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A virtual element method for the Steklov eigenvalue problem

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The aim of this paper is to develop a virtual element method for the two-dimensional Steklov eigenvalue problem. We propose a discretization by means of the virtual elements presented in [L. Beirão da Veiga *et al.*, Basic principles of virtual element methods, *Math. Models Methods Appl. Sci.* **23** (2013) 199–214]. Under standard assumptions on the computational domain, we establish that the resulting scheme provides a correct approximation of the spectrum and prove optimal-order error estimates for the eigenfunctions and a double order for the eigenvalues. We also prove higher-order error estimates for the computation of the eigensolutions on the boundary, which in some Steklov problems (computing sloshing modes, for instance) provides the quantity of main interest (the free surface of the liquid). Finally, we report some numerical tests supporting the theoretical results.

Keywords: Virtual element method; Steklov eigenvalue problem; error estimates.

AMS Subject Classification: 65N25, 65N30, 74S99

1. Introduction

Very recently, a new evolution of the *Mimetic Finite Difference Method* was proposed in Ref. 7 under the name of *Virtual Element Method* (VEM). This approach takes the steps from the main ideas of modern mimetic schemes but follows from a Galerkin discretization of the problem and therefore can be fully interpreted as a generalization of the finite element method. Thus, VEM couples the flexibility of mimetic methods with the theoretical and applicative background of finite elements. Since VEM is very recent, the current published literature is still very limited.[1,3,7–10,16]

This paper deals with the solution of an eigenvalue problem by means of VEM. In particular, we have chosen the Steklov eigenvalue problem, which involves the Laplace operator but is characterized by the presence of the eigenvalue in the boundary condition. The reason of this choice is that the analysis turns out simpler, since the right-hand side involves only boundary terms whose approximation by virtual elements can be seen as a classical interpolation.

The numerical approximation of eigenvalue problems is object of great interest from both, the practical and theoretical points of view. We refer to Ref. 13 and the references therein for the state-of-the-art in this subject area. In particular, the Steklov eigenvalue problem appears in many applications. For instance, we mention the study of the vibration modes of a structure in contact with an incompressible fluid (see Ref. 11) and the analysis of the stability of mechanical oscillators immersed in a viscous media (see Ref. 28). One of its main applications arises from the dynamics of liquids in moving containers, i.e. sloshing problems (see Refs. 12, 17–19, 22 and 31).

Among the existing techniques to solve this problem, various finite element methods have been introduced and analyzed. For instance, conforming finite element discretizations have been considered in Refs. 2 and 14, while Refs. 33 and 27 deal with nonconforming finite elements. Other numerical treatments for the Steklov eigenvalue problem, including *a posteriori* error analysis can be found in Refs. 4, 5, 21, 23, 32 and the references cited therein. Traditionally, finite element methods rely on triangular (simplicial) or quadrilateral meshes. However, in complex simulations, it can be convenient to use more general polygonal meshes.

The aim of this paper is to introduce and analyze a virtual element method which applies to general polygonal (even non-convex) meshes for the solution of the two-dimensional Steklov eigenvalue problem. We begin with a variational formulation of the spectral problem. We propose a discretization based on the approach introduced in Ref. 7 for the Laplace equation. By using the abstract spectral approximation theory (see Ref. 6), under rather mild assumptions on the polygonal meshes, we establish that the resulting scheme provides a correct approximation of the spectrum and prove optimal-order error estimates for the eigenfunctions and a double order for the eigenvalues.

The outline of this paper is as follows: we introduce in Sec. 2 the variational formulation of the Steklov eigenvalue problem, define a solution operator and establish its spectral characterization. In Sec. 3, we introduce the virtual element discrete formulation and describe the spectrum of a discrete solution operator. In Sec. 4, we prove that the numerical scheme provides a correct spectral approximation and establish optimal-order error estimates for the eigenvalues and eigenfunctions. We also prove an improved error estimate for the eigenfunctions on the free boundary, which allows computing a quantity of typical interest in sloshing problems. Finally, in Sec. 5, we report a set of numerical experiments that allow us to assess the convergence properties of the method and to check whether the experimental rates of convergence agree with the theoretical ones.

Throughout the paper we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

2. The Spectral Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\partial\Omega$. Let Γ_0 and Γ_1 be disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ and $|\Gamma_0| \neq 0$. We denote by n the outward unit normal vector to $\partial\Omega$ and by ∂_n the normal derivative.

We consider the following eigenvalue problem:

Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \partial_n w = \begin{cases} \lambda w & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{cases} \end{cases}$$

By testing the first equation above with $v \in H^1(\Omega)$ and integrating by parts, we arrive at the following equivalent weak formulation.

Problem 1. *Find* $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, *such that*

$$\int_\Omega \nabla w \cdot \nabla v = \lambda \int_{\Gamma_0} wv \quad \forall\, v \in H^1(\Omega).$$

Since the bilinear form on the left-hand side is not $H^1(\Omega)$-elliptic, it is convenient to use a shift argument to rewrite this eigenvalue problem in the following form:

Problem 2. *Find* $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, *such that*

$$\widehat{a}(w, v) = (\lambda + 1) b(w, v) \quad \forall\, v \in H^1(\Omega),$$

where

$$\widehat{a}(w, v) := a(w, v) + b(w, v), \quad w, v \in H^1(\Omega),$$

$$a(w, v) := \int_\Omega \nabla w \cdot \nabla v, \qquad w, v \in H^1(\Omega),$$

$$b(w, v) := \int_{\Gamma_0} wv, \qquad w, v \in H^1(\Omega)$$

are bounded bilinear symmetric forms.

