi_h)\mathrm{d}of_i(\phi_h), \tag{3.8}$$

where the parameter $\sigma_{\mathbb{K}}^K > 0$ is a multiplicative factor to take into account the inverse of the permeability tensor. In particular, we define $\sigma_{\mathbb{K}}^K$ as the arithmetic mean of the mean values of the diagonal elements of tensor $\mathbb{K}^{-1}$ (see Beirão da Veiga *et al.*, 2016b).

The following result establishes that $s_\Delta^K(\cdot,\cdot)$ and $s_{\mathbf{curl}}^K(\cdot,\cdot)$ satisfy the stability property (3.7).

Proposition 3.5 The bilinear forms defined in (3.8) satisfy the stability property (3.7).

Proof. The proof follows using the same arguments presented in Cangiani *et al.* (2017b, Proposition 5.3) (see also Brezzi & Marini, 2013, Section 4.4). First, we have that the stabilizing bilinear forms $s_\Delta^K(\cdot,\cdot)$ and $s_{\mathbf{curl}}^K(\cdot,\cdot)$ are the Euclidean scalar product (multiplied by positive constants) associated to the degrees of freedom; therefore, these forms are positive definite. In addition, by considering that the degrees of freedom (cf. $\mathbf{D}_1 - \mathbf{D}_5$) scale like 1 (see Remark 3.1), and that $A_\Delta^K(\cdot,\cdot)$ scales like h_K^{-2}, it is easy to see that the bilinear form $s_\Delta^K(\cdot,\cdot)$ scales like h_K^{-2}, too. Using the same arguments we have that $s_{\mathbf{curl}}^K(\cdot,\cdot)$ scales as $A_{\mathbf{curl}}^K(\cdot,\cdot)$. □

Then, we set the following global bilinear form,

$$A^h(\psi_h, \phi_h) := \sum_{K\in\mathcal{T}_h} A^{h,K}(\psi_h, \phi_h) \qquad \forall\ \psi_h, \phi_h \in W_h, \tag{3.9}$$

where

$$A^{h,K}(\psi_h, \phi_h) := A^{h,K}_{\mathbf{curl}}(\psi_h, \phi_h) + \nu A^{h,K}_{\Delta}(\psi_h, \phi_h) \qquad \forall\ \psi_h, \phi_h \in W_h^k(K), \tag{3.10}$$

with $A^{h,K}_{\Delta}(\cdot,\cdot)$ and $A^{h,K}_{\mathbf{curl}}(\cdot,\cdot)$ are the local bilinear forms on $W_h^k(K) \times W_h^k(K)$ defined by

$$A^{h,K}_{\Delta}(\psi_h, \phi_h) := A^K_{\Delta}\big(\Pi_K^{k,\Delta}\psi_h, \Pi_K^{k,\Delta}\phi_h\big) + s^K_{\Delta}\big(\psi_h - \Pi_K^{k,\Delta}\psi_h, \phi_h - \Pi_K^{k,\Delta}\phi_h\big), \tag{3.11}$$

$$A^{h,K}_{\mathbf{curl}}(\psi_h, \phi_h) := \int_K \mathbb{K}^{-1}\boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\psi_h \cdot \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\phi_h + s^K_{\mathbf{curl}}\big(\psi_h - \Pi_K^{k,\nabla^\perp}\psi_h, \phi_h - \Pi_K^{k,\nabla^\perp}\phi_h\big). \tag{3.12}$$

The following result establishes the usual consistency and stability properties for the discrete local forms.

PROPOSITION 3.6 For $k \geq 2$ the local bilinear forms $A^{h,K}_{\Delta}(\cdot,\cdot)$, $A^{h,K}_{\mathbf{curl}}(\cdot,\cdot)$ and $A^{h,K}(\cdot,\cdot)$ (defined in (3.11), (3.12) and (3.10), respectively) on each element K, satisfy

- *Consistency*: for all $h > 0$ and for all $K \in \mathcal{T}_h$ we have that

$$A^{h,K}_{\Delta}(q, \phi_h) = A^K_{\Delta}(q, \phi_h) \qquad \forall\ q \in \mathcal{P}_k(K), \qquad \forall\ \phi_h \in W_h^k(K). \tag{3.13}$$

- *Stability and boundedness*: there exist positive constants $\alpha_i, i = 1, \ldots, 6$, independent of K, such that:

$$\alpha_1 A^K_{\Delta}(\phi_h, \phi_h) \leq A^{h,K}_{\Delta}(\phi_h, \phi_h) \leq \alpha_2 A^K_{\Delta}(\phi_h, \phi_h) \qquad \forall\ \phi_h \in W_h^k(K), \tag{3.14}$$

$$\alpha_3 A^K_{\mathbf{curl}}(\phi_h, \phi_h) \leq A^{h,K}_{\mathbf{curl}}(\phi_h, \phi_h) \leq \alpha_4 A^K_{\mathbf{curl}}(\phi_h, \phi_h) \qquad \forall\ \phi_h \in W_h^k(K), \tag{3.15}$$

$$\alpha_5 A^K(\phi_h, \phi_h) \leq A^{h,K}(\phi_h, \phi_h) \leq \alpha_6 A^K(\phi_h, \phi_h) \qquad \forall\ \phi_h \in W_h^k(K). \tag{3.16}$$

Proof. The proof follows standard arguments in the VEM literature (see Beirão da Veiga *et al.*, 2013, 2016a; Antonietti *et al.*, 2016; Cangiani *et al.*, 2017b). □

Since $\mathbb{K}^{-1}$ is a full tensor the bilinear form $A^{h,K}_{\mathbf{curl}}(\cdot,\cdot)$ does not satisfy the consistency property. In order to overcome this drawback we have the following auxiliary results, which will be useful to prove the theoretical results.

LEMMA 3.7 Let $K \in \mathcal{T}_h$ and let $\mathbb{T}$ be a smooth and symmetric tensor field defined on K. For any $\mathbf{p}, \mathbf{q}$ smooth enough vector fields defined on K, we have

$$
\begin{aligned}
(\mathbb{T}\,\mathbf{p}, \mathbf{q})_{0,K} - \left(\mathbb{T}\,\boldsymbol{\Pi}_{k-1}^K \mathbf{p}, \boldsymbol{\Pi}_{k-1}^K \mathbf{q}\right)_{0,K} \leq\ & \|\mathbb{T}\,\mathbf{p} - \boldsymbol{\Pi}_{k-1}^K(\mathbb{T}\,\mathbf{p})\|_{0,K}\|\mathbf{q} - \boldsymbol{\Pi}_{k-1}^K\mathbf{q}\|_{0,K} \\
&+ \|\mathbb{T}\,\mathbf{q} - \boldsymbol{\Pi}_{k-1}^K(\mathbb{T}\,\mathbf{q})\|_{0,K}\|\mathbf{p} - \boldsymbol{\Pi}_{k-1}^K\mathbf{p}\|_{0,K} \\
&+ C_{\mathbb{T}}\,\|\mathbf{p} - \boldsymbol{\Pi}_{k-1}^K\mathbf{p}\|_{0,K}\|\mathbf{q} - \boldsymbol{\Pi}_{k-1}^K\mathbf{q}\|_{0,K},
\end{aligned}
$$

where $C_{\mathbb{T}} > 0$ is a constant depending on $\mathbb{T}$.

Proof. For simplicity we consider the following notation: $\overline{\mathbf{p}} := \boldsymbol{\Pi}_{k-1}^K \mathbf{p}$ and $\overline{\mathbf{q}} := \boldsymbol{\Pi}_{k-1}^K \mathbf{q}$. Then using the symmetry of $\mathbb{T}$, adding and subtracting suitable terms and using the properties of the projection $\boldsymbol{\Pi}_{k-1}^K$, we have that

$$
\begin{aligned}
&(\mathbb{T}\,\mathbf{p}, \mathbf{q})_{0,K} - (\mathbb{T}\,\overline{\mathbf{p}}, \overline{\mathbf{q}})_{0,K} \\
&\quad = (\mathbb{T}\,\mathbf{p}, \mathbf{q})_{0,K} - (\mathbb{T}\,\mathbf{p}, \overline{\mathbf{q}})_{0,K} + (\mathbb{T}\,\mathbf{p}, \overline{\mathbf{q}})_{0,K} - (\overline{\mathbf{p}}, \mathbb{T}\,\overline{\mathbf{q}})_{0,K} \\
&\quad = (\mathbb{T}\,\mathbf{p}, \mathbf{q} - \overline{\mathbf{q}})_{0,K} + (\mathbf{p} - \overline{\mathbf{p}}, \mathbb{T}\,\overline{\mathbf{q}})_{0,K} \\
&\quad = (\mathbb{T}\,\mathbf{p}, \mathbf{q} - \overline{\mathbf{q}})_{0,K} - \left(\overline{\mathbb{T}\,\mathbf{p}}, \mathbf{q} - \overline{\mathbf{q}}\right)_{0,K} + (\mathbf{p} - \overline{\mathbf{p}}, \mathbb{T}\,\overline{\mathbf{q}})_{0,K} - \left(\mathbf{p} - \overline{\mathbf{p}}, \overline{\mathbb{T}\,\mathbf{q}}\right)_{0,K} \\
&\quad = \left(\mathbb{T}\,\mathbf{p} - \overline{\mathbb{T}\,\mathbf{p}}, \mathbf{q} - \overline{\mathbf{q}}\right)_{0,K} + \left(\mathbf{p} - \overline{\mathbf{p}}, \mathbb{T}\,\overline{\mathbf{q}} - \overline{\mathbb{T}\,\mathbf{q}}\right)_{0,K} \\
&\quad = \left(\mathbb{T}\,\mathbf{p} - \overline{\mathbb{T}\,\mathbf{p}}, \mathbf{q} - \overline{\mathbf{q}}\right)_{0,K} + \left(\mathbf{p} - \overline{\mathbf{p}}, \mathbb{T}\,\overline{\mathbf{q}} - \overline{\mathbb{T}\,\mathbf{q}} - \mathbb{T}\,\mathbf{q} + \mathbb{T}\,\mathbf{q}\right)_{0,K} \\
&\quad = \left(\mathbb{T}\,\mathbf{p} - \overline{\mathbb{T}\,\mathbf{p}}, \mathbf{q} - \overline{\mathbf{q}}\right)_{0,K} + \left(\mathbf{p} - \overline{\mathbf{p}}, \mathbb{T}\,\mathbf{q} - \overline{\mathbb{T}\,\mathbf{q}}\right)_{0,K} - (\mathbf{p} - \overline{\mathbf{p}}, \mathbb{T}\,(\mathbf{q} - \overline{\mathbf{q}}))_{0,K}.
\end{aligned}
$$

Then, the result follows from the Cauchy–Schwarz inequality with $C_{\mathbb{T}} = \|\mathbb{T}\|_{L^\infty(K)^{2\times 2}}$. □

As an immediate consequence of Lemma 3.7 we have the following result.

LEMMA 3.8 For all $K \in \mathcal{T}_h$ and for all $\varphi_h, \phi_h \in W_h^k(K)$, we have

$$
\begin{aligned}
&A_{\mathbf{curl}}^K(\varphi_h, \phi_h) - A_{\mathbf{curl}}^{h,K}(\varphi_h, \phi_h) \\
&\qquad \leq \|\mathbb{K}^{-1}\,\mathbf{curl}\,\varphi_h - \boldsymbol{\Pi}_{k-1}^K(\mathbb{K}^{-1}\mathbf{curl}\,\varphi_h)\|_{0,K}\|\mathbf{curl}\,\phi_h - \boldsymbol{\Pi}_{k-1}^K\mathbf{curl}\,\phi_h\|_{0,K} \\
&\qquad\quad + \|\mathbb{K}^{-1}\,\mathbf{curl}\,\phi_h - \boldsymbol{\Pi}_{k-1}^K(\mathbb{K}^{-1}\mathbf{curl}\,\phi_h)\|_{0,K}\|\mathbf{curl}\,\varphi_h - \boldsymbol{\Pi}_{k-1}^K\mathbf{curl}\,\varphi_h\|_{0,K} \\
&\qquad\quad + C_{\mathbb{K}}\|\mathbf{curl}\,\phi_h - \boldsymbol{\Pi}_{k-1}^K\mathbf{curl}\,\phi_h\|_{0,K}\|\mathbf{curl}\,\varphi_h - \boldsymbol{\Pi}_{k-1}^K\mathbf{curl}\,\varphi_h\|_{0,K} \\
&\qquad\quad + s_{\mathbf{curl}}^K(\varphi_h - \Pi_K^{k,\nabla^\perp}\varphi_h, \phi_h - \Pi_K^{k,\nabla^\perp}\phi_h),
\end{aligned}
$$

where $C_{\mathbb{K}} > 0$ is a constant depending on the tensor $\mathbb{K}^{-1}$.

The next step consists in constructing a computable approximation of the right-hand side (2.7). With this aim, we define, for each element K, the following computable (using the sets of degrees of freedom

$\mathbf{D}_1 - \mathbf{D}_5$) term:

$$F^{h,K}(\phi_h) := \int_K \boldsymbol{\Pi}_{k-1}^K \mathbf{f} \cdot \mathbf{curl}\, \phi_h \equiv \int_K \mathbf{f} \cdot \boldsymbol{\Pi}_{k-1}^K \mathbf{curl}\, \phi_h \qquad \forall\, \phi_h \in W_h^k(K).$$

Thus, we introduce the following element as an approximation of (2.7):

$$F^h(\phi_h) := \sum_{K \in \mathcal{T}_h} F^{h,K}(\phi_h) \qquad \forall\, \phi_h \in W_h. \tag{3.17}$$

REMARK 3.9 If $\mathbf{f}$ is a smooth function then using integration by parts in (2.7) gives $(\mathbf{f}, \mathbf{curl}\, \phi)_{0,\Omega} = (\mathrm{rot}\, \mathbf{f}, \phi)_{0,\Omega}$ $\forall\, \phi \in W$. As a consequence, it is possible to consider an alternative right-hand side as follows: for each $k \geq 2$ and for each $K \in \mathcal{T}_h$, we define

$$\widetilde{F}^{h,K}(\phi_h) := \int_K \Pi_{k-2}^K(\mathrm{rot}\, \mathbf{f})\, \phi_h \equiv \int_K \mathrm{rot}\, \mathbf{f}\, \Pi_{k-2}^K \phi_h \qquad \forall\, \phi_h \in W_h^k(K).$$

Then, it is possible to consider the following alternative global right-hand side: $\widetilde{F}^h : W_h \to \mathbb{R}$ defined by

$$\widetilde{F}^h(\phi_h) := \sum_{K \in \mathcal{T}_h} \widetilde{F}^{h,K}(\phi_h) \qquad \forall\, \phi_h \in W_h. \tag{3.18}$$

We note that $\widetilde{F}^h(\cdot)$ is fully computable using the degrees of freedom $\mathbf{D}_1 - \mathbf{D}_5$, since Π_{k-2}^K is computable.