Next, we define the solution operator associated with Problem 2:

$$T : H^1(\Omega) \to H^1(\Omega),$$

$$f \mapsto Tf := u,$$

where $u \in H^1(\Omega)$ is the solution of the corresponding source problem:

$$\widehat{a}(u, v) = b(f, v) \quad \forall\, v \in H^1(\Omega). \tag{2.1}$$

The following lemma allows us to establish the well-posedness of this source problem.

Lemma 2.1. *There exists a constant* $\alpha > 0$, *depending on* Ω, *such that*

$$\widehat{a}(v, v) \geq \alpha \|v\|_{1,\Omega}^2 \quad \forall\, v \in H^1(\Omega).$$

Proof. The result follows immediately from the generalized Poincaré inequality. □

We deduce from Lemma 2.1 that the linear operator T is well defined and bounded. Notice that $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$ solves Problem 2 (and hence Problem 1) if and only if $Tw = \mu w$ with $\mu \neq 0$ and $w \neq 0$, in which case $\mu := \frac{1}{1+\lambda}$. Moreover, it is easy to check that T is self-adjoint with respect to the inner product $\widehat{a}(\cdot, \cdot)$ in $H^1(\Omega)$. Indeed, given $f, g \in H^1(\Omega)$,

$$\widehat{a}(Tf, g) = b(f, g) = b(g, f) = \widehat{a}(Tg, f) = \widehat{a}(f, Tg).$$

The following is an additional regularity result for the solution of problem (2.1) and consequently, for the eigenfunctions of T.

Lemma 2.2. *There exists* $r_\Omega > \frac{1}{2}$ *such that the following results hold*:

(i) *for all* $f \in H^1(\Omega)$ *and for all* $r \in [\frac{1}{2}, r_\Omega)$, *the solution* u *of problem* (2.1) *satisfies* $u \in H^{1+r_1}(\Omega)$ *with* $r_1 := \min\{r, 1\}$ *and there exists* $C > 0$ *such that*

$$\|u\|_{1+r_1,\Omega} \leq C \|f\|_{1,\Omega};$$

(ii) *if* w *is an eigenfunction of Problem* 1 *with eigenvalue* λ, *for all* $r \in [\frac{1}{2}, r_\Omega)$, $w \in H^{1+r}(\Omega)$ *and there exists* $C > 0$ (*depending on* λ) *such that*

$$\|w\|_{1+r,\Omega} \leq C \|w\|_{1,\Omega}.$$

Proof. The proof of (i) follows from the classical regularity result for the Laplace equation with Neumann boundary conditions (cf. Ref. 25). The proof of (ii) follows from the same arguments and the fact that w is the solution of problem (2.1) with $f = \lambda w$, combined with a bootstrap trick. □

The constant $r_\Omega > \frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. If Ω is convex, then $r_\Omega > 1$, whereas, otherwise, $r_\Omega := \frac{\pi}{\omega}$ with ω being the largest re-entrant angle of Ω (see Ref. 25). Hence, because of the compact inclusion $H^{1+r}(\Omega) \hookrightarrow H^1(\Omega)$, T is a compact operator. Therefore, we have the following spectral characterization result.

Theorem 2.1. *The spectrum of* T *decomposes as follows*: $\mathrm{sp}(T) = \{0, 1\} \cup \{\mu_k\}_{k \in \mathbb{N}}$, *where*:

(i) *$\mu = 1$ is an eigenvalue of T and its associated eigenspace is the space of constant functions in Ω;*
(ii) *$\mu = 0$ is an infinite-multiplicity eigenvalue of T and its associated eigenspace is $H^1_{\Gamma_0}(\Omega) := \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_0\}$;*
(iii) *$\{\mu_k\}_{k\in\mathbb{N}} \subset (0,1)$ is a sequence of finite-multiplicity eigenvalues of T which converge to 0 and their corresponding eigenspaces lie in $H^{1+r}(\Omega)$.*

Proof. Properties (i) and (ii) are easy to check. Property (iii) follows from the classical spectral characterization of compact operators and Lemma 2.2(ii). □

3. The Discrete Problem

In this section, first we recall the mesh construction and the assumptions considered in Ref. 7 for the virtual element method. Then, we will introduce a virtual element discretization of Problems 1 and 2 and provide a spectral characterization of the resulting discrete eigenvalue problems.

Let $\{\mathcal{T}_h\}_h$ be a family of decompositions of Ω into polygons K. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e. $h := \max_{K\in\Omega} h_K$.

For the analysis, we will make as in Ref. 7 the following assumptions.

- **A0.1.** Every mesh $\mathcal{T}_h$ consists of a finite number of *simple* polygons (i.e. open simply connected sets with non-self-intersecting polygonal boundaries).
- **A0.2.** There exists $\gamma > 0$ such that, for all meshes $\mathcal{T}_h$, each polygon $K \in \mathcal{T}_h$ is star-shaped with respect to a ball of radius greater than or equal to γh_K.
- **A0.3.** There exists $\widehat{\gamma} > 0$ such that, for all meshes $\mathcal{T}_h$, for each polygon $K \in \mathcal{T}_h$, the distance between any two of its vertices is greater than or equal to $\widehat{\gamma} h_K$.

We consider now a simple polygon K and, for $k \in \mathbb{N}$, we define

$$\mathbb{B}_k(\partial K) := \{v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e) \text{ for all edges } e \subset \partial K\}.$$

We then consider the finite-dimensional space defined as follows:

$$V_k^K := \{v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v|_K \in \mathbb{P}_{k-2}(K)\},$$

where, for $k = 1$, we have used the convention that $\mathbb{P}_{-1}(K) := \{0\}$. We choose in this space the degrees of freedom introduced in Sec. 4.1 of Ref. 7. Finally, for every decomposition $\mathcal{T}_h$ of Ω into simple polygons K and for a fixed $k \in \mathbb{N}$, we define

$$V_h := \{v \in H^1(\Omega) : v|_K \in V_k^K\}.$$

In what follows, we will also use the broken H^1-seminorm

$$|v|^2_{1,h} := \sum_{K\in\mathcal{T}_h} \|\nabla v\|^2_{0,K},$$

which is well defined for every $v \in L^2(\Omega)$ such that $v|_K \in H^1(K)$ for all polygon $K \in \mathcal{T}_h$.

In order to construct the discrete scheme, we need some preliminary definitions. First, we split the bilinear form $\widehat{a}(\cdot,\cdot)$ as follows:

$$\widehat{a}(u,v) = \sum_{K\in\mathcal{T}_h} a^K(u,v) + b(u,v), \quad u,v \in H^1(\Omega),$$

where

$$a^K(u,v) := \int_K \nabla u \cdot \nabla v, \quad u,v \in H^1(\Omega). \tag{3.1}$$

To compute the local matrix a^K for $u, v \in V_h$, we must have into account that due to the implicit space definition, we would not know how to compute the bilinear form exactly. Nevertheless, the final output will be a local matrix on each element K whose associated bilinear form is exact whenever one of the two entries is a polynomial of degree k. This will allow us to retain the optimal approximation properties of the space V_h.

With this end, for any $K \in \mathcal{T}_h$ and for any sufficiently regular function φ, we define first:

$$\overline{\varphi} := \frac{1}{N_K}\sum_{i=1}^{N_K} \varphi(P_i), \tag{3.2}$$

where P_i, $1 \le i \le N_K$, are the vertices of K. Now, we define the projector $\Pi_k^K : V_k^K \to \mathbb{P}_k(K) \subseteq V_k^K$ for each $v \in V_k^K$ as the solution of

$$a^K(\Pi_k^K v, q) = a^K(v,q) \quad \forall\, q \in \mathbb{P}_k(K), \tag{3.3a}$$

$$\overline{\Pi_k^K v} = \overline{v}. \tag{3.3b}$$

Remark 3.1. Equation (3.3b) is only needed for the problem above to be well-posed. However, it is not used at all on the forthcoming analysis. Therefore, it could be substituted by any other appropriate compatible average of φ on ∂K, for instance,

$$\overline{\varphi} := \frac{1}{|\partial K|}\int_{\partial K} \varphi,$$

which makes sense for any $\varphi \in H^1(K)$.

On the other hand, let $S^K(\cdot,\cdot)$ be any symmetric positive definite bilinear form to be chosen as to satisfy

$$c_0 a^K(v,v) \le S^K(v,v) \le c_1 a^K(v,v) \quad \forall\, v \in V_k^K \quad \text{with} \quad \Pi_k^K v = 0, \tag{3.4}$$

for some positive constants c_0 and c_1 independent of K. Then, set

$$a_h(u_h,v_h) := \sum_{K\in\mathcal{T}_h} a_h^K(u_h,v_h), \quad u_h, v_h \in V_h,$$

where $a_h^K(\cdot,\cdot)$ is the bilinear form defined on $V_k^K \times V_k^K$ by

$$a_h^K(u,v) := a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v), \quad u, v \in V_k^K. \tag{3.5}$$

The following properties of the bilinear form $a_h^K(\cdot,\cdot)$ have been established in Theorem 4.1 of Ref. 7.

- *k-Consistency*:

$$a_h^K(p, v_h) = a^K(p, v_h) \quad \forall\, p \in \mathbb{P}_k(K), \quad \forall\, v_h \in V_k^K.$$

- *Stability*: There exist two positive constants α_* and α^*, independent of K, such that:

$$\alpha_* a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* a^K(v_h, v_h) \quad \forall\, v_h \in V_k^K. \tag{3.6}$$

Now, we are in a position to write the virtual element discretization of Problem 1.

Problem 3. *Find* $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, *such that*

$$a_h(w_h, v_h) = \lambda_h b(w_h, v_h) \quad \forall\, v_h \in V_h.$$

We use again a shift argument to rewrite this discrete eigenvalue problem in the following convenient equivalent form.

Problem 4. *Find* $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, *such that*

$$\widehat{a}_h(w_h, v_h) = (\lambda_h + 1) b(w_h, v_h) \quad \forall\, v_h \in V_h,$$

where

$$\widehat{a}_h(w_h, v_h) := a_h(w_h, v_h) + b(w_h, v_h), \quad w_h, v_h \in V_h.$$

We observe that by virtue of (3.6) and the trace theorem, the bilinear form $\widehat{a}_h(\cdot,\cdot)$ is bounded. Moreover, as shown in the following lemma, it is also uniformly elliptic.

Lemma 3.1. *There exists a constant* $\beta > 0$, *independent of* h, *such that*

$$\widehat{a}_h(v_h, v_h) \ge \beta \|v_h\|_{1,\Omega}^2 \quad \forall\, v_h \in V_h.$$

Proof. Thanks to (3.6) and Lemma 2.1, it is easy to check that the above inequality holds with $\beta := \alpha \min\{\alpha_*, 1\}$. □

The discrete version of the operator T is then given by

$$T_h : H^1(\Omega) \to H^1(\Omega),$$
$$f \mapsto T_h f := u_h,$$

where $u_h \in V_h$ is the solution of the corresponding discrete source problem

$$\widehat{a}_h(u_h, v_h) = b(f, v_h) \quad \forall\, v_h \in V_h.$$

Because of Lemma 3.1, the linear operator T_h is well defined and bounded uniformly with respect to h. Once more, as in the continuous case, $(\lambda_h, w_h) \in \mathbb{R}\times V_h$ solves Problem 4 (and hence Problem 3) if and only if $T_h w_h = \mu_h w_h$ with $\mu_h \neq 0$ and $w_h \neq 0$, in which case $\mu_h := \frac{1}{1+\lambda_h}$. Moreover, $T_h|_{V_h} : V_h \to V_h$ is self-adjoint with respect to $\widehat{a}_h(\cdot,\cdot)$. Indeed, given $f, g \in V_h$,

$$\widehat{a}_h(T_h f, g) = b(f,g) = b(g,f) = \widehat{a}_h(T_h g, f) = \widehat{a}_h(f, T_h g).$$

As a consequence, we have the following spectral characterization.