3.4 *The discrete virtual schemes*

Now, we use the discrete forms and the results of the previous sections to write two discrete VEM for the approximation of the unique solution of the Brinkman problem presented in (2.3).

The virtual element discretization reads as follows: find $\psi_h \in W_h$, such that

$$A^h(\psi_h, \phi_h) = F^h(\phi_h) \qquad \forall\, \phi_h \in W_h, \tag{3.19}$$

where $A^h(\cdot,\cdot)$ is the discrete bilinear form defined in (3.9) and $F^h(\cdot)$ introduced in (3.17). We note that as a consequence of (3.16) the bilinear form $A^h(\cdot,\cdot)$ is bounded. Moreover, it is also uniformly elliptic, as shown the following result.

LEMMA 3.10 There exists a constant $\widetilde{\alpha} > 0$, independent of ν and h, such that

$$A^h(\phi_h, \phi_h) \geq \widetilde{\alpha}\, \|\phi_h\|_W^2 \qquad \forall\, \phi_h \in W_h.$$

Proof. Thanks to (3.14) and (3.15) it is easy to check that the above inequality holds with $\widetilde{\alpha} := \max\{\alpha_3 \lambda_1, \alpha_1\} > 0$. □

As an immediate consequence of Lemma 3.10 we have the following theorem:

THEOREM 3.11 Discrete formulation (3.19) admits a unique solution $\psi_h \in W_h$, which satisfies the following continuous dependence on the data

$$\|\psi_h\|_W \le C\|\mathbf{f}\|_{0,\Omega},$$

where the positive constant C is independent of ν.

REMARK 3.12 By considering the alternative right-hand side proposed in Remark 3.9, it is possible to write the following well-posed discrete formulation: find $\widetilde{\psi}_h \in W_h$, such that

$$A^h(\widetilde{\psi}_h, \phi_h) = \widetilde{F}^h(\phi_h) \qquad \forall\ \phi_h \in W_h. \tag{3.20}$$

We are going to analyze in detail the virtual element discretization (3.19) and summarize the results for the VE discretization (3.20) (see Remark 4.25).

4. Convergence analysis

In the present section we develop an error analysis for the discrete virtual element schemes presented in Section 3.4. For the forthcoming analysis we will assume that the mesh assumptions **A**1 and **A**2, introduced in Section 3.1, are satisfied.

We will use the following broken H^ℓ-seminorm, for each integer $\ell > 0$:

$$|\phi|^2_{\ell,h} := \sum_{K\in\mathcal{T}_h} |\phi|^2_{\ell,K},$$

which is well defined for every $\phi \in L^2(\Omega)$ such that $\phi|_K \in H^\ell(K)$ for all polygon $K \in \mathcal{T}_h$.

Moreover, we recall the following approximation result, which is derived by interpolation between Sobolev spaces (see for instance Girault & Raviart, 1986, Theorem I.1.4) from the analogous result for integer values. In its turn the result for integer values is stated in Beirão da Veiga *et al.* (2013, Proposition 4.2) and follows from the classical Scott–Dupont theory (see Brenner & Scott, 2008, and Antonietti *et al.*, 2016, Proposition 3.1):

PROPOSITION 4.1 If the assumption **A**2 is satisfied then there exists a constant $C > 0$, such that for every $\phi \in H^\delta(K)$, there exists $\phi_\pi \in \mathcal{P}_k(K)$, $k \ge 0$, such that

$$|\phi - \phi_\pi|_{\ell,K} \le Ch_K^{\delta-\ell}|\phi|_{\delta,K}, \quad 0 \le \delta \le k+1,\ \ell = 0, 1, \ldots, [\delta],$$

where $[\delta]$ denoting largest integer equal or smaller than $\delta \in \mathbb{R}$.

We are going to use the following standard approximation property (see Brenner & Scott, 2008; Cáceres & Gatica, 2017):

LEMMA 4.2 There exists a constant $C > 0$, independent of h_K, such that for all $\mathbf{v} \in [H^\delta(K)]^2$

$$\|\mathbf{v} - \boldsymbol{\Pi}^K_{k-1}\mathbf{v}\|_{0,K} \le Ch_K^\delta|\mathbf{v}|_{\delta,K} \quad 0 \le \delta \le k, \quad k \ge 2.$$

Now, we start with the following bound for a dual norm.

PROPOSITION 4.3 Let $k \geq 2$ and $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^{k-1}(K)]^2$ for each $K \in \mathcal{T}_h$. Let $F(\cdot)$ and $F^h(\cdot)$ the functionals defined in (2.7) and (3.17), respectively. Then, we have the following estimates:

$$\|F - F^h\|_{W_h'} := \sup_{\substack{\phi_h \in W_h \\ \phi_h \neq 0}} \frac{|F(\phi_h) - F^h(\phi_h)|}{\|\phi_h\|_W} \leq Ch^{k-1}|\mathbf{f}|_{k-1,h}.$$

Proof. Let $\phi_h \in W_h$, then using the definition of $F(\cdot)$ and $F^h(\cdot)$ (see (2.7) and (3.17), respectively), orthogonality property of the projector $\boldsymbol{\Pi}_{k-1}^K$, the Cauchy–Schwarz inequality and Lemma 4.2, we have

$$\begin{aligned} |F(\phi_h) - F^h(\phi_h)| &\leq \sum_{K \in \mathcal{T}_h} \|\mathbf{f} - \boldsymbol{\Pi}_{k-1}^K \mathbf{f}\|_{0,K} \|\mathbf{curl}\, \phi_h - \boldsymbol{\Pi}_{k-1}^K \mathbf{curl}\, \phi_h\|_{0,K} \\ &\leq C \sum_{K \in \mathcal{T}_h} h_K^{k-1} |\mathbf{f}|_{k-1,K} |\phi_h|_{1,K} \\ &\leq Ch^{k-1} |\mathbf{f}|_{k-1,h} \|\phi_h\|_W. \end{aligned}$$

Therefore,

$$\|F - F^h\|_{W_h'} \leq Ch^{k-1}|\mathbf{f}|_{k-1,h}.$$

□

In order to develop the error estimates, from now on, we make the following assumption for the permeability tensor: $\mathbb{K}^{-1} \in W^{1+s,\infty}(\Omega)^{2\times 2}$.

The next step is to establish the following result.

LEMMA 4.4 Let $k \geq 2$ and $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^{k-1}(K)]^2$ for each $K \in \mathcal{T}_h$. Let ψ and ψ_h be the unique solutions of problems (2.3) and (3.19), respectively. Then, there exist $s > 1/2$ and a positive constant C, independent of ν and h, such that

$$\begin{aligned} \|\psi - \psi_h\|_W \leq C \Big(&\|F - F^h\|_{W_h'} + \|\psi - \psi_I\|_W + |\psi - \psi_\pi|_{1,h} \\ &+ |\psi - \psi_\pi|_{2,h} + h^{\min\{1+s,k\}} \|\psi\|_{2+s,\Omega} \Big), \end{aligned}$$

for all $\psi_I \in W_h$ and for all $\psi_\pi \in L^2(\Omega)$ such that $\psi_\pi|_K \in \mathcal{P}_k(K)$ for all polygon $K \in \mathcal{T}_h$.

Proof. Let $\psi_I \in W_h$. We set $v_h := \psi_h - \psi_I$. Thus,

$$\|\psi - \psi_h\|_W \leq \|\psi - \psi_I\|_W + \|v_h\|_W. \tag{4.1}$$

Now, thanks to Lemma 3.10, adding and subtracting the term $A(\psi, v_h)$ and using the definition of the continuous and discrete problems (2.3) and (3.19), respectively, we have

$$\begin{aligned}
\widetilde{\alpha}\|v_h\|_W^2 &\leq A^h(v_h, v_h) \\
&= A^h(\psi_h, v_h) - A^h(\psi_I, v_h) \\
&= F^h(v_h) - A^h(\psi_I, v_h) \\
&= F^h(v_h) - F(v_h) + A(\psi, v_h) - A^h(\psi_I, v_h) \\
&= F^h(v_h) - F(v_h) + \sum_{K\in\mathcal{T}_h} \left\{ \nu A_\Delta^K(\psi, v_h) + A_{\mathbf{curl}}^K(\psi, v_h) \right\} \\
&\quad - \sum_{K\in\mathcal{T}_h} \left\{ \nu A_\Delta^{h,K}(\psi_I, v_h) + A_{\mathbf{curl}}^{h,K}(\psi_I, v_h) \right\} \\
&= F^h(v_h) - F(v_h) + \sum_{K\in\mathcal{T}_h} \left\{ \nu A_\Delta^K(\psi - \psi_\pi, v_h) - \nu A_\Delta^{h,K}(\psi_I - \psi_\pi, v_h) \right\} \\
&\quad + \sum_{K\in\mathcal{T}_h} \left\{ A_{\mathbf{curl}}^K(\psi, v_h) - A_{\mathbf{curl}}^{h,K}(\psi_I, v_h) \right\},
\end{aligned} \tag{4.2}$$

where we have added and subtracted $\psi_\pi \in L^2(\Omega)$ such that $\psi_\pi|_K \in \mathcal{P}_k(K)$ for all $K \in \mathcal{T}_h$ for $k \geq 2$, in the last step.

Now, we are going to analyze the last two terms of (4.2). By using the continuity of bilinear forms $A_\Delta^K(\cdot,\cdot)$ and $A_\Delta^{h,K}(\cdot,\cdot)$ and the triangular inequality, we have that

$$\begin{aligned}
\nu\left\{ A_\Delta^K(\psi - \psi_\pi, v_h) + A_\Delta^{h,K}(\psi_I - \psi_\pi, v_h) \right\} &\leq C\nu \left(|\psi - \psi_\pi|_{2,K} + |\psi_I - \psi_\pi|_{2,K} \right) |v_h|_{2,K} \\
&\leq C\nu \left(|\psi - \psi_\pi|_{2,K} + |\psi - \psi_I|_{2,K} \right) |v_h|_{2,K}.
\end{aligned} \tag{4.3}$$

On the other hand, for the last term in (4.2), adding and subtracting the term $A_{\mathbf{curl}}^K(\psi_I, v_h)$, we have

$$\begin{aligned}
&A_{\mathbf{curl}}^K(\psi, v_h) - A_{\mathbf{curl}}^{h,K}(\psi_I, v_h) \\
&\qquad = A_{\mathbf{curl}}^K(\psi, v_h) - A_{\mathbf{curl}}^K(\psi_I, v_h) + A_{\mathbf{curl}}^K(\psi_I, v_h) - A_{\mathbf{curl}}^{h,K}(\psi_I, v_h) \\
&\qquad = A_{\mathbf{curl}}^K(\psi - \psi_I, v_h) + \left\{ A_{\mathbf{curl}}^K(\psi_I, v_h) - A_{\mathbf{curl}}^{h,K}(\psi_I, v_h) \right\}.
\end{aligned} \tag{4.4}$$

Next, thanks to the continuity of $A_{\mathbf{curl}}^K(\cdot,\cdot)$, it follows that

$$A_{\mathbf{curl}}^K(\psi - \psi_I, v_h) \leq C|\psi - \psi_I|_{1,K}|v_h|_{1,K}, \tag{4.5}$$

and the second term in (4.4) is bounded using Lemma 3.8 as follows:

$$\begin{aligned}
A^{K}_{\mathbf{curl}}(\psi_I, v_h) - A^{h,K}_{\mathbf{curl}}(\psi_I, v_h) \\
\leq \|\mathbb{K}^{-1}\,\mathbf{curl}\,\psi_I - \boldsymbol{\Pi}^{K}_{k-1}(\mathbb{K}^{-1}\mathbf{curl}\,\psi_I)\|_{0,K}\|\mathbf{curl}\,v_h - \boldsymbol{\Pi}^{K}_{k-1}\mathbf{curl}\,v_h\|_{0,K} \\
+ \|\mathbb{K}^{-1}\,\mathbf{curl}\,v_h - \boldsymbol{\Pi}^{K}_{k-1}(\mathbb{K}^{-1}\mathbf{curl}\,v_h)\|_{0,K}\|\mathbf{curl}\,\psi_I - \boldsymbol{\Pi}^{K}_{k-1}\mathbf{curl}\,\psi_I\|_{0,K} \\
+ C\|\mathbf{curl}\,\psi_I - \boldsymbol{\Pi}^{K}_{k-1}\mathbf{curl}\,\psi_I\|_{0,K}\|\mathbf{curl}\,v_h - \boldsymbol{\Pi}^{K}_{k-1}\mathbf{curl}\,v_h\|_{0,K} \\
+ s^{K}_{\mathbf{curl}}(\psi_I - \Pi^{k,\nabla^{\perp}}_{K}\psi_I, v_h - \Pi^{k,\nabla^{\perp}}_{K}v_h).
\end{aligned}$$

Adding and subtracting appropriate terms, using triangular inequality, the estimate in Lemma 4.2 and standard argument, we obtain

$$A^{K}_{\mathbf{curl}}(\psi_I, v_h) - A^{h,K}_{\mathbf{curl}}(\psi_I, v_h) \leq C\left(|\psi-\psi_I|_{1,K} + |\psi-\psi_\pi|_{1,K} + h_K^{\min\{1+s,k\}}|\psi|_{2+s,K}\right)|v_h|_{1,K}. \quad (4.6)$$

Now, from (4.2) using the triangular and Cauchy–Schwarz inequalities and (4.3), (4.4), (4.5) and (4.6), we have

$$\begin{aligned}
\widetilde{\alpha}\|v_h\|_W^2 &\leq \|F-F^h\|_{W_h'}\|v_h\|_W + \sum_{K\in\mathcal{T}_h} C\nu\left(|\psi-\psi_\pi|_{2,K} + |\psi-\psi_I|_{2,K}\right)|v_h|_{2,K} \\
&\quad + \sum_{K\in\mathcal{T}_h} C\left(|\psi-\psi_I|_{1,K} + h_K^{\min\{1+s,k\}}|\psi|_{2+s,K} + |\psi-\psi_\pi|_{1,K}\right)|v_h|_{1,K} \\
&\leq C\|F^h-F\|_{W_h'}\|v_h\|_W + C\sum_{K\in\mathcal{T}_h}\left(|\psi-\psi_\pi|_{1,K} + \sqrt{\nu}|\psi-\psi_\pi|_{2,K}\right)\left(\sqrt{\nu}|v_h|_{2,K} + |v_h|_{1,K}\right) \\
&\quad + C\sum_{K\in\mathcal{T}_h}\left(\sqrt{\nu}|\psi-\psi_I|_{2,K} + |\psi-\psi_I|_{1,K} + h_K^{\min\{1+s,k\}}\|\psi\|_{2+s,K}\right)\left(\sqrt{\nu}|v_h|_{2,K} + |v_h|_{1,K}\right),
\end{aligned}$$

applying the Cauchy–Schwarz inequality in the second and third terms of the above estimate, we get

$$\begin{aligned}
\widetilde{\alpha}\|v_h\|_W^2 \leq C\Big(&\|F-F^h\|_{W_h'} + \|\psi-\psi_I\|_W + |\psi-\psi_\pi|_{1,h} \\
&+ \sqrt{\nu}|\psi-\psi_\pi|_{2,h} + h^{\min\{1+s,k\}}\|\psi\|_{2+s,\Omega}\Big)\|v_h\|_W. \quad (4.7)
\end{aligned}$$

Therefore, the proof follows from (4.1) and (4.7). □

The next step is to find an appropriate term $\psi_I \in W_h$ that can be used in Lemma 4.4 to prove convergence of our discrete scheme. Thus, we have the following result.