Theorem 3.1. *The spectrum of $T_h|_{V_h}$ consists of $M_h := \dim(V_h)$ eigenvalues, repeated according to their respective multiplicities. It decomposes as follows:* $\mathrm{sp}(T_h|_{V_h}) = \{0,1\} \cup \{\mu_{hk}\}_{k=1}^{N_h}$, *where*:

(i) *the eigenspace associated with $\mu_h = 1$ is the space of constant functions in Ω;*
(ii) *the eigenspace associated with $\mu_h = 0$ is $Z_h := V_h \cap H^1_{\Gamma_0}(\Omega) = \{q_h \in V_h : q_h = 0$ on $\Gamma_0\}$;*
(iii) $\mu_{hk} \subset (0,1)$, $k = 1,\ldots,N_h := M_h - \dim(Z_h) - 1$, *are non-defective eigenvalues repeated according to their respective multiplicities.*

4. Spectral Approximation

To prove that T_h provides a correct spectral approximation of T, we will resort to the classical theory for compact operators (see Ref. 6), which is based on the convergence in norm of T_h to T as $h \to 0$. With the aim of proving this, the first step is to establish the following result.

Lemma 4.1. *There exists $C > 0$ such that, for all $f \in H^1(\Omega)$, if $u = Tf$ and $u_h = T_h f$, then*

$$\|(T - T_h)f\|_{1,\Omega} = \|u - u_h\|_{1,\Omega} \leq C(\|u - u_I\|_{1,\Omega} + |u - u_\pi|_{1,h}),$$

for all $u_I \in V_h$ and for all $u_\pi \in L^2(\Omega)$ such that $u_\pi|_K \in \mathbb{P}_k(K)\ \forall K \in \mathcal{T}_h$.

Proof. Let $f \in H^1(\Omega)$. For $u_I \in V_h$, we set $v_h := u_h - u_I$ and thanks to Lemma 3.1, the definition (3.5) of a_h^K and those of T and T_h, we have

$$\begin{aligned}
\beta\|v_h\|^2_{1,\Omega} &\leq \widehat{a}_h(v_h, v_h) = \widehat{a}_h(u_h, v_h) - \widehat{a}_h(u_I, v_h) \\
&= b(f, v_h) - \sum_{K\in\mathcal{T}_h} a_h^K(u_I, v_h) - b(u_I, v_h) \\
&= b(f, v_h) - b(u_I, v_h) \\
&\quad - \sum_{K\in\mathcal{T}_h} (a_h^K(u_I - u_\pi, v_h) + a^K(u_\pi - u, v_h) + a^K(u, v_h)) \\
&= b(u - u_I, v_h) - \sum_{K\in\mathcal{T}_h} (a_h^K(u_I - u_\pi, v_h) + a^K(u_\pi - u, v_h)).
\end{aligned}$$

Therefore, from the trace theorem, (3.6) and the boundedness of $a_h^K(\cdot,\cdot)$ and $a^K(\cdot,\cdot)$,

$$\begin{aligned}\beta\|v_h\|_{1,\Omega}^2 &\leq \|u-u_I\|_{0,\Gamma_0}\|v_h\|_{0,\Gamma_0}\\ &\quad+\sum_{K\in\mathcal{T}_h}(\alpha^*|u_I-u_\pi|_{1,K}|v_h|_{1,K}+|u_\pi-u|_{1,K}|v_h|_{1,K})\\ &\leq \|u-u_I\|_{1,\Omega}\|v_h\|_{1,\Omega}\\ &\quad+\sum_{K\in\mathcal{T}_h}(\alpha^*|u_I-u|_{1,K}|v_h|_{1,K}+(\alpha^*+1)|u-u_\pi|_{1,K}|v_h|_{1,K})\\ &\leq C(\|u-u_I\|_{1,\Omega}+|u-u_\pi|_{1,h})\|v_h\|_{1,\Omega}.\end{aligned}$$

Hence, the proof follows from the triangular inequality. □

The next step is to find appropriate terms u_I and u_π that can be used in the above lemma to prove the claimed convergence. For the latter we have the following proposition, which is derived by interpolation between Sobolev spaces (see for instance Theorem I.1.4 in Ref. 24) from the analogous result for integer values of s. In its turn, the result for integer values is stated in Proposition 4.2 of Ref. 7 and follows from the classical Scott–Dupont theory (see Ref. 15).

Proposition 4.1. *If the assumption* **A0.2** *is satisfied, then there exists a constant* C, *depending only on* k *and* γ, *such that for every* s *with* $0 \leq s \leq k$ *and for every* $v \in H^{1+s}(K)$, *there exists* $v_\pi \in \mathbb{P}_k(K)$ *such that*

$$\|v-v_\pi\|_{0,K}+h_K|v-v_\pi|_{1,K}\leq Ch_K^{1+s}\|v\|_{1+s,K}.$$

For the term $u_I \in V_h$ in Lemma 4.1, we have the following result which is an extension of Proposition 4.3 in Ref. 7 to less regular functions.

Proposition 4.2. *If the assumptions* **A0.2** *and* **A0.3** *are satisfied, then, for each* s *with* $0 \leq s \leq k$, *there exists a constant* C, *depending only on* k, γ *and* $\widehat{\gamma}$, *such that for every* $v \in H^{1+s}(\Omega)$, *there exists* $v_I \in V_h$ *that satisfies*

$$\|v-v_I\|_{0,\Omega}+h|v-v_I|_{1,\Omega}\leq Ch^{1+s}\|v\|_{1+s,\Omega}.$$

Proof. Let $v \in H^{1+s}(\Omega)$, $0 \leq s \leq k$. Since we are assuming **A0.2**, let $v_\pi \in L^2(\Omega)$ be defined on each $K \in \mathcal{T}_h$ so that $v_\pi|_K \in \mathbb{P}_k(K)$ and the estimate of Proposition 4.1 holds true.

For each polygon $K \in \mathcal{T}_h$, consider the triangulation $\mathcal{T}_h^K$ obtained by joining each vertex of K with the midpoint of the ball with respect to which K is starred. Let $\widehat{\mathcal{T}}_h := \bigcup_{K\in\mathcal{T}_h}\mathcal{T}_h^K$. Since we are also assuming **A0.3**, $\{\widehat{\mathcal{T}}_h\}_h$ is a shape-regular family of triangulations of Ω.

Let v_{c} be the Clément interpolant of degree k of v over $\widehat{\mathcal{T}}_h$ (cf. Ref. 20). Then, $v_{\mathrm{c}} \in H^1(\Omega)$ and the following error estimate follows by interpolation between

Sobolev spaces from the analogous result for integer values of s (which in turn has been proved in Ref. 20):

$$\|v - v_{\mathrm{c}}\|_{0,\Omega} + h|v - v_{\mathrm{c}}|_{1,\Omega} \le Ch^{1+s}\|v\|_{1+s,\Omega}. \tag{4.1}$$

Now, for each $K \in \mathcal{T}_h$, we define $v_I|_K \in H^1(K)$ as the solution of the following problem:

$$\begin{cases} -\Delta v_I = -\Delta v_\pi & \text{in } K, \\ v_I = v_{\mathrm{c}} & \text{on } \partial K. \end{cases}$$

Note that $v_I|_K \in V_k^K$. Moreover, although v_I is defined locally, since on the boundary of each element it coincides with v_{c} which belongs to $H^1(\Omega)$, we have that also v_I belongs to $H^1(\Omega)$ and, hence, $v_I \in V_h$.

According to the above definition we have that

$$\begin{cases} -\Delta(v_\pi - v_I) = 0 & \text{in } K, \\ v_\pi - v_I = v_\pi - v_{\mathrm{c}} & \text{on } \partial K, \end{cases}$$

and, hence, it is easy to check that

$$|v_\pi - v_I|_{1,K} = \inf\{|z|_{1,K}, z \in H^1(K) : z = v_\pi - v_{\mathrm{c}} \text{ on } \partial K\} \le |v_\pi - v_{\mathrm{c}}|_{1,K}.$$

Therefore,

$$\begin{aligned} |v - v_I|_{1,K} &\le |v - v_\pi|_{1,K} + |v_\pi - v_I|_{1,K} \le |v - v_\pi|_{1,K} + |v_\pi - v_{\mathrm{c}}|_{1,K} \\ &\le 2|v - v_\pi|_{1,K} + |v - v_{\mathrm{c}}|_{1,K}, \end{aligned}$$

which together with Proposition 4.1 and (4.1) lead to

$$|v - v_I|_{1,\Omega} \le Ch^s\|v\|_{1+s,\Omega}. \tag{4.2}$$

On the other hand, for all $K \in \mathcal{T}_h$, each triangle $T \in \mathcal{T}_h^K$ has one edge on ∂K. Hence, since $v_I = v_{\mathrm{c}}$ on ∂K, a scaling argument and the classical Poincaré inequality yield

$$\|v_{\mathrm{c}} - v_I\|_{0,T} \le Ch_K|v_{\mathrm{c}} - v_I|_{1,T}.$$

Thus, from the above inequality, (4.1) and (4.2), we have

$$\begin{aligned} \|v - v_I\|_{0,\Omega} &\le \|v - v_{\mathrm{c}}\|_{0,\Omega} + \|v_{\mathrm{c}} - v_I\|_{0,\Omega} \le \|v - v_{\mathrm{c}}\|_{0,\Omega} + Ch|v_{\mathrm{c}} - v_I|_{1,\Omega} \\ &\le \|v - v_{\mathrm{c}}\|_{0,\Omega} + Ch|v - v_{\mathrm{c}}|_{1,\Omega} + Ch|v - v_I|_{1,\Omega} \\ &\le Ch^{1+s}\|v\|_{1+s,\Omega}, \end{aligned}$$

which together with (4.2) allow us to conclude the proof. □

The following result yields the convergence in norm of T_h to T as $h \to 0$.

Lemma 4.2. *For all* $r \in [\frac{1}{2}, r_\Omega)$, *let* $r_1 := \min\{r, 1\}$ *as defined in Lemma* 2.2(i). *Then, there exists* $C > 0$ *such that*

$$\|(T - T_h)f\|_{1,\Omega} \le Ch^{r_1}\|f\|_{1,\Omega} \quad \forall f \in H^1(\Omega).$$

Proof. The result follows from Lemma 4.1, Propositions 4.1 and 4.2, and Lemma 2.2(i). □

4.1. *Error estimates*

As a direct consequence of Lemma 4.2, standard results about spectral approximation (see Ref. 26, for instance) show that isolated parts of $\mathrm{sp}(T)$ are approximated by isolated parts of $\mathrm{sp}(T_h)$. More precisely, let $\mu \in (0,1)$ be an isolated eigenvalue of T with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. Then, there exist m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which converge to μ. Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces.