PROPOSITION 4.5 Assume that **A**1 and **A**2 are satisfied. Then, for each $\phi \in H^{\delta}(\Omega)$, there exist $\phi_I \in W_h$ and $C > 0$, independent of h, such that

$$\|\phi-\phi_I\|_{\ell,\Omega} \leq Ch^{\delta-\ell}|\phi|_{\delta,\Omega}, \quad \ell = 0,1,2, \quad 2 \leq \delta \leq k+1, \quad k \geq 2.$$

Proof. The proof follows repeating the arguments from Beirão da Veiga *et al.* (2019a, Proposition 4.2) (see also Antonietti *et al.*, 2016, Proposition 3.1). □

The following theorem establishes the convergence of our scheme.

THEOREM 4.6 Let $k \geq 2$ and $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^{k-1}(K)]^2$ for each $K \in \mathcal{T}_h$. Let $\psi \in W$ and $\psi_h \in W_h$ be the unique solutions to the continuous and discrete problems (2.3) and (3.19), respectively. Then, there exist $s > 1/2$ and a positive constant C, independent of ν and h, such that

$$\|\psi - \psi_h\|_W \leq Ch^{\min\{s,k-1\}} \big(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\big).$$

Proof. The result follows from Lemma 4.4 and Propositions 4.1, 4.3 and 4.5. □

4.1 *Error estimates in H^1 and L^2*

In this section we prove estimates in H^1- and L^2-norms for the stream function using duality arguments. The following preliminary result will be useful to show the estimate in H^1 and the proof follows standard argument.

LEMMA 4.7 Let $k \geq 2$ and let $\varphi_\pi, \phi_\pi \in \mathcal{P}_k(K)$, then the bilinear forms $A^{h,K}(\cdot,\cdot)$, $A^K(\cdot,\cdot)$, $A^{h,K}_{\mathbf{curl}}(\cdot,\cdot)$ and $A^K_{\mathbf{curl}}(\cdot,\cdot)$ on each element K, satisfy

$$\begin{aligned} A^{h,K}(\varphi_h,\phi_h) - A^K(\varphi_h,\phi_h) &= A^{h,K}(\varphi_h-\varphi_\pi,\phi_h-\phi_\pi) - A^K(\varphi_h-\varphi_\pi,\phi_h-\phi_\pi) \\ &\quad + A^{h,K}_{\mathbf{curl}}(\varphi_h-\varphi_\pi,\phi_h) - A^K_{\mathbf{curl}}(\varphi_h-\varphi_\pi,\phi_h) \\ &\quad + A^{h,K}_{\mathbf{curl}}(\varphi_\pi,\phi_h) - A^K_{\mathbf{curl}}(\varphi_\pi,\phi_h) \quad \forall\, \varphi_h,\phi_h \in W^k_h(K). \end{aligned} \tag{4.8}$$

The following result establishes an error estimate in H^1-norm for the stream function. We are going to use a duality argument. With this aim we make the following assumption in Theorems 4.9 and 4.10.

ASSUMPTION 4.8 Constants s and C in Theorem 2.3 are independent of ν.

THEOREM 4.9 Let $k \geq 2$ and $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^{k-1}(K)]^2$ for each $K \in \mathcal{T}_h$. Let $\psi \in W$ and $\psi_h \in W_h$ be the unique solutions to the continuous and discrete problems (2.3) and (3.19), respectively. Then, there exist $\tilde{s} \in (1/2, 1]$, $s > 1/2$ and a constant $C > 0$, independent of ν and h, such that

$$\|\psi - \psi_h\|_{1,\Omega} \leq Ch^{\tilde{s}+\min\{s,k-1\}} \big(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\big).$$

Proof. Let $\phi \in W$ be the solution of the auxiliary variational problem: find ϕ such that

$$A(\phi, v) = \int_\Omega \nabla(\psi - \psi_h)\cdot\nabla v \qquad \forall\, v \in W, \tag{4.9}$$

where $A(\cdot,\cdot)$ is the bilinear form defined in (2.4).

From Theorem 2.3 there exists $\tilde{s} \in (1/2, 1]$ and $C > 0$ (cf. Assumption 4.8) such that $\phi \in H^{2+\tilde{s}}(\Omega)$ and

$$\|\phi\|_{2+\tilde{s},\Omega} \leq C|\psi - \psi_h|_{1,\Omega}. \tag{4.10}$$

Now, let $\phi_I \in W_h$ be such that Proposition 4.5 holds true. Taking $v := (\psi - \psi_h) \in W$ as test function in (4.9), using the symmetry of the bilinear form and adding and subtracting ϕ_I, we obtain

$$\begin{aligned}
|\psi - \psi_h|^2_{1,\Omega} &\le A(\psi - \psi_h, \phi) \\
&= A(\psi - \psi_h, \phi - \phi_I) + A(\psi - \psi_h, \phi_I) \\
&= A(\psi - \psi_h, \phi - \phi_I) + A(\psi, \phi_I) - A(\psi_h, \phi_I) \\
&= A(\psi - \psi_h, \phi - \phi_I) + F(\phi_I) - A(\psi_h, \phi_I) \\
&= A(\psi - \psi_h, \phi - \phi_I) + [F(\phi_I) - F^h(\phi_I)] + [A^h(\psi_h, \phi_I) - A(\psi_h, \phi_I)] \\
&=: T_1 + T_2 + T_3, \qquad (4.11)
\end{aligned}$$

where we have used the definition of the continuous (see (2.3)) and discrete problems (see (3.19)).

Now, we bound each term T_1, T_2, T_3. We start with T_1 as follows:

$$\begin{aligned}
T_1 &= A(\psi - \psi_h, \phi - \phi_I) \\
&\le C\|\psi - \psi_h\|_W \|\phi - \phi_I\|_W \\
&\le \|\psi - \psi_h\|_W \left(|\phi - \phi_I|^2_{1,\Omega} + \nu|\phi - \phi_I|^2_{2,\Omega}\right)^{1/2} \\
&\le Ch^{\min\{s,k-1\}}(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}) \left(Ch^{2(1+\tilde{s})}\|\phi\|^2_{2+\tilde{s},\Omega} + C\nu h^{2\tilde{s}}\|\phi\|^2_{2+\tilde{s},\Omega}\right)^{1/2} \\
&\le Ch^{\min\{s,k-1\}}(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega})(1 + \sqrt{\nu})h^{\tilde{s}}\|\phi\|_{2+\tilde{s},\Omega} \\
&\le C(1 + \sqrt{\nu})h^{\tilde{s}+\min\{s,k-1\}}(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega})|\psi - \psi_h|_{1,\Omega}, \qquad (4.12)
\end{aligned}$$

where we have used the continuity of bilinear form $A(\cdot,\cdot)$, Theorem 4.6 and Proposition 4.5.

For T_2 we use the definition of the functionals $F(\cdot)$ and $F^h(\cdot)$, applying the Cauchy–Schwarz inequality and Lemma 4.2 as follows:

$$\begin{aligned}
T_2 = F(\phi_I) - F^h(\phi_I) &= \sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f} - \boldsymbol{\Pi}^K_{k-1}\mathbf{f}) \cdot (\mathbf{curl}\,\phi_I - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\phi_I) \\
&\le \sum_{K\in\mathcal{T}_h} \|\mathbf{f} - \boldsymbol{\Pi}^K_{k-1}\mathbf{f}\|_{0,K}\|\mathbf{curl}\,\phi_I - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\phi_I\|_{0,K} \\
&\le \sum_{K\in\mathcal{T}_h} Ch_K^{k-1}|\mathbf{f}|_{k-1,K}\big(\|\mathbf{curl}\,\phi_I - \mathbf{curl}\,\phi\|_{0,K} \\
&\qquad + \|\mathbf{curl}\,\phi - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\phi\|_{0,K} + \|\boldsymbol{\Pi}^K_{k-1}(\mathbf{curl}\,(\phi - \phi_I))\|_{0,K}\big) \\
&\le \sum_{K\in\mathcal{T}_h} Ch_K^{k-1}|\mathbf{f}|_{k-1,K}\left(C\|\phi - \phi_I\|_{1,K} + Ch_K^{1+\tilde{s}}\|\phi\|_{2+\tilde{s},K}\right) \\
&\le Ch^{\tilde{s}+\min\{s,k-1\}}\left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)|\psi - \psi_h|_{1,\Omega}, \qquad (4.13)
\end{aligned}$$

where we have also used the Cauchy–Schwarz inequality, Proposition 4.5 and (4.10).

Now, we continue with the term T_3. Let $\psi_\pi, \phi_\pi \in \mathcal{P}_k(K)$ such that Proposition 4.1 holds true. Using (4.8) we have

$$\begin{aligned} T_3 &= \sum_{K\in\mathcal{T}_h} [A^{h,K}(\psi_h - \psi_\pi, \phi_I - \phi_\pi) + A^K(\psi_\pi - \psi_h, \phi_I - \phi_\pi)] \\ &\quad + \sum_{K\in\mathcal{T}_h} [A^{h,K}_{\mathbf{curl}}(\psi_h - \psi_\pi, \phi_\pi) - A^K_{\mathbf{curl}}(\psi_h - \psi_\pi, \phi_\pi)] \\ &\quad + \sum_{K\in\mathcal{T}_h} [A^{h,K}_{\mathbf{curl}}(\psi_\pi, \phi_I) - A^K_{\mathbf{curl}}(\psi_\pi, \phi_I)] \\ &=: T_{3a} + T_{3b} + T_{3c}. \end{aligned} \tag{4.14}$$

We bound each term on the right-hand side above. To do that we introduce

$$\|\varphi\|^2_{W,K} := \left(|\varphi|^2_{1,K} + \nu|\varphi|^2_{2,K}\right) \qquad \forall\, \varphi \in H^2(K).$$

Then, the first term can be bounded as follows: using the continuity of the bilinear forms $A^{h,K}(\cdot,\cdot)$ and $A^K(\cdot,\cdot)$, we have

$$\begin{aligned} T_{3a} &\le \sum_{K\in\mathcal{T}_h} C\|\psi_h - \psi_\pi\|_{W,K}\|\phi_I - \phi_\pi\|_{W,K} \\ &\le \sum_{K\in\mathcal{T}_h} C(\|\psi_h - \psi\|_{W,K} + \|\psi - \psi_\pi\|_{W,K})(\|\phi_I - \phi\|_{W,K} + \|\phi - \phi_\pi\|_{W,K}) \\ &\le \sum_{K\in\mathcal{T}_h} C(\|\psi_h - \psi\|_{W,K} + C(1+\sqrt{\nu})h_K^{\min\{s,k-1\}}\|\psi\|_{2+s,K})\left(\|\phi_I - \phi\|_{W,K} + C(1+\sqrt{\nu})h_K^{\tilde{s}}\|\phi\|_{2+\tilde{s},K}\right), \end{aligned}$$

where we have used Proposition 4.1. Now, using the Cauchy–Schwarz inequality, Theorem 4.6 and (4.10), we obtain

$$\begin{aligned} T_{3a} &\le C(\|\psi_h - \psi\|_W + C(1+\sqrt{\nu})h^{\min\{s,k-1\}}\|\psi\|_{2+s,\Omega})\left(C(1+\sqrt{\nu})h^{\tilde{s}}\|\phi\|_{2+\tilde{s},\Omega} + \|\phi_I - \phi\|_W\right) \\ &\le C(\|\psi_h - \psi\|_W + (1+\sqrt{\nu})h^{\min\{s,k-1\}}\|\psi\|_{2+s,\Omega})\left(C(1+\sqrt{\nu})h^{\tilde{s}}\|\phi\|_{2+\tilde{s},\Omega} + C(1+\sqrt{\nu})h^{\tilde{s}}\|\phi\|_{2+\tilde{s},\Omega}\right) \\ &\le C(1+\sqrt{\nu})h^{\min\{s,k-1\}}\left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)h^{\tilde{s}}\|\phi\|_{2+\tilde{s},\Omega} \\ &\le C(1+\sqrt{\nu})h^{\tilde{s}+\min\{s,k-1\}}\left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)|\psi - \psi_h|_{1,\Omega}. \end{aligned} \tag{4.15}$$

Now, we will bound the second term on the right-hand side of (4.14). By using Lemma 3.8 and the fact that $\phi_\pi \in \mathcal{P}_k(K)$, we have

$$\begin{aligned} T_{3b} \leq \sum_{K\in\mathcal{T}_h} & \|\mathbb{K}^{-1}\,\mathbf{curl}\,(\psi_h-\psi_\pi) - \boldsymbol{\Pi}^K_{k-1}(\mathbb{K}^{-1}\mathbf{curl}\,(\psi_h-\psi_\pi))\|_{0,K}\|\mathbf{curl}\,\phi_\pi - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\phi_\pi\|_{0,K} \\ & + \|\mathbb{K}^{-1}\,\mathbf{curl}\,\phi_\pi - \boldsymbol{\Pi}^K_{k-1}(\mathbb{K}^{-1}\mathbf{curl}\,\phi_\pi)\|_{0,K}\|\mathbf{curl}\,(\psi_h-\psi_\pi) - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,(\psi_h-\psi_\pi)\|_{0,K} \\ & + C\|\mathbf{curl}\,(\psi_h-\psi_\pi) - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,(\psi_h-\psi_\pi)\|_{0,K}\|\mathbf{curl}\,\phi_\pi - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,\phi_\pi\|_{0,K} \\ & + s^K_{\mathbf{curl}}\left((\psi_h-\psi_\pi) - \Pi^{k,\nabla^\perp}_K(\psi_h-\psi_\pi), \phi_\pi - \Pi^{k,\nabla^\perp}_K\phi_\pi\right) \\ = \sum_{K\in\mathcal{T}_h} & \|\mathbb{K}^{-1}\,\mathbf{curl}\,\phi_\pi - \boldsymbol{\Pi}^K_{k-1}(\mathbb{K}^{-1}\mathbf{curl}\,\phi_\pi)\|_{0,K}\|\mathbf{curl}\,(\psi_h-\psi_\pi) - \boldsymbol{\Pi}^K_{k-1}\mathbf{curl}\,(\psi_h-\psi_\pi)\|_{0,K} \\ \leq \sum_{K\in\mathcal{T}_h} & C\left(|\phi_\pi-\phi|_{1,K} + h_K^{1+\tilde{s}}\|\phi\|_{2+\tilde{s},K}\right)\left(|\psi_h-\psi|_{1,K} + |\psi-\psi_\pi|_{1,K}\right) \\ \leq\ & Ch^{\tilde{s}+\min\{s,k-1\}}\left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)|\psi-\psi_h|_{1,\Omega}, \end{aligned} \tag{4.16}$$

where we have also used the stability properties of projector $\boldsymbol{\Pi}^K_{k-1}$, triangular inequality, Proposition 4.1 and the Cauchy–Schwarz inequality.