We recall the definition of the *gap* $\widehat{\delta}$ between two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $H^1(\Omega)$:

$$\widehat{\delta}(\mathcal{X}, \mathcal{Y}) := \max\{\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\},$$

where

$$\delta(\mathcal{X}, \mathcal{Y}) := \sup_{x \in \mathcal{X}: \|x\|_{1,\Omega}=1} \left(\inf_{y \in \mathcal{Y}} \|x - y\|_{1,\Omega} \right).$$

The following error estimates for the approximation of eigenvalues and eigenfunctions hold true.

Theorem 4.1. *There exists a strictly positive constant C such that*

$$\widehat{\delta}(\mathcal{E}, \mathcal{E}_h) \le C\gamma_h,$$
$$|\mu - \mu_h^{(i)}| \le C\gamma_h, \quad i = 1, \ldots, m,$$

where

$$\gamma_h := \sup_{f \in \mathcal{E}: \|f\|_{1,\Omega}=1} \|(T - T_h)f\|_{1,\Omega}.$$

Proof. As a consequence of Lemma 4.2, T_h converges in norm to T as h goes to zero. Then, the proof follows as a direct consequence of Theorems 7.1 and 7.3 from Ref. 6. □

The theorem above yields error estimates depending on γ_h. The next step is to show an optimal-order estimate for this term.

Theorem 4.2. *For all $r \in [\frac{1}{2}, r_\Omega)$ there exists a positive constant C such that*

$$\|(T - T_h)f\|_{1,\Omega} \le Ch^{\min\{r,k\}} \|f\|_{1,\Omega} \quad \forall\, f \in \mathcal{E}, \tag{4.3}$$

and, consequently,

$$\gamma_h \le Ch^{\min\{r,k\}}. \tag{4.4}$$

Proof. The proof is identical to that of Lemma 4.2, but using now the additional regularity from Lemma 2.2(ii). □

The error estimate for the eigenvalue $\mu \in (0,1)$ of T leads to an analogous estimate for the approximation of the eigenvalue $\lambda = \frac{1}{\mu} - 1$ of Problem 1 by means of the discrete eigenvalues $\lambda_h^{(i)} := \frac{1}{\mu_h^{(i)}} - 1$, $1 \le i \le m$, of Problem 3. However, the order of convergence in Theorem 4.1 is not optimal for μ and, hence, not optimal for λ either. Our next goal is to improve this order.

Theorem 4.3. *For all $r \in [\frac{1}{2}, r_\Omega)$, there exists a strictly positive constant C such that*

$$|\lambda - \lambda_h^{(i)}| \le C h^{2\min\{r,k\}}.$$

Proof. Let w_h be such that $(\lambda_h^{(i)}, w_h)$ is a solution of Problem 3 with $\|w_h\|_{1,\Omega} = 1$. According to Theorem 4.1, there exists a solution (λ, w) of Problem 1 such that

$$\|w - w_h\|_{1,\Omega} \le C\gamma_h. \tag{4.5}$$

From the symmetry of the bilinear forms and the facts that $a(w,v) = \lambda b(w,v)$ for all $v \in H^1(\Omega)$ (cf. Problem 1) and $a_h(w_h, v_h) = \lambda_h^{(i)} b(w_h, v_h)$ for all $v_h \in V_h$ (cf. Problem 3), we have

$$\begin{aligned} &a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) \\ &\quad = a(w_h, w_h) - \lambda b(w_h, w_h) \\ &\quad = [a(w_h, w_h) - a_h(w_h, w_h)] - (\lambda - \lambda_h^{(i)}) b(w_h, w_h), \end{aligned}$$

from which we obtain the following identity:

$$\begin{aligned} (\lambda_h^{(i)} - \lambda) b(w_h, w_h) &= a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) \\ &\quad + [a_h(w_h, w_h) - a(w_h, w_h)]. \end{aligned} \tag{4.6}$$

The next step is to bound each term on the right-hand side above. The first and the second ones are easily bounded from the continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the trace theorem, (4.5) and (4.4):

$$|a(w - w_h, w - w_h)| + \lambda |b(w - w_h, w - w_h)| \le C h^{2\min\{r,k\}}. \tag{4.7}$$

For the third term, we use (3.4) and (3.3a) to write:

$$\begin{aligned} &|a_h(w_h, w_h) - a(w_h, w_h)| \\ &\quad = \left| \sum_{K \in \mathcal{T}_h} [a^K(\Pi_k^K w_h, \Pi_k^K w_h) + S^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)] \right. \\ &\qquad \left. - \sum_{K \in \mathcal{T}_h} a^K(w_h, w_h) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \sum_{K\in\mathcal{T}_h} [a^K(\Pi_k^K w_h, \Pi_k^K w_h) - a^K(w_h, w_h)] \right| \\
&\quad + \sum_{K\in\mathcal{T}_h} c_1 a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) \\
&= \sum_{K\in\mathcal{T}_h} [a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)] \\
&\quad + \sum_{K\in\mathcal{T}_h} c_1 a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) \\
&= \sum_{K\in\mathcal{T}_h} (1 + c_1) a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h).
\end{aligned}$$

Therefore, from the definition of $a^K(\cdot,\cdot)$ (cf. (3.1)) and the fact that Π_k^K is the projector defined by (3.3a), we obtain

$$\begin{aligned}
|a_h(w_h, w_h) - a(w_h, w_h)| &\leq C \sum_{K\in\mathcal{T}_h} |w_h - \Pi_k^K w_h|_{1,K}^2 \\
&\leq C \sum_{K\in\mathcal{T}_h} |w_h - \Pi_k^K w|_{1,K}^2 \\
&\leq C \sum_{K\in\mathcal{T}_h} (|w_h - w|_{1,K} + |w - \Pi_k^K w|_{1,K})^2.
\end{aligned}$$

Now, also from (3.3a) it is immediate to check that

$$|w - \Pi_k^K w|_{1,K} \leq |w - w_\pi|_{1,K} \quad \forall\, w_\pi \in \mathbb{P}_k(K).$$

Then, from the last two inequalities, Proposition 4.1, (4.5) and (4.4), we obtain

$$|a_h(w_h, w_h) - a(w_h, w_h)| \leq Ch^{2\min\{r,k\}}.$$

On the other hand, by virtue of Lemma 3.1 and the fact that $\lambda_h^{(i)} \to \lambda$ as h goes to zero, we know that there exists $C > 0$ such that

$$b(w_h, w_h) = \frac{\widehat{a}_h(w_h, w_h)}{\lambda_h^{(i)} + 1} \geq \frac{\beta\|w_h\|_{1,\Omega}^2}{\lambda_h^{(i)} + 1} \geq \frac{\beta}{C} > 0.$$

By using this estimate to bound the left-hand side of (4.6) from below, together with the previous one and (4.7) for an upper bound of the right-hand side, we conclude that

$$|\lambda - \lambda_h^{(i)}| \leq Ch^{2\min\{r,k\}}$$

and we end the proof. □

4.2. *Error estimates for the eigenfunctions on* Γ_0

Our next goal is to improve the error estimate for the trace of the eigenfunctions in the $L^2(\Gamma_0)$-norm. With this end, we will resort to a duality technique. Given

$u \in H^1(\Omega)$ and $u_h \in V_h$, let $v \in H^1(\Omega)$ be the solution of the following problem:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_n v + v = \begin{cases} u - u_h & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{cases} \end{cases}$$

By testing the first equation above with functions in $H^1(\Omega)$ and integrating by parts, we obtain

$$\widehat{a}(v,z) := \int_\Omega \nabla v \cdot \nabla z + \int_{\Gamma_0} vz = \int_{\Gamma_0} (u-u_h)z =: b(u-u_h, z) \quad \forall\, z \in H^1(\Omega). \tag{4.8}$$

Therefore, $v = T(u-u_h)$, so that according to Lemma 2.2(i), for all $r \in [\frac{1}{2}, r_\Omega)$, $v \in H^{1+r_1}(\Omega)$ (recall that $r_1 := \min\{r, 1\}$) and

$$\|v\|_{1+r_1,\Omega} \le C\|u-u_h\|_{1,\Omega}. \tag{4.9}$$

The improved error estimate will be a consequence of the following result.

Lemma 4.3. *Let $f \in \mathcal{E}$ be an eigenfunction of the operator T. If $u = Tf$ and $u_h = T_h f$, then, for all $r \in [\frac{1}{2}, r_\Omega)$, there exists $C > 0$ such that*

$$\|(T-T_h)f\|_{0,\Gamma_0} = \|u-u_h\|_{0,\Gamma_0} \le Ch^{r_1/2+\min\{r,k\}}\|f\|_{1,\Omega}.$$

Proof. Let v be as defined above and $v_I \in V_h$ so that the estimate of Proposition 4.2 holds true. Testing (4.8) with $z = (u-u_h) \in H^1(\Omega)$, we obtain

$$\|u-u_h\|_{0,\Gamma_0}^2 = \widehat{a}(u-u_h, v) = \widehat{a}(u-u_h, v-v_I) + \widehat{a}(u-u_h, v_I). \tag{4.10}$$

To bound the first term on the right-hand side above, we use the continuity of the bilinear form $\widehat{a}(\cdot,\cdot)$, Proposition 4.2 and (4.9):

$$\begin{aligned} \widehat{a}(u-u_h, v-v_I) &\le C\|u-u_h\|_{1,\Omega}\|v-v_I\|_{1,\Omega} \\ &\le Ch^{r_1}\|u-u_h\|_{1,\Omega}\|v\|_{1+r_1,\Omega} \le Ch^{r_1}\|u-u_h\|_{1,\Omega}^2. \end{aligned} \tag{4.11}$$

For the second term, we use that $\widehat{a}(u, v_h) = b(f, v_h) = \widehat{a}_h(u_h, v_h)$ for all $v_h \in V_h$ to write

$$\begin{aligned} \widehat{a}(u-u_h, v_I) &= \widehat{a}_h(u_h, v_I) - \widehat{a}(u_h, v_I) = \sum_{K\in\mathcal{T}_h} (a_h^K(u_h, v_I) - a^K(u_h, v_I)) \\ &= \sum_{K\in\mathcal{T}_h} (a^K(\Pi_k^K u_h, \Pi_k^K v_I) + S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) - a^K(u_h, v_I)) \\ &= \sum_{K\in\mathcal{T}_h} (a^K(\Pi_k^K u_h - u_h, v_I - \Pi_k^K v_I) + S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I)), \end{aligned} \tag{4.12}$$

where we have used (3.3a) to derive the last equality.