Finally, we bound the third term on the right-hand side of (4.14). We proceed as in the previous estimate to obtain that

$$\begin{aligned} T_{3c} &\leq \sum_{K\in\mathcal{T}_h} \|\mathbb{K}^{-1}\,\mathbf{curl}\,\psi_\pi - \boldsymbol{\Pi}^K_{k-1}(\mathbb{K}^{-1}\mathbf{curl}\,\psi_\pi)\|_{0,K}\|\mathbf{curl}\,\phi_I - \boldsymbol{\Pi}^K_{k-1}(\mathbf{curl}\,\phi_I)\|_{0,K} \\ &\leq \sum_{K\in\mathcal{T}_h} C(h_K^{\min\{1+s,k\}}\|\psi\|_{2+s,K})(h_K^{1+\tilde{s}}\|\phi\|_{2+\tilde{s},K}) \\ &\leq Ch^{\tilde{s}+\min\{s,k-1\}}\left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)|\psi-\psi_h|_{1,\Omega}. \end{aligned} \tag{4.17}$$

Then, from (4.15), (4.16) and (4.17), we get

$$T_3 \leq Ch^{\tilde{s}+\min\{s,k-1\}}\left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)|\psi-\psi_h|_{1,\Omega}. \tag{4.18}$$

Finally, the proof follows from (4.11), (4.12), (4.13) and (4.18). □

The following theorem establishes an error estimate in L^2-norm.

THEOREM 4.10 Let $\mathbf{f} \in [L^2(\Omega)]^2$ and let $\psi \in W$ and $\psi_h \in W_h$ be the unique solutions to the continuous and discrete problems (2.3) and (3.19), respectively. Then,

(a) If $k = 2$ and $\mathbf{f}|_K \in [H^1(K)]^2$ for all $K \in \mathcal{T}_h$ then there exist $s > 1/2$, $\tilde{s} \in (1/2, 1]$ and a constant $C > 0$, independent of ν and h, such that

$$\|\psi-\psi_h\|_{0,\Omega} \leq Ch^{\tilde{s}+\min\{s,1\}}\left(|\mathbf{f}|_{1,h} + \|\psi\|_{2+s,\Omega}\right).$$

(b) If $k \geq 3$ and $\mathbf{f}|_K \in [H^{k-1}(K)]^2$ for all $K \in \mathcal{T}_h$ then there exist $s > 1/2$, $\gamma \in (1/2, 2]$ and a constant $C > 0$, independent of ν and h, such that

$$\|\psi - \psi_h\|_{0,\Omega} \leq Ch^{\gamma+\min\{s,k-1\}} \left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right).$$

Proof. The proof of (a) follows from Theorem 4.9. In fact we have

$$\|\psi - \psi_h\|_{0,\Omega} \leq C|\psi - \psi_h|_{1,\Omega} \leq Ch^{\tilde{s}+\min\{s,1\}} \left(|\mathbf{f}|_{1,h} + \|\psi\|_{2+s,\Omega}\right),$$

where $C > 0$ is a constant independent of ν and h.

On the other hand, if $k \geq 3$, let $\phi \in W$ be the solution of the following auxiliary variational problem:

$$A(\phi, v) = G(v) \qquad \forall\, v \in W, \tag{4.19}$$

where $A(\cdot,\cdot)$ is the bilinear form defined in (2.4) and $G(\cdot)$ is the functional defined by

$$G(v) := \int_\Omega (\psi - \psi_h)v \qquad \forall\, v \in W.$$

From Theorem 2.3 there exists $\gamma \in (1/2, 2]$ and $C > 0$ (cf. Assumption 4.8) such that $\phi \in H^{2+\gamma}(\Omega)$ and

$$\|\phi\|_{2+\gamma,\Omega} \leq C\|\psi - \psi_h\|_{0,\Omega}.$$

Now, taking $v := (\psi - \psi_h) \in W$ as a test function in (4.19), we obtain

$$\|\psi - \psi_h\|^2_{0,\Omega} = A(\psi - \psi_h, \phi),$$

where we have used the symmetry of $A(\cdot,\cdot)$.

Let $\phi_I \in W_h$ such that Proposition 4.5 holds true. In particular, we have

$$\|\phi - \phi_I\|_{2,\Omega} \leq Ch^\gamma \|\phi\|_{2+\gamma,\Omega} \quad \text{and} \quad \|\phi - \phi_I\|_{1,\Omega} \leq Ch^{1+\gamma}\|\phi\|_{2+\gamma,\Omega}.$$

Repeating the arguments used to obtain estimate (4.11), we get

$$\|\psi - \psi_h\|^2_{0,\Omega} \leq B_1 + B_2 + B_3,$$

where

$$B_1 := A(\psi - \psi_h, \phi - \phi_I), \quad B_2 := F(\phi_I) - F^h(\phi_I) \quad \text{and} \quad B_3 := A^h(\psi_h, \phi_I) - A(\psi_h, \phi_I).$$

Now, repeating the steps used in the proof of Theorem 4.9, we can estimate the terms B_1, B_2 and B_3 to obtain that

$$\|\psi - \psi_h\|_{0,\Omega} \leq Ch^{\gamma+\min\{s,k-1\}} \left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right), \quad k \geq 3.$$

The proof is complete. □

4.2 *Recovering the velocity field*

The solution of the discrete problem (3.19) delivers the stream function of the velocity field. In addition, it is possible to obtain the remaining quantities of interest: velocity $\mathbf{u}$ and pressure p.

We begin with the velocity field: if $\psi \in W$ the unique solution of problem (2.3), then

$$\mathbf{u} = \mathbf{curl}\, \psi. \tag{4.20}$$

At the discrete level, we compute a discrete velocity as a post-processing of the stream function $\psi_h \in W_h$ as follows: if ψ_h is the unique solution of problem (3.19) then the function

$$\mathbf{u}_h := \mathbf{curl}\, \psi_h, \tag{4.21}$$

is an approximation of the velocity.

Now, we establish the accuracy of the discrete velocity. With this aim we introduce the following ν-dependent norm:

$$|||\mathbf{v}|||_{1,\Omega} := \left(\|\mathbf{v}\|_{0,\Omega}^2 + \nu|\mathbf{v}|_{1,\Omega}^2\right)^{1/2} \qquad \forall\ \mathbf{v} \in [H^1(\Omega)]^2.$$

THEOREM 4.11 Let $k \geq 2$ and $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^{k-1}(K)]^2$ for each $K \in \mathcal{T}_h$. Let ψ and ψ_h be the unique solutions of problem (2.3) and problem (3.19), respectively. Then, there exist $s > 1/2$ and a positive constant C, independent of ν and h, such that

$$|||\mathbf{u} - \mathbf{u}_h|||_{1,\Omega} \leq Ch^{\min\{s,k-1\}} \left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right).$$

Proof. From (4.20), (4.21) and Theorem 4.6, we have

$$\begin{aligned}
|||\mathbf{u} - \mathbf{u}_h|||_{1,\Omega}^2 &= \|\mathbf{curl}\, \psi - \mathbf{curl}\, \psi_h\|_{0,\Omega}^2 + \nu|\mathbf{curl}\, \psi - \mathbf{curl}\, \psi_h|_{1,\Omega}^2 \\
&\leq C\|\psi - \psi_h\|_W^2 \\
&\leq Ch^{2\min\{s,k-1\}} \left(|\mathbf{f}|_{k-1,h} + \|\psi\|_{2+s,\Omega}\right)^2.
\end{aligned}$$

The proof is complete. □

REMARK 4.12 Recently, it has been presented in Vacca (2018) a VEM, of arbitrary order $\ell \geq 2$, to solve the Darcy and Brinkman problems in terms of the velocity $\mathbf{u}$ and the pressure p fields. We are going to compare the computational cost (in terms of degrees of freedom) between the method from Vacca (2018) and our C^1 scheme to obtain the same accuracy $\mathcal{O}(h^\ell)$ for the velocity field. Assuming a sufficiently smooth solution for the Brinkman problem, it has been established in Vacca (2018, Section 3 and Theorem 5.2) that the computational cost in terms of degrees of freedom is given by

$R_2 := \dim(\mathbf{V}_h^\ell) + \dim(Q_h^\ell)$ with

$$\dim(\mathbf{V}_h^\ell) := N_K\left(\frac{\ell(\ell+1)}{2} - 1 + \frac{(\ell-2)(\ell-1)}{2}\right) + 2\left(N_V + (\ell-1)N_E\right),$$
$$\dim(Q_h^\ell) := N_K\frac{\ell(\ell+1)}{2} - 1,$$

where N_E is the number of internal edges, N_V is the number of internal vertices and N_K the number of elements of $\mathcal{T}_h$. It can be observed from (3.6) that for $k = \ell + 1$ and using the Euler formula ($N_V - N_E + N_K - 1 = 0$), that the computational cost of our scheme is smaller than the method studied in Vacca (2018). In particular, to obtain an $\mathcal{O}(h^2)$ for the velocity field, $R_2 = 5N_K + 2(N_V + N_E) - 1$, while our scheme needs $R_1 = 3N_V + N_E$ (cf. (3.6)). In addition, for our method, the resulting linear system is positive definite, as opposed to the one in Vacca (2018) that is undefinite; this allows for more efficient methods such as Cholesky factorization or conjugate gradient.

4.3 *Recovering the pressure field*

In this section we present a novel strategy to recover fluid pressure. We will write a generalized Poisson problem with data coming from the stream function. Then, we propose a discrete virtual scheme, based on the C^0 enhanced virtual element space from Ahmad *et al.* (2013). We will also establish an error estimate for the fluid pressure in H^1-norm, under the assumptions that Ω is convex and that the family of polygonal meshes $\mathcal{T}_h$ is quasi-uniform.

In order to recover the pressure we consider the following Hilbert space:

$$\widetilde{H}^1(\Omega) := \left\{q \in H^1(\Omega) : (q,1)_{0,\Omega} = 0\right\}.$$

By using the identity $-\Delta\mathbf{u} = \mathbf{curl}\,(\mathrm{rot}\,\mathbf{u}) - \nabla(\mathrm{d}iv\,\mathbf{u})$ in the momentum equation of (2.1), we obtain

$$\begin{aligned}\mathbf{f} &= \mathbb{K}^{-1}\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p\\ &= \mathbb{K}^{-1}\mathbf{u} + \nu\,(\mathbf{curl}\,(\mathrm{rot}\,\mathbf{u}) - \nabla(\mathrm{d}iv\,\mathbf{u})) + \nabla p\\ &= \mathbb{K}^{-1}\mathbf{u} + \nu\,\mathbf{curl}\,(\mathrm{rot}\,\mathbf{u}) + \nabla p,\end{aligned}$$

where we have used that $\mathrm{d}iv\,\mathbf{u} = 0$ in Ω (cf. (2.1)). The above equality can be rewritten as follows:

$$\begin{aligned}\nabla p &= \mathbf{f} - \mathbb{K}^{-1}\mathbf{curl}\,\psi - \nu\,\mathbf{curl}\,(\mathrm{rot}\,\mathbf{curl}\,\psi)\\ &= \mathbf{f} - \mathbb{K}^{-1}\mathbf{curl}\,\psi + \nu\,\mathbf{curl}\,(\Delta\psi),\end{aligned} \tag{4.22}$$

where we have used the fact that $\mathbf{u} = \mathbf{curl}\,\psi$ and the identity $\mathrm{rot}(\mathbf{curl}\,\psi) = -\Delta\psi$.

Then, by testing (4.22) with ∇q for $q \in \widetilde{H}^1(\Omega)$, we get the following variational problem: find $p \in \widetilde{H}^1(\Omega)$ such that

$$B_\nabla(p,q) = G^\psi(q) \qquad \forall\, q \in \widetilde{H}^1(\Omega), \tag{4.23}$$

where $B_{\nabla} : \widetilde{H}^1(\Omega) \times \widetilde{H}^1(\Omega) \to \mathbb{R}$ is defined by

$$B_{\nabla}(p,q) := \int_{\Omega} \nabla p \cdot \nabla q \qquad \forall\, p,q \in \widetilde{H}^1(\Omega) \tag{4.24}$$

and $G^{\psi} : \widetilde{H}^1(\Omega) \to \mathbb{R}$ is the functional defined by:

$$G^{\psi}(q) := \int_{\Omega} \mathbf{f} \cdot \nabla q - \int_{\Omega} \mathbb{K}^{-1} \mathbf{curl}\, \psi \cdot \nabla q + \nu \int_{\Omega} \mathbf{curl}\,(\Delta\psi) \cdot \nabla q \qquad \forall\, q \in \widetilde{H}^1(\Omega). \tag{4.25}$$

Since Ω is convex we have that $\psi \in H^3(\Omega)$, thus, as a consequence of the generalized Poincaré inequality and the Lax–Milgram Theorem, problem (4.23) is well posed.

In what follows, we will write a lowest order discrete virtual scheme associated to (4.23) in order to build a discrete pressure over the same polygonal mesh $\mathcal{T}_h$ used to solve the stream function discrete formulation (3.19). We observe that, following the arguments in this section, it is possible to write a high order virtual scheme of order $\ell := k-2$ (Ahmad *et al.* (2013); Beirão da Veiga *et al.* (2016a); Cangiani *et al.* (2017b)), with $k \geq 3$ being the order for the C^1 VEM (3.19).