Now, from the symmetry of $S^K(\cdot,\cdot)$, inequality (3.4) and the definition of $a^K(\cdot,\cdot)$, we have that $S^K(v_h, z_h) \leq c_1|v_h|_{1,K}|z_h|_{1,K}$ for all $v_h, z_h \in V_k^K$ such that $\Pi_k^K v_h = \Pi_k^K z_h = 0$. We use this inequality to bound the second term on the right-hand side of (4.12):

$$\sum_{K\in\mathcal{T}_h} S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) \leq c_1 \sum_{K\in\mathcal{T}_h} |u_h - \Pi_k^K u_h|_{1,K}|v_I - \Pi_k^K v_I|_{1,K}. \tag{4.13}$$

Using that Π_k^K is the projector defined by (3.3a), we have that

$$\begin{aligned} |u_h - \Pi_k^K u_h|_{1,K} &\leq |u_h - \Pi_k^K u|_{1,K} \\ &\leq |u_h - u|_{1,K} + |u - u_\pi|_{1,K} \quad \forall\, u_\pi \in \mathbb{P}_k(K), \end{aligned}$$

and, analogously,

$$|v_I - \Pi_k^K v_I|_{1,K} \leq |v_I - v|_{1,K} + |v - v_\pi|_{1,K} \quad \forall\, v_\pi \in \mathbb{P}_k(K).$$

Substituting these inequalities into (4.13) and using (4.3), Proposition 4.1 and Lemma 2.2(ii) (since $f \in \mathcal{E}$) for the former and Propositions 4.1 and 4.2 and (4.9) for the latter, we obtain

$$\sum_{K\in\mathcal{T}_h} S^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) \leq Ch^{r_1+\min\{r,k\}}\|f\|_{1,\Omega}\|u - u_h\|_{1,\Omega}.$$

By repeating the same steps as above, we obtain a similar bound for the first term on the right-hand side of (4.12):

$$\sum_{K\in\mathcal{T}_h} a^K(u_h - \Pi_k^K u_h, v_I - \Pi_k^K v_I) \leq Ch^{r_1+\min\{r,k\}}\|f\|_{1,\Omega}\|u - u_h\|_{1,\Omega}.$$

Hence,

$$\widehat{a}(u - u_h, v_I) \leq Ch^{r_1+\min\{r,k\}}\|f\|_{1,\Omega}\|u - u_h\|_{1,\Omega}.$$

The proof follows by substituting this inequality and (4.11) into (4.10) and using (4.3). □

The next step is to define a solution operator on the space $L^2(\Gamma_0)$:

$$\begin{aligned} \widetilde{T} : L^2(\Gamma_0) &\to L^2(\Gamma_0), \\ \widetilde{f} &\mapsto \widetilde{T}\widetilde{f} := u|_{\Gamma_0}, \end{aligned}$$

where $u \in H^1(\Omega)$ is the solution of the following problem:

$$\widehat{a}(u, v) = \int_{\Gamma_0} \widetilde{f} v \quad \forall\, v \in H^1(\Omega). \tag{4.14}$$

It is easy to check that the operator $\widetilde{T}$ is compact and self-adjoint. We also define the corresponding discrete solution operator:

$$\begin{aligned} \widetilde{T}_h : L^2(\Gamma_0) &\to L^2(\Gamma_0), \\ \widetilde{f} &\mapsto \widetilde{T}_h\widetilde{f} := u_h|_{\Gamma_0}, \end{aligned}$$

where $u_h \in V_h$ is the solution of the discrete problem

$$\widehat{a}_h(u_h, v_h) = \int_{\Gamma_0} \widetilde{f} v_h \quad \forall\, v_h \in V_h. \tag{4.15}$$

The spectra of T and $\widetilde{T}$ coincide. In fact, it is immediate to check that if $Tw = \mu w$, with $w \neq 0$ and $\mu \neq 0$, then $w|_{\Gamma_0} \neq 0$ and $\widetilde{T}(w|_{\Gamma_0}) = \mu w|_{\Gamma_0}$. Conversely, if $\widetilde{T}\widetilde{w} = \mu\widetilde{w}$, with $\widetilde{w} \neq 0$ and $\mu \neq 0$, then there exists $w \in H^1(\Omega)$, such that $Tw = \mu w$ and $w|_{\Gamma_0} = \widetilde{w}$. The same arguments allow us to show that the spectra of T_h and $\widetilde{T}_h$ also coincide and their respective eigenfunctions are related in the same way as those of T and $\widetilde{T}$.

To prove that the operators $\widetilde{T}_h$ converge in norm to $\widetilde{T}$, we will use the following additional regularity estimate analogous to that in Lemma 2.2 but that only involves $\|f\|_{0,\Gamma_0}$.

Lemma 4.4. *For all $s \in (0, \frac{1}{2})$, there exists $C > 0$ such that, for all $f \in L^2(\Gamma_0)$, the solution u of problem (4.14) satisfies $u \in H^{1+s}(\Omega)$ and*

$$\|u\|_{1+s,\Omega} \leq C\|f\|_{0,\Gamma_0}.$$

Proof. The proof is a consequence of Theorem 4 in Ref. 29. □

Now, we are able to conclude the convergence in norm of $\widetilde{T}_h$ to $\widetilde{T}$.

Lemma 4.5. *For all $s \in (0, \frac{1}{2})$, there exists $C > 0$ such that*

$$\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\|_{0,\Gamma_0} \leq Ch^s\|\widetilde{f}\|_{0,\Gamma_0}.$$

Proof. Given $\widetilde{f} \in L^2(\Gamma_0)$, let $u \in H^1(\Omega)$ and $u_h \in V_h$ be the solutions of problems (4.14) and (4.15), respectively, so that $\widetilde{T}\widetilde{f} = u|_{\Gamma_0}$ and $\widetilde{T}_h\widetilde{f} = u_h|_{\Gamma_0}$. The arguments used in the proof of Lemma 4.1 can be repeated in this case yielding

$$\|u - u_h\|_{1,\Omega} \leq C(\|u - u_I\|_{1,\Omega} + |u - u_\pi|_{1,h}),$$

with u_I and u_π as in that lemma. Thus, the result follows from Propositions 4.1 and 4.2, and Lemma 4.4. □

As a consequence of this lemma, a