Now, we split the bilinear form $B_{\nabla}(\cdot,\cdot)$, as follows:

$$B_{\nabla}(p,q) = \sum_{K\in\mathcal{T}_h} B^K_{\nabla}(p,q) = \sum_{K\in\mathcal{T}_h} \int_K \nabla p \cdot \nabla q \qquad \forall\, p,q \in \widetilde{H}^1(\Omega).$$

Now, for each polygon $K \in \mathcal{T}_h$, we introduce the space

$$\mathbb{B}_1(\partial K) := \left\{ q_h \in C^0(\partial K) : q_h|_e \in \mathcal{P}_1(e) \quad \forall\, e \subset \partial K \right\}.$$

Then, we consider the finite-dimensional space $\widehat{H}_h(K)$, defined as

$$\widehat{H}_h(K) := \left\{ q_h \in H^1(K) : q_h|_e \in \mathbb{B}_1(\partial K),\ \Delta q_h \in \mathcal{P}_0(K) \right\}.$$

The following set of linear operator is defined for all $q_h \in \widehat{H}_h(K)$:

- $\mathbf{P}_1$: the values of $q_h(V_i)$ for each vertex V_i of K.

We define the projector $\Pi^{\nabla}_K : \widehat{H}_h(K) \longrightarrow \mathcal{P}_1(K) \subseteq \widehat{H}_h(K)$ for each $q_h \in \widehat{H}_h(K)$ as the solution of

$$B^K_{\nabla}\big(\Pi^{\nabla}_K q_h, r\big) = B^K_{\nabla}(q_h, r) \qquad \forall\, r \in \mathcal{P}_1(K),$$
$$\widehat{\Pi^{\nabla}_K q_h} = \widehat{q}_h,$$

where $\widehat{q_h}$ is defined in (3.2). We have that the operator Π_K^∇ is explicitly computable using the set $\mathbf{P}_1$ (see Beirão da Veiga *et al.*, 2016a). We introduce our local virtual space:

$$H_h(K) := \left\{ q_h \in \widehat{H}_h(K) : \left(q_h - \Pi_K^\nabla q_h, 1\right)_{0,K} = 0 \right\}.$$

Moreover, it is easy to see that the set $\mathbf{P}_1$ constitutes a set of degrees of freedom for $H_h(K)$ (see Beirão da Veiga *et al.*, 2016a).

We can now present the global virtual space to approximate the fluid pressure: for each decomposition $\mathcal{T}_h$ of Ω into simple polygons K, we define

$$H_h := \left\{ q_h \in \widetilde{H}^1(\Omega) : q_h|_K \in H_h(K) \right\}.$$

A set of degrees of freedom for H_h is given by the values of q_h at the vertices of $\mathcal{T}_h$.

Now, we will continue with the construction of the discrete bilinear form and the linear functional of problem (4.23). To do that, for each $K \in \mathcal{T}_h$, we consider the L^2-projection onto the space $[\mathcal{P}_0(K)]^2$. For $\mathbf{v} \in [L^2(K)]^2$, $\boldsymbol{\Pi}_0^K \mathbf{v} \in [\mathcal{P}_0(K)]^2$ is the unique function such that

$$\int_K \left(\mathbf{v} - \boldsymbol{\Pi}_0^K \mathbf{v}\right) \cdot \mathbf{q} = 0 \qquad \forall\ \mathbf{q} \in [\mathcal{P}_0(K)]^2. \tag{4.26}$$

Remark 4.13 For each $q_h \in H_h$ the function $\boldsymbol{\Pi}_0^K \nabla q_h$ is computable using the degrees of freedom $\mathbf{P}_1$ (see Beirão da Veiga *et al.*, 2016a).

Let $s_\nabla^K(\cdot,\cdot)$ be any symmetric positive definite bilinear form such that

$$c_4 B_\nabla^K(q_h, q_h) \le s_\nabla^K(q_h, q_h) \le c_5 B_\nabla^K(q_h, q_h) \qquad \forall\ q_h \in H_h(K), \text{ with } \Pi_K^\nabla q_h = 0, \tag{4.27}$$

for some positive constants c_4 and c_5 independent of K. A classical choice for the stabilizing bilinear form $s_\nabla^K(\cdot,\cdot)$ satisfying (4.27) is given by the Euclidean scalar product associated to the degrees of freedom $\mathbf{P}_1$ (see Beirão da Veiga *et al.*, 2013; Cangiani *et al.*, 2017b):

$$s_\nabla^K(p_h, q_h) := \sum_{i=1}^{N_V^K} p_h(V_i) q_h(V_i), \tag{4.28}$$

where V_i are the vertices of K, with $1 \le i \le N_V^K$. A proof that the bilinear form defined in (4.28) satisfies the property (4.27) is given in Cangiani *et al.* (2017b, Proposition 5.3).

Then, we set

$$B_\nabla^h(p_h, q_h) := \sum_{K \in \mathcal{T}_h} B_\nabla^{h,K}(p_h, q_h) \qquad \forall\ p_h, q_h \in H_h,$$

where

$$B_{\nabla}^{h,K}(p_h,q_h) := \int_K \boldsymbol{\Pi}_0^K \nabla p_h \cdot \boldsymbol{\Pi}_0^K \nabla q_h + s_{\nabla}^K\big(p_h - \Pi_K^{\nabla} p_h, q_h - \Pi_K^{\nabla} q_h\big), \tag{4.29}$$

for all $p_h, q_h \in H_h(K)$. The following result gives us consistency and stability properties of the local discrete bilinear form $B_{\nabla}^{h,K}(\cdot,\cdot)$.

Proposition 4.14 The local bilinear forms $B_{\nabla}^{K}(\cdot,\cdot)$ and $B_{\nabla}^{h,K}(\cdot,\cdot)$ defined in (4.24) and (4.29), respectively, satisfy the following properties:

- Consistency: for each $h > 0$ and each $K \in \mathcal{T}_h$, we have

$$B_{\nabla}^{h,K}(q_h, r) = B_{\nabla}^{K}(q_h, r) \qquad \forall\, r \in \mathcal{P}_1(K), \quad \forall\, q_h \in H_h(K). \tag{4.30}$$

- Stability: there exist positive constants α_7, α_8, independent of h_K and K, such that

$$\alpha_7 B_{\nabla}^{K}(q_h, q_h) \le B_{\nabla}^{h,K}(q_h, q_h) \le \alpha_8 B_{\nabla}^{K}(q_h, q_h) \qquad \forall\, q_h \in H_h(K). \tag{4.31}$$

The next step consists of constructing an approximation of the right-hand side (4.25), which depends on the stream function ψ and the source term $\mathbf{f}$. With this aim, from now on, we assume that the discrete problem (3.19) has been solved with $k = 3$. So, $\psi_h \in W_h$ is available and satisfies the error bound in Theorem 4.6.

First, for any $K \in \mathcal{T}_h$, we consider the L^2-projection onto $\mathcal{P}_1(K)$: For $v \in L^2(K)$, $\Pi_1^K v \in \mathcal{P}_1(K)$ is the unique function such that

$$\big(v - \Pi_1^K v, r\big)_{0,K} = 0 \qquad \forall\, r \in \mathcal{P}_1(K). \tag{4.32}$$

Now, for each $K \in \mathcal{T}_h$, we define the following discrete linear functional:

$$\begin{aligned} G^{\psi_h,K}(q_h) := \int_K \mathbf{f} \cdot \boldsymbol{\Pi}_0^K \nabla q_h - \int_K \mathbb{K}^{-1} \boldsymbol{\Pi}_2^K \mathbf{curl}\, \psi_h \cdot \boldsymbol{\Pi}_0^K \nabla q_h \\ + \nu \int_K \mathbf{curl}\, (\Pi_1^K(\Delta \psi_h)) \cdot \boldsymbol{\Pi}_0^K \nabla q_h \qquad \forall\, q_h \in H_h(K), \end{aligned}$$

where $\boldsymbol{\Pi}_2^K$ is the projection defined in (3.4) (with $k = 3$) and Π_1^K is the projection defined in (4.32). We observe that both functions are fully computable for $\psi_h \in W_h$.

We define

$$G^{\psi_h}(q_h) := \sum_{K \in \mathcal{T}_h} G^{\psi_h,K}(q_h) \qquad \forall\, q_h \in H_h. \tag{4.33}$$

Thanks to Remark 4.13 we have that $G^{\psi_h}(\cdot)$ is computable using the degrees of freedom in H_h.

Therefore, we propose the following virtual element discretization to recover the fluid pressure: find $p_h \in H_h$ such that

$$B^h_{\nabla}(p_h, q_h) = G^{\psi_h}(q_h) \qquad \forall\, q_h \in H_h. \tag{4.34}$$

We observe that by virtue of (4.31) the bilinear form $B^h_{\nabla}(\cdot,\cdot)$ is bounded. Moreover, the following result states that $B^h_{\nabla}(\cdot,\cdot)$ is elliptic, thanks to the generalized Poincaré inequality.

LEMMA 4.15 There exists a constant $\lambda > 0$, independent of ν and h, such that

$$B^h_{\nabla}(q_h, q_h) \geq \lambda \left\| q_h \right\|^2_{1,\Omega} \qquad \forall\, q_h \in H_h.$$

Next, we will prove that the linear functional defined in (4.33) is bounded. To do that we consider the following approximation result (see Cangiani *et al.*, 2017b).

PROPOSITION 4.16 If the assumption **A**2 is satisfied then there exists a constant $C > 0$, such that for every $v \in H^2(K)$, there exists $v_\pi \in \mathcal{P}_1(K)$, such that

$$\|v - v_\pi\|_{0,K} + h_K|v - v_\pi|_{1,K} \leq C h_K^2 |v|_{2,K}.$$

For the projections $\boldsymbol{\Pi}^K_0$ and Π^K_1 defined in (4.26) and (4.32), respectively, we have the following approximation result (see Brenner & Scott, 2008; Beirão da Veiga *et al.*, 2016a; Gatica *et al.*, 2018a).

PROPOSITION 4.17 Let $\boldsymbol{\Pi}^K_0$ and Π^K_1 be the projections defined in (4.26) and (4.32), respectively. If the assumption **A**2 is satisfied then the following approximation properties hold true: there exist constants $\hat{C}, \tilde{C} > 0$, independent of h_K, such that

$$\|v - \Pi^K_1 v\|_{0,K} \leq \hat{C} h^\delta_K |v|_{\delta,K} \quad \forall\, v \in H^\delta(K), \qquad 0 \leq \delta \leq 2,$$
$$\|\mathbf{v} - \boldsymbol{\Pi}^K_0 \mathbf{v}\|_{0,K} \leq \tilde{C} h^\epsilon_K |\mathbf{v}|_{\epsilon,K} \quad \forall\, \mathbf{v} \in [H^\epsilon(K)]^2, \quad 0 \leq \epsilon \leq 1.$$

Now, we consider the following interpolation result on the virtual space H_h (see Mora *et al.*, 2015; Cangiani *et al.*, 2017a,b).

PROPOSITION 4.18 If the assumptions **A**1 and **A**2 are satisfied then there exists a constant $C > 0$, independent of h, such that for each $v \in H^2(\Omega)$ there exists $v_I \in H_h$, such that

$$\|v - v_I\|_{0,\Omega} + h|v - v_I|_{1,\Omega} \leq C h^2 |v|_{2,\Omega}.$$

In order to establish the well posedness of (4.34) we will assume that the family of polygonal meshes $\mathcal{T}_h$ is quasi-uniform.

A3: For each $h > 0$ and for each $K \in \mathcal{T}_h$ there exists a constant $\widehat{c} > 0$, independent of h, such that $h_K \geq \widehat{c}\, h$.

We have the following inverse inequality on a polygon (see Chen & Huang, 2018, Lemma 3.1).

LEMMA 4.19 If the assumptions **A1** and **A2** are satisfied then there exists $\widetilde{C} > 0$, independent of h, such that

$$|q|_{1,K} \leq \widetilde{C} h_K^{-1} \|q\|_{0,K} \qquad \forall\, q \in \mathcal{P}_\ell(K), \quad \ell \geq 0.$$

The following result establishes that the functional $G^{\psi_h}(\cdot)$ defined in (4.33) is bounded.

PROPOSITION 4.20 Let $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^2(K)]^2$ for all $K \in \mathcal{T}_h$. Let ψ be the unique solution of the problem (2.3). If the assumptions **A**1, **A**2 and **A**3 are satisfied and $\psi \in H^3(\Omega)$, then the functional $G^{\psi_h} : H_h \to \mathbb{R}$ defined in (4.33) is linear and bounded.

Proof. By using triangular and Cauchy–Schwarz inequalities and the stability of the projections $\boldsymbol{\Pi}_2^K$ and $\boldsymbol{\Pi}_0^K$ (see Lemma 4.2 and Proposition 4.17), we have

$$|G^{\psi_h}(q_h)| \leq C\|\mathbf{f}\|_{0,\Omega}\|q_h\|_{1,\Omega} + C\|\psi_h\|_{1,\Omega}\|q_h\|_{1,\Omega} + \sum_{K\in\mathcal{T}_h} \nu \int_K |\mathbf{curl}\,(\Pi_1^K(\Delta\psi_h)) \cdot \boldsymbol{\Pi}_0^K \nabla q_h|. \tag{4.35}$$

Now, adding and subtracting the term $\mathbf{curl}\,(\Delta\psi)\cdot\nabla q_h$ in the last term on the right-hand side above, and using the definition of $\boldsymbol{\Pi}_0^K$ together with the triangular and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}
\int_K |\mathbf{curl}\,(\Pi_1^K(\Delta\psi_h)) \cdot \boldsymbol{\Pi}_0^K \nabla q_h| &\leq \int_K \left|\left(\mathbf{curl}\,(\Pi_1^K \Delta\psi_h) - \mathbf{curl}\,(\Delta\psi)\right)\cdot\nabla q_h\right| + \int_K |\mathbf{curl}\,(\Delta\psi)\cdot\nabla q_h| \\
&\leq \left\|\mathbf{curl}\,\left(\Delta\psi - \Pi_1^K\Delta\psi_h\right)\right\|_{0,K}\|\nabla q_h\|_{0,K} + \|\mathbf{curl}\,(\Delta\psi)\|_{0,K}\|\nabla q_h\|_{0,K} \\
&\leq C|\Delta\psi - \Pi_1^K\Delta\psi_h|_{1,K}\|q_h\|_{1,K} + C|\psi|_{3,K}\|\nabla q_h\|_{0,K} \\
&\leq C\big(\left|\Delta\psi - \Pi_1^K\Delta\psi\right|_{1,K} + \left|\Pi_1^K\Delta\psi - \Pi_1^K\Delta\psi_h\right|_{1,K} + C|\psi|_{3,K}\big)\|q_h\|_{1,K} \\
&\leq C\big(C\left|\Delta\psi\right|_{1,K} + \left|\Pi_1^K(\Delta\psi - \Delta\psi_h)\right|_{1,K} + C|\psi|_{3,K}\big)\|q_h\|_{1,K} \\
&\leq C\big(|\psi|_{3,K} + \left|\Pi_1^K(\Delta\psi - \Delta\psi_h)\right|_{1,K}\big)\|q_h\|_{1,K},
\end{aligned} \tag{4.36}$$

where we have also added the term $\Pi_1^K\Delta\psi$ and used the stability properties of the projector Π_1^K (see Proposition 4.17).

Now, we use the inverse inequality in Lemma 4.19, Assumption **A**3 and Theorem 4.6, to obtain

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \nu \int_K |\mathbf{curl}\,(\Pi_1^K(\Delta\psi_h)) \cdot \boldsymbol{\Pi}_0^K \nabla q_h| &\leq C \sum_{K\in\mathcal{T}_h} \left(\nu|\psi|_{3,K} + \widetilde{C}\nu h_K^{-1}\|\Pi_1^K(\Delta\psi - \Delta\psi_h)\|_{0,K}\right)\|q_h\|_{1,K} \\
&\leq C \sum_{K\in\mathcal{T}_h} \left(\nu|\psi|_{3,K} + \widetilde{C} h_K^{-1}\nu\|\Delta\psi - \Delta\psi_h\|_{0,K}\right)\|q_h\|_{1,K} \\
&\leq C \sum_{K\in\mathcal{T}_h} \left(\nu|\psi|_{3,K} + \widetilde{C}\nu h_K^{-1}|\psi - \psi_h|_{2,K}\right)\|q_h\|_{1,K}
\end{aligned}$$

$$\leq C\left(\nu\|\psi\|_{3,\Omega} + h_K^{-1}\|\psi - \psi_h\|_W\right)\|q_h\|_{1,\Omega}$$
$$\leq C\left(|\mathbf{f}|_{2,h} + \|\psi\|_{3,\Omega}\right)\|q_h\|_{1,\Omega}, \tag{4.37}$$

where the constant $C > 0$ depends on the constant $\widehat{c}$ in Assumption **A**3. Finally, from (4.35) and (4.37), we get

$$|G^{\psi_h}(q_h)| \leq C\left(\|\psi_h\|_{1,\Omega} + \|\psi\|_{3,\Omega} + |\mathbf{f}|_{2,h}\right)\|q_h\|_{1,\Omega} \leq C\left(\|\mathbf{f}\|_{0,\Omega} + |\mathbf{f}|_{2,h}\right)\|q_h\|_{1,\Omega}.$$

□

As a consequence of the last result and the Lax–Milgram Theorem we have the following result.

THEOREM 4.21 Let $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^2(K)]^2$ for all $K \in \mathcal{T}_h$. Let ψ and ψ_h be the unique solutions of (2.3) and (3.19), respectively. Suppose that **A**1, **A**2 and **A**3 are satisfied and that $\psi \in H^3(\Omega)$. Then, problem (4.34) admits a unique solution $p_h \in H_h$ and there exists $C > 0$, independent of ν and h, such that

$$\|p_h\|_{1,\Omega} \leq C\left(\|\mathbf{f}\|_{0,\Omega} + |\mathbf{f}|_{2,h}\right).$$

In what follows we will establish the order of convergence of the discrete scheme (4.34). We begin with the following result, which proof follows the same arguments in Beirão da Veiga *et al.* (2013); Cangiani *et al.* (2017b).

PROPOSITION 4.22 Let p and p_h be the unique solutions of problems (4.23) and (4.34), respectively. If the assumptions **A**1, **A**2 and **A**3 are satisfied then there exists $C > 0$, independent of ν and h, such that

$$\|p - p_h\|_{1,\Omega} \leq C\left(\|G^\psi - G^{\psi_h}\|_{H_h'} + \|p - p_I\|_{1,\Omega} + |p - p_\pi|_{1,h}\right),$$

for all $p_I \in H_h$ and for each $p_\pi \in L^2(\Omega)$ such that $p_\pi|_K \in \mathcal{P}_1(K)$ for all $K \in \mathcal{T}_h$, where

$$\|G^\psi - G^{\psi_h}\|_{H_h'} := \sup_{\substack{q_h \in H_h \\ q_h \neq 0}} \frac{|G^\psi(q_h) - G^{\psi_h}(q_h)|}{\|q_h\|_{1,\Omega}}.$$

Now, we will bound the term $\|G^\psi - G^{\psi_h}\|_{H_h'}$, under the assumptions **A**3 and $\psi \in H^4(\Omega)$.

PROPOSITION 4.23 Let $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^2(K)]^2$ $\forall K \in \mathcal{T}_h$, ψ and ψ_h be the unique solutions of the problem (2.3) and problem (3.19), respectively. Let $G^\psi(\cdot)$ and $G^{\psi_h}(\cdot)$ the functional defined in (4.25) and (4.33), respectively. Suppose that **A**1, **A**2 and **A**3 are satisfied and that $\psi \in H^4(\Omega)$, then we have the following estimate

$$\|G^\psi - G^{\psi_h}\|_{H_h'} \leq Ch\left(\|\mathbf{f}\|_{0,\Omega} + |\mathbf{f}|_{1,h} + |\mathbf{f}|_{2,h}\right).$$

Proof. Let $q_h \in H_h$, then using the definition of $G^{\psi}(\cdot)$ and $G^{\psi_h}(\cdot)$, and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\left|G^{\psi}(q_h) - G^{\psi_h}(q_h)\right| \leq & \sum_{K\in\mathcal{T}_h} \left| \int_K \mathbf{f}\cdot \left(\nabla q_h - \boldsymbol{\Pi}_0^K \nabla q_h\right)\right| \\
& + \sum_{K\in\mathcal{T}_h} \left| \int_K \mathbb{K}^{-1}\mathbf{curl}\,\psi \cdot \nabla q_h - \mathbb{K}^{-1}\boldsymbol{\Pi}_2^K \mathbf{curl}\,\psi_h \cdot \boldsymbol{\Pi}_0^K \nabla q_h \right| \\
& + \sum_{K\in\mathcal{T}_h} \nu \left| \int_K \mathbf{curl}\,(\Delta\psi_h)\cdot\nabla q_h - \mathbf{curl}\,\left(\Pi_1^K \Delta\psi\right)\cdot \boldsymbol{\Pi}_0^K \nabla q_h \right| \\
:= & T_1 + T_2 + T_3.
\end{aligned}$$

Now, using standard arguments in the virtual element literature (see Beirão da Veiga *et al.* (2013, 2016a); Cangiani *et al.* (2017b)), we have that

$$T_1 \leq Ch|\mathbf{f}|_{1,h}\left\|q_h\right\|_{1,\Omega}, \tag{4.38}$$

and

$$T_2 \leq Ch\left(\|\psi\|_{4,\Omega} + |\mathbf{f}|_{1,h} + \|\psi_h\|_{2,\Omega}\right)\|q_h\|_{1,\Omega}. \tag{4.39}$$

Finally, to estimate the term T_3, we proceed as in (4.36) and (4.37).

$$\begin{aligned}
T_3 &= \sum_{K\in\mathcal{T}_h} \nu \left| \int_K \mathbf{curl}\,(\Delta\psi)\cdot\nabla q_h - \mathbf{curl}\,\left(\Pi_1^K\Delta\psi_h\right)\cdot\nabla q_h \right| \\
&= \sum_{K\in\mathcal{T}_h} \nu \left| \int_K \mathbf{curl}\,\left(\Delta\psi - \Pi_1^K\Delta\psi_h\right)\cdot\nabla q_h \right| \\
&\leq \sum_{K\in\mathcal{T}_h} \nu \left\| \mathbf{curl}\,\left(\Delta\psi - \Pi_1^K\Delta\psi_h\right)\right\|_{0,K} \|\nabla q_h\|_{0,K} \\
&\leq \sum_{K\in\mathcal{T}_h} C\nu|\Delta\psi - \Pi_1^K\Delta\psi_h|_{1,K}\|q_h\|_{1,K} \\
&\leq \sum_{K\in\mathcal{T}_h} C\left(\nu\left|\Delta\psi - \Pi_1^K\Delta\psi\right|_{1,K} + \nu\left|\Pi_1^K\Delta\psi - \Pi_1^K\Delta\psi_h\right|_{1,K}\right)\|q_h\|_{1,K} \\
&\leq \sum_{K\in\mathcal{T}_h} C\left(C\nu h_K\,|\Delta\psi|_{2,K} + \nu\left|\Pi_1^K(\Delta\psi - \Delta\psi_h)\right|_{1,K}\right)\|q_h\|_{1,K} \\
&\leq \sum_{K\in\mathcal{T}_h} C\left(\nu h\|\psi\|_{4,K} + \nu\left|\Pi_1^K(\Delta\psi - \Delta\psi_h)\right|_{1,K}\right)\|q_h\|_{1,K}.
\end{aligned} \tag{4.40}$$

Now, using Lemma 4.19, assumption **A**3 and Theorem 4.6, we have that

$$\begin{aligned}
\nu|\Pi_1^K(\Delta\psi - \Delta\psi_h)|_{1,K} &\leq \widetilde{C}\nu h_K^{-1}\|\Pi_1^K(\Delta\psi - \Delta\psi_h)\|_{0,K} \leq \widetilde{C}\nu h_K^{-1}\|\Delta\psi - \Delta\psi_h\|_{0,K} \\
&\leq \widetilde{C}\nu h_K^{-1}\|\Delta(\psi - \psi_h)\|_{0,\Omega} \leq \widetilde{C}h_K^{-1}\|\psi - \psi_h\|_W \\
&\leq C\widetilde{C}h_K^{-1}h^2\left(|\mathbf{f}|_{2,h} + \|\psi\|_{4,\Omega}\right) \\
&\leq \frac{C\widetilde{C}}{\widehat{c}}h\left(|\mathbf{f}|_{2,h} + \|\psi\|_{4,\Omega}\right).
\end{aligned} \tag{4.41}$$

Then from (4.40), (4.41) and the Cauchy–Schwarz inequality, we get

$$T_3 \leq Ch\left(\|\psi\|_{4,\Omega} + |\mathbf{f}|_{2,h}\right)\|q_h\|_{1,\Omega}. \tag{4.42}$$

Therefore, using the estimates (4.38), (4.39) and (4.42), we obtain

$$\begin{aligned}
|G^{\psi}(q_h) - G^{\psi_h}(q_h)| &\leq Ch\left(\|\psi_h\|_{2,\Omega} + \|\psi\|_{4,\Omega} + |\mathbf{f}|_{1,h} + |\mathbf{f}|_{2,h}\right)\|q_h\|_{1,\Omega} \\
&\leq Ch\left(\|\mathbf{f}\|_{0,\Omega} + |\mathbf{f}|_{1,h} + |\mathbf{f}|_{2,h}\right)\|q_h\|_{1,\Omega}.
\end{aligned}$$

The proof is complete. □

The following theorem provides the rate of convergence of our virtual scheme (4.34). The proof follows from Propositions 4.22, 4.23, 4.16 and 4.18.

THEOREM 4.24 Let $\mathbf{f} \in [L^2(\Omega)]^2$ such that $\mathbf{f}|_K \in [H^2(K)]^2$ for all $K \in \mathcal{T}_h$. Let ψ, ψ_h, p and p_h be the unique solutions of problems (2.3), (3.19), (4.23) and (4.34), respectively. Suppose that **A**1, **A**2 and **A**3 are satisfied and that $\psi \in H^4(\Omega)$. Then, there exists $C > 0$, independent of ν and h, such that

$$\|p - p_h\|_{1,\Omega} \leq Ch\left(\|\mathbf{f}\|_{0,\Omega} + |\mathbf{f}|_{1,h} + |\mathbf{f}|_{2,h}\right).$$

Now, we state in the following remark the approximation properties of the VEM (3.20).

REMARK 4.25 We note that for the alternative discretization problem (3.20), we can prove analogous rate of convergences as in Theorems 4.6, 4.9, 4.10 and 4.24. We do not include proofs to avoid repeating the steps of the results. We will present a numerical test to confirm the error estimates in this case.

5. Numerical results

In this section we present some numerical experiments to test the practical performance of the proposed virtual element discretizations (3.19) and (4.34) and in order to confirm the theoretical results. We will test the method for the cases $k = 2$ and $k = 3$.

We have tested the method by using different families of meshes (see Fig. 1):

- $\mathcal{T}_h^1$: Triangular meshes;
- $\mathcal{T}_h^2$: Trapezoidal meshes;
- $\mathcal{T}_h^3$: Sequence of Centroidal Voronoi Tessellation;
- $\mathcal{T}_h^4$: Hexagonal meshes.

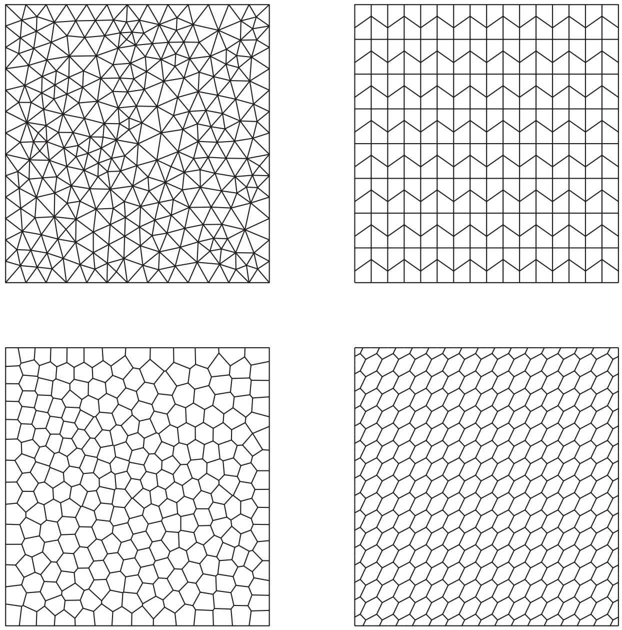

FIG. 1. Sample meshes. $\mathcal{T}_h^1$ (top left), $\mathcal{T}_h^2$ (top right), $\mathcal{T}_h^3$ (bottom left), $\mathcal{T}_h^4$ (bottom right).

In order to test the convergence properties of the proposed scheme we introduce the following quantities:

$$\mathtt{e}_W(\psi)=\mathtt{error}(\psi,W):=\left(\sum_{K\in\mathcal{T}_h}|\psi-\Pi_K^{k,\Delta}\psi_h|_{1,K}^2+\nu|\psi-\Pi_K^{k,\Delta}\psi_h|_{2,K}^2\right)^{1/2},$$

$$\mathtt{e}_i(\psi)=\mathtt{error}(\psi,H^i):=\left(\sum_{K\in\mathcal{T}_h}|\psi-\Pi_K^{k,\Delta}\psi_h|_{i,K}^2\right)^{1/2},\quad i=0,1,$$

$$\mathtt{e}_1(p)=\mathtt{error}(p,H^1):=\left(\sum_{K\in\mathcal{T}_h}\left|p-\Pi_K^{\nabla}p_h\right|_{1,K}^2\right)^{1/2},$$

$$\mathtt{e}_1(\mathbf{u})=\mathtt{error}(\mathbf{u},H^1):=\left(\sum_{K\in\mathcal{T}_h}|||\mathbf{u}-\boldsymbol{\Pi}_{k-1}^K\mathbf{u}_h|||_{1,K}^2\right)^{1/2}.$$

We will compute experimental rates of convergence for each individual error as follows:

$$\mathtt{r}_i(\cdot):=\frac{\log(\mathtt{e}_i(\cdot)/\mathtt{e}_i'(\cdot))}{\log(h/h')},\quad i=0,1,W,$$

where h, h' denote two consecutive mesh sizes with their respective errors $\mathtt{e}_i$ and $\mathtt{e}_i'$.

TABLE 1 *Test* 1. *Errors and experimental rates for the stream function* ψ_h *and velocity* $\mathbf{u}_h$, *with* $k = 2$, *using the meshes* $\mathcal{T}_h^1$ *and different values of* ν

ν	d*ofs*	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$
					$k = 2$					
	156	1/8	2.9811e–3	—	6.1713e–2	—	1.5025e–0	—	1.5813e–0	—
	717	1/16	7.6129e–4	1.96	1.5555e–2	1.98	7.6189e–1	0.97	8.2123e–1	0.94
1e0	3075	1/32	1.6005e–4	2.24	4.0753e–3	1.93	3.9142e–1	0.96	4.1752e–1	0.97
	12567	1/64	4.3471e–5	1.88	9.7274e–4	2.06	1.9080e–1	1.03	2.0155e–1	1.05
	50445	1/128	9.6899e–6	2.16	2.4631e–4	1.98	9.6036e–2	0.99	1.0202e–1	0.98
	156	1/8	3.8465e–3	—	4.4890e–2	—	6.7910e–2	—	6.9083e–2	—
	717	1/16	5.0859e–4	2.91	1.1968e–2	1.90	2.7103e–2	1.32	2.8244e–2	1.29
1e-3	3075	1/32	4.2636e–5	3.57	3.5945e–3	1.73	1.2622e–2	1.10	1.3318e–2	1.08
	12567	1/64	5.3640e–6	1.99	1.0142e–3	1.82	6.0360e–3	1.06	6.3852e–3	1.06
	50445	1/128	1.3833e–6	1.95	2.6831e–4	1.91	3.0356e–3	0.99	3.2208e–3	0.98
	156	1/8	3.7868e–3	—	4.4383e–2	—	4.4417e–2		4.4417e–2	—
	717	1/16	5.5121e–4	2.78	1.0773e–2	2.04	1.0808e–2	2.03	1.0808e–2	2.03
1e-6	3075	1/32	5.6753e–5	3.27	2.3948e–3	2.16	2.4349e–3	2.15	2.4347e–3	2.15
	12567	1/64	5.8777e–6	3.27	5.1550e–4	2.21	5.5714e–4	2.12	5.5702e–4	2.12
	50445	1/128	8.0838e–7	2.86	1.2773e–4	2.01	1.6571e–4	1.74	1.6591e–4	1.74

5.1 *Test 1: convergence history*

In this numerical test we solve the Brinkman problem (2.1) on the square domain $\Omega := (0,1)^2$, with different values for the viscosity ν and with the following tensor:

$$\mathbb{K}^{-1}(x,y) := \begin{pmatrix} \sin(2\pi x) + 1.1 & 10^{-6} \\ 10^{-6} & \sin(2\pi y) + 1.1 \end{pmatrix}.$$

In addition, we take the load term $\mathbf{f}$ in such a way that the analytical solution is given by:

$$\mathbf{u}(x,y) = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix}, \qquad p(x,y) = x^3y^3 - \frac{1}{16},$$
$$\psi(x,y) = 100x^2(1-x)^2y^2(1-y)^2.$$

We report in Table 1, the errors and the orders of convergence for the stream function ψ_h obtained with the VEM (3.19) and for the post-process velocity $\mathbf{u}_h$ (cf. (4.21)). We take different values of ν; 1e0, 1e-3, 1e-6. The polynomial degree is given by $k = 2$ and we consider the sequences of meshes $\mathcal{T}_h^1$. In this case it can be clearly seen that, the method converges with orders predicted in Theorems 4.6, 4.9, 4.10 and 4.11 for the stream function and velocity.

In Table 2 we report the errors and the orders of convergence for the stream function ψ_h obtained with the VEM (3.19), for the post-process velocity $\mathbf{u}_h$ (cf. (4.21)), and for the pressure p_h obtained with the VEM (4.34). In this case the polynomial degree is given by $k = 3$ and we take different values of ν; 1e-3, 1e-6 and we consider the sequences of meshes $\mathcal{T}_h^4$.

We note that the results reported in Table 2 confirm, for both methods, the convergence rates predicted in Theorems 4.6, 4.9, 4.10, 4.11 for the stream function and velocity, and the first-order convergence rate in the discrete H^1-norm (in agreement with Theorem 4.24) for the pressure.

TABLE 2 *Test* 1. *Errors and experimental rates for the stream function* ψ_h*, velocity field* $\mathbf{u}_h$ *and the pressure* p_h*, with* $k = 3$*, using the meshes* $\mathcal{T}_h^4$ *and different values of* ν

ν	dofs	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$	$\mathtt{e}_1(p)$	$\mathtt{r}_1(p)$
						$k = 3$						
	592	1/8	6.9697e–4	—	7.4549e–3	—	1.1202e–2	—	1.1154e–2	—	1.2461e–1	—
1e-3	2336	1/16	3.1100e–5	4.48	7.1764e–4	3.37	2.2583e–3	2.31	2.3192e–3	2.26	6.7001e–2	0.89
	9280	1/32	1.3219e–6	4.45	8.6184e–5	3.05	5.5778e–4	2.01	5.8701e–4	1.98	3.4758e–2	0.94
	36992	1/64	7.0976e–8	4.21	1.0872e–5	2.98	1.4126e–4	1.98	1.5043e–4	1.96	1.7706e–2	0.97
	592	1/8	7.6336e–4	—	8.2932e–3	—	8.2978e–3	—	8.2976e–3	—	1.2455e–1	—
1e-6	2336	1/16	3.8478e–5	4.31	8.3266e–4	3.31	8.3571e–4	3.31	8.3577e–4	3.31	6.6998e–2	0.89
	9280	1/32	1.9187e–6	4.32	9.5227e–5	3.12	9.6922e–5	3.10	9.7015e–5	3.10	3.4757e–2	0.94
	36992	1/64	9.8283e–8	4.28	1.1422e–5	3.05	1.2289e–5	2.97	1.2349e–5	2.97	1.7706e–3	0.97

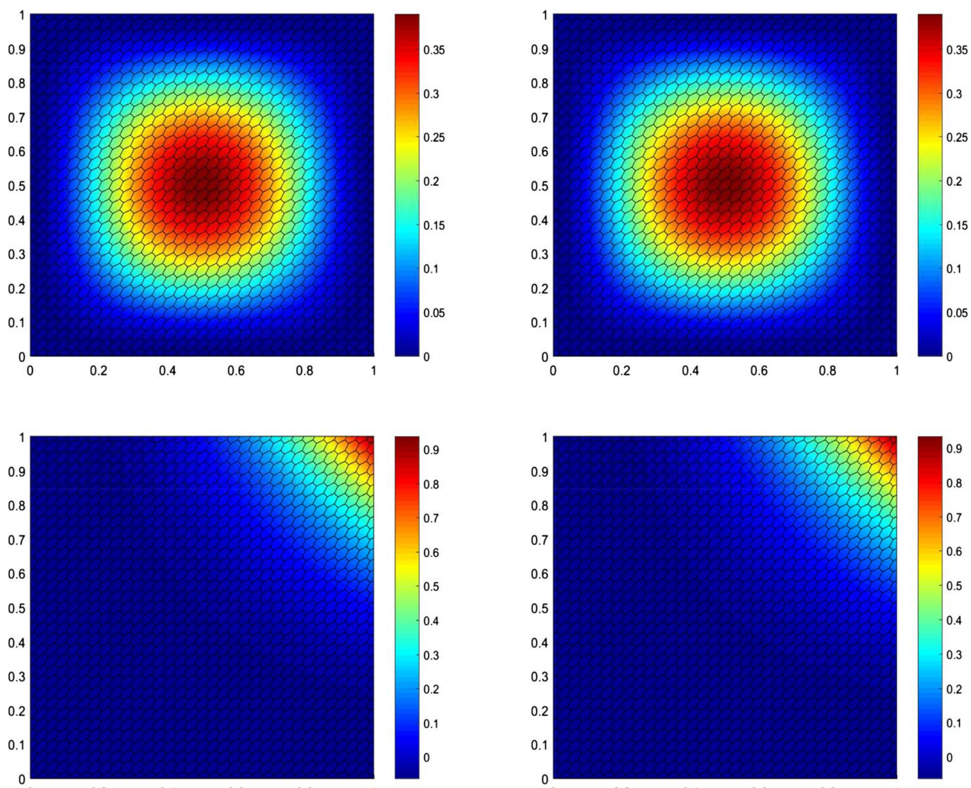

FIG. 2. Test 1. Exact and approximate solutions ψ; ψ_h; p; p_h (top left, top right, bottom left, bottom right, respectively) using the VE methods (3.19) and (4.34) with $\mathcal{T}_h^4$, $h = 1/32$, $k = 3$ and $\nu = 1\mathrm{e}-6$.

Figure 2 shows plots of the exact (left) and computed (right) stream function and pressure obtained with the VEMs analyzed in this paper, using the meshes $\mathcal{T}_h^4$, with $h = 1/32$, $\nu = 1\mathrm{e}-6$ and polynomial degree $k = 3$.

TABLE 3 *Test 2. Errors and experimental rates for the stream function* ψ_h*, velocity field* $\mathbf{u}_h$ *and the pressure* p_h*, with* $k = 3$*, using the meshes* $\mathcal{T}_h^3$ *and different values of* ν

ν	d*ofs*	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$	$\mathtt{e}_1(p)$	$\mathtt{r}_1(p)$
						$k = 3$						
	455	1/8	1.9311e–5	—	3.0283e–4	—	4.4374e–3	—	4.7685e–3	—	5.2516e–2	—
	2083	1/16	1.0992e–6	4.13	3.9817e–5	2.92	1.1299e–3	1.97	1.3179e–3	1.85	2.5695e–2	1.03
2e-4	8751	1/32	6.0995e–8	4.17	5.4004e–6	2.88	2.8968e–4	1.96	3.6877e–4	1.83	1.2814e–2	1.00
	35927	1/64	4.2882e–9	3.83	6.8727e–7	2.97	7.3705e–5	1.97	9.6097e–5	1.94	6.6766e–2	0.94
	145521	1/128	2.1703e–10	4.30	7.9006e–8	3.12	1.7623e–5	2.06	2.3125e–5	2.05	3.2555e–3	1.03
	455	1/8	1.9384e–4	—	3.7720e–3	—	1.2516e–2	—	1.2628e–2	—	8.7157e–2	—
	2083	1/16	9.4345e–6	4.36	5.3400e–4	2.82	3.6302e–3	1.78	3.6767e–3	1.78	4.7249e–2	0.88
2e-8	8751	1/32	8.2511e–7	3.51	8.6277e–5	2.62	1.0713e–3	1.76	1.0868e–3	1.75	2.4709e–2	0.93
	35927	1/64	7.4428e–8	3.47	1.0898e–5	2.98	2.7792e–4	1.94	2.9335e–4	1.88	1.3366e–2	0.88
	145521	1/128	3.7298e–9	4.31	1.2437e–6	3.13	6.6977e–5	2.05	7.6593e–5	1.93	6.4157e–3	1.05
	455	1/8	1.2177e–3	—	3.4590e–2	—	4.5075e–2	—	4.5142e–2	—	7.4545e–2	—
	2083	1/16	2.5824e–4	2.23	1.0359e–2	1.73	1.8725e–2	1.26	1.8746e–2	1.26	6.1927e–2	0.26
2e-12	8751	1/32	3.5689e–3	2.85	2.2864e–3	2.17	6.7606e–3	1.46	6.7605e–3	1.47	4.1795e–2	0.56
	35927	1/64	3.4144e–6	3.38	3.2606e–4	2.80	1.8847e–3	1.84	1.8864e–3	1.84	2.5123e–2	0.73
	145521	1/128	1.8212e–7	4.22	3.7525e–5	3.11	4.7370e–4	1.99	4.7676e–4	1.98	1.2654e–2	0.98

5.2 *Test 2: ν-dependent solution*

The aim of this numerical example is to test the convergence properties of the proposed VE methods (3.19) and (4.34) by considering the following ν-dependent solution (Mardal *et al.*, 2002):

$$\mathbf{u}(x,y) = \begin{pmatrix} -x\, e^{-xy/\sqrt{\nu}} \\ y\, e^{-xy/\sqrt{\nu}} \end{pmatrix}, \qquad p(x,y) = \sqrt{\nu}\, e^{-x/\sqrt{\nu}} - \nu\left(1 - e^{-1/\sqrt{\nu}}\right),$$

$$\psi(x,y) = \sqrt{\nu}\, e^{-xy/\sqrt{\nu}}.$$

We have taken the load term $\mathbf{f}$ and the boundary conditions according to the above solution. In addition, we consider $\Omega = (0,1)^2$, the permeability tensor $\mathbb{K} := \mathbb{I}$ and different values of the viscosity ν.

In Table 3 we report the errors and the orders of convergence for the stream function ψ_h obtained with the VEM (3.19), for the post-process velocity $\mathbf{u}_h$ (cf. (4.21)), and for the pressure p_h obtained with the VEM (4.34). In this case the polynomial degree is given by $k = 3$, we take different values of ν and we consider the sequences of meshes $\mathcal{T}_h^3$.

It can be seen from Table 3 that, for both methods and for all the values of the viscosity, the convergence rates predicted in Theorems 4.6, 4.9, 4.10, 4.11 and 4.24 are attained for all quantities.

Figure 3 shows plots of the exact (left) and computed (right) stream function and pressure obtained with the VEMs analyzed in this paper, using the meshes $\mathcal{T}_h^3$, with $h = 1/32$, $\nu = 2\mathrm{e}-4$ and polynomial degree $k = 3$.

In Fig. 4, we depict approximate velocity field $\mathbf{u}_h$ obtained from the discrete stream function by using (4.21).

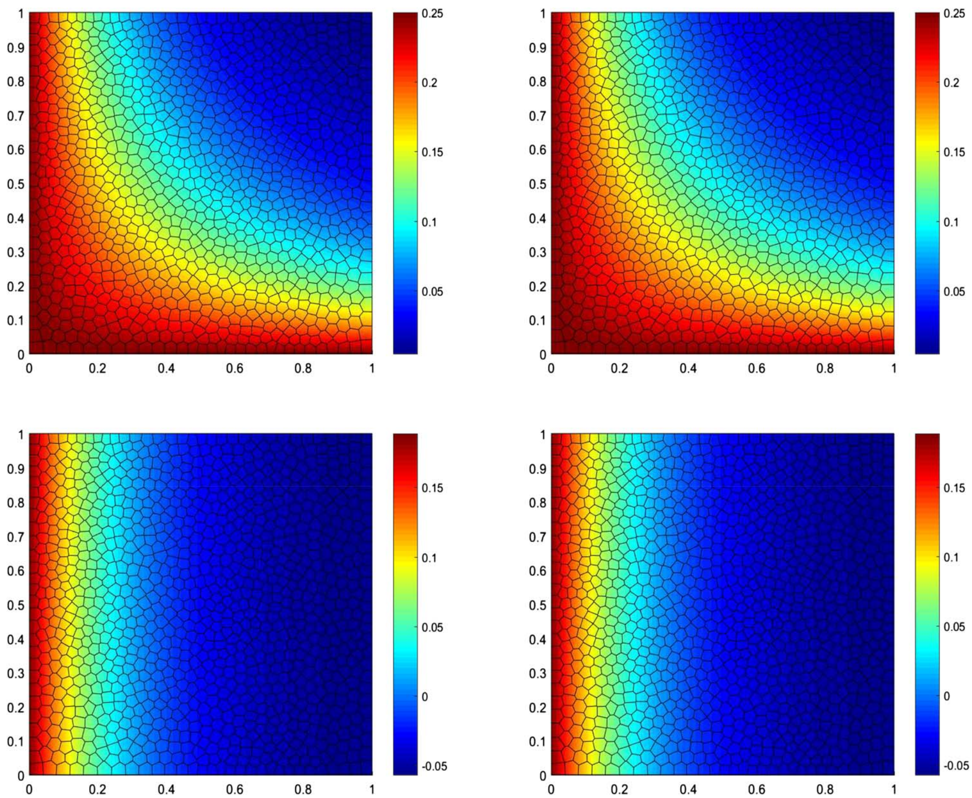

FIG. 3. Test 2. Exact and approximate solutions ψ; ψ_h; p; p_h (top left, top right, bottom left, bottom right, respectively) using the VE methods (3.19) and (4.34) with $\mathcal{T}_h^3$, $h = 1/32$, $k = 3$ and $\nu = 2\mathrm{e}-4$.

5.3 *Test 3: alternative right-hand side*

In this numerical example we test the convergence properties of the proposed VE methods (3.20) and (4.34). We note that the VEM (3.20) is defined by considering the right-hand side (3.18). With this aim we take the square domain $\Omega := (0, 1)^2$, $\nu = 1$ and $\mathbb{K} := \mathbb{I}$. In addition, we take the load term $\mathbf{f}$ in such a way that the analytical solution is given by:

$$\mathbf{u}(x, y) = \frac{2}{\pi^2}\begin{pmatrix} e^{x^2+y^2}\sin(2\pi x)(y\cos(2\pi y) - \pi\sin(2\pi y)) \\ -e^{x^2+y^2}\cos(2\pi y)(\pi\cos(2\pi x) + x\sin(2\pi x)) \end{pmatrix}, \quad p(x, y) = \sin(x) - \sin(y),$$

$$\psi(x, y) = \frac{1}{\pi^2}\sin(2\pi x)\cos(2\pi y)\, e^{x^2+y^2}.$$

We report in Table 4 the errors and the orders of convergence for the stream function $\widetilde{\psi}_h$ obtained with the VEM (3.20) and for the post-process velocity $\mathbf{u}_h$ (cf. (4.21)). In this case the polynomial degree is given by $k = 2$ and we consider the sequences of meshes $\mathcal{T}_h^2$.

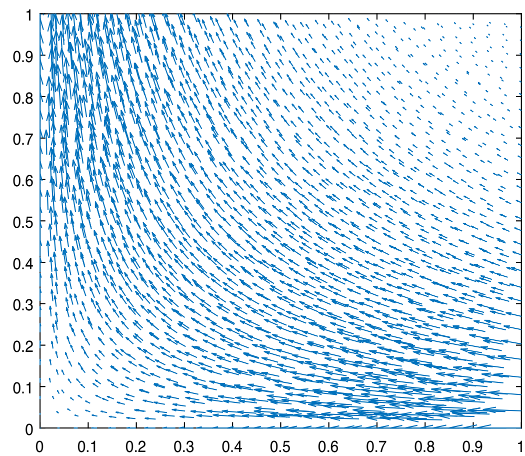

FIG. 4. Test 2. Velocity field obtained from the discrete stream function by using (4.21) and the meshes $\mathcal{T}_h^3$, with $h = 1/32$, $k = 3$ and $\nu = 2\mathrm{e} - 4$.

TABLE 4 *Test* 3. *Errors and experimental rates for the stream function* $\widetilde{\psi}_h$ *and for the velocity* $\mathbf{u}_h$, *with* $k = 2$ *and using the meshes* $\mathcal{T}_h^2$

ν	d*ofs*	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$
					$k = 2$					
	147	1/8	1.0602e–2	—	2.1878e–1	—	3.3961e–0	—	4.0146e–0	—
	675	1/16	2.9717e–3	1.83	7.8144e–2	1.48	1.7432e–0	0.96	2.2643e–0	0.82
1e0	2883	1/32	8.4340e–4	1.81	2.3046e–2	1.76	8.6434e–1	1.00	1.1862e–0	0.93
	11907	1/64	2.2162e–4	1.92	6.1001e–3	1.91	4.2919e–1	1.00	6.0103e–1	0.98
	48387	1/128	5.6244e–5	1.97	1.5525e–3	1.97	2.1416e–1	1.00	3.0163e–1	0.99

In Table 5 we report the errors and the orders of convergence for the stream function $\widetilde{\psi}_h$ obtained with the VEM (3.20) and for the post-process velocity $\mathbf{u}_h$ (cf. (4.21)). We have also computed the pressure p_h by using (4.34) with the discrete stream function $\widetilde{\psi}_h$. In this case the polynomial degree is given by $k = 3$ and we consider the sequences of meshes $\mathcal{T}_h^2$.

Once again it can be clearly seen from Tables 4 and 5 that the methods converge with orders predicted in Theorems 4.6, 4.9, 4.10, 4.11 and 4.24 (see Remark 4.25).

5.4 *Test 4: mesh allowing small edges*

The aim of this final test is to analyze the influence of the mesh assumptions. In this test we compare the performance of VEM when the geometric assumption **A**1 is violated. With this end we solve the Brinkman problem (2.1) on the square domain $\Omega := (0, 1)^2$ and using the family of meshes $\mathcal{T}_h^5$ presented in Fig. 5. We note that the family of meshes $\mathcal{T}_h^5$ has been obtained by gluing two different polygonal meshes. We observe that very small edges appear on the interface of the resulting mesh.

TABLE 5 *Test* 3. *Errors and experimental rates for the stream function* $\widetilde{\psi}_h$*, the velocity* $\mathbf{u}_h$ *and the pressure* p_h*, with* $k = 3$ *and using the meshes* $\mathcal{T}_h^2$

ν	d*ofs*	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$	$\mathtt{e}_1(p)$	$\mathtt{r}_1(p)$
						$k = 3$						
	259	1/8	1.4556e–3	—	1.9235e–2	—	7.1466e–1	—	1.3691e–0	—	4.8476e–0	—
	1155	1/16	9.4798e–5	3.94	2.1420e–3	3.16	1.7901e–1	1.99	3.2453e–1	2.07	1.7332e–0	1.48
1e0	4867	1/32	5.9324e–6	3.99	2.5265e–4	3.08	4.4834e–2	1.99	7.1707e–2	2.17	5.5800e–1	1.63
	19971	1/64	3.6742e–7	4.01	3.1003e–5	3.02	1.1273e–2	1.99	1.6340e–2	2.13	1.8064e–1	1.62
	80899	1/128	2.2808e–8	4.00	3.8572e–6	3.00	2.8321e–3	1.99	3.8780e–3	2.07	6.3333e–2	1.51

We consider the viscosity $\nu = 1$ and $\mathbb{K} = \mathbb{I}$. In addition, we take the load term $\mathbf{f}$ in such a way that the analytical solution is given by:

$$\mathbf{u}(x,y) = \frac{1}{2}\begin{pmatrix} \sin^2(2\pi x)\sin(2\pi y)\cos(2\pi y) \\ -\sin^2(2\pi y)\sin(2\pi x)\cos(2\pi x)\end{pmatrix}, \qquad p(x,y) = \frac{1}{\pi}\sin(2\pi x)\cos(2\pi y),$$

$$\psi(x,y) = \frac{1}{8\pi}\sin^2(2\pi x)\sin^2(2\pi y).$$

We report in Tables 6 and 7 the errors and the orders of convergence for the virtual element schemes (3.19) and (4.34) for $k = 2, 3$, respectively. We note that the convergence rates are in agreement with the rates predicted in Theorems 4.6, 4.9, 4.10, 4.11 and 4.24 for the stream function, velocity and pressure. Even though our theoretical analysis has been strongly developed under assumption **A**1, this numerical example shows that the results of Section 4 should hold true for more general mesh assumptions (Beirão da Veiga *et al.*, 2017; Brenner & Sung, 2018). However, further research is needed in this direction.

6. Conclusions

In this paper we have proposed and analyzed a C^1-VEM of high order for the numerical approximation of the Brinkman equations formulated in terms of the stream function. We have shown that the proposed scheme is well posed by using the framework of the classical Lax–Milgram theory. We derived optimal convergence rates (and robust with respect to viscosity) in viscosity dependent H^2-norm, and using duality argument we also established error estimates in H^1- and L^2-norms. Using the discrete stream function we compute a discrete velocity field by means of a post-process, and error estimates in H^1-norm has been obtained. In addition, we have presented a novel strategy to approximate the fluid pressure, which is based on a discrete virtual scheme for a second-order variational problem with datum coming from the discrete stream function and the load term $\mathbf{f}$. Under the assumptions of convexity and that the family of polygonal meshes $\mathcal{T}_h$ is quasi-uniform, we have written an error estimate in H^1-norm for pressure. The key features of the proposed method are the possibility to use general polygonal meshes, the matrix associated to the linear system turns out to be positive definite and the possibility to recover further variables of interest (velocity and pressure) in a simply way. Possible extensions of this work include the following: (a) weakening of the mesh assumptions, (b) write error estimates with constants independent of the permeability tensor and (c) the study of new VEMs for other stream function formulations of fluid flow problems with pressure recovery.

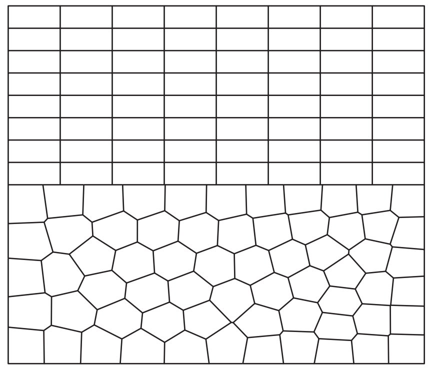

FIG. 5. Mesh with small edges $\mathcal{T}_h^5$.

TABLE 6 *Test 4. Errors and experimental rates for the stream function ψ_h and velocity $\mathbf{u}_h$, with $k = 2$, using the meshes $\mathcal{T}_h^5$*

ν	d*ofs*	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$
					$k = 2$					
	99	1/4	1.2766e–2	—	1.0977e–1	—	1.7734e–0	—	1.9631e–0	—
	489	1/8	6.4358e–3	0.98	3.5125e–2	1.64	9.5290e–1	0.89	1.1574e–0	0.76
1e0	2139	1/16	2.1045e–3	1.51	1.1597e–2	1.59	4.8232e–1	0.98	5.7148e–1	1.01
	8889	1/32	5.5807e–4	1.91	3.7060e–3	1.64	2.3768e–1	1.02	2.6782e–1	1.09
	36165	1/64	1.4552e–4	1.93	1.0230e–3	1.85	1.1811e–1	1.00	1.3198e–1	1.02
	146028	1/128	3.6836e–5	1.98	2.6276e–4	1.96	5.8845e–2	1.00	6.5573e–2	1.00

TABLE 7 *Test 4. Errors and experimental rates for the stream function ψ_h, the velocity $\mathbf{u}_h$ and the pressure p_h, with $k = 3$ and using the meshes $\mathcal{T}_h^5$*

ν	d*ofs*	h	$\mathtt{e}_0(\psi)$	$\mathtt{r}_0(\psi)$	$\mathtt{e}_1(\psi)$	$\mathtt{r}_1(\psi)$	$\mathtt{e}_W(\psi)$	$\mathtt{r}_W(\psi)$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$	$\mathtt{e}_1(p)$	$\mathtt{r}_1(p)$
						$k = 3$						
	163	1/4	2.7991e–3	—	4.2324e–2	—	8.9129e–1	—	1.1630e–0	—	2.5324e–0	—
	779	1/8	3.6920e–4	2.92	5.4587e–3	2.95	2.3872e–1	1.90	4.5266e–1	1.36	1.4643e–0	0.79
1e0	3363	1/16	2.9259e–5	3.65	5.2466e–4	3.37	5.8672e–2	2.02	9.1625e–2	2.30	8.7644e–1	0.74
	13899	1/32	2.0252e–6	3.85	5.3678e–5	3.28	1.4091e–2	2.05	1.8704e–2	2.29	3.6235e–1	1.27
	56411	1/64	1.3089e–7	3.95	6.0607e–6	3.14	3.4257e–3	2.04	4.2121e–3	2.15	1.3941e–1	1.37
	227471	1/128	8.2684e–9	3.98	7.3151e–7	3.05	8.4692e–4	2.01	1.0130e–3	2.05	5.7776e–2	1.27

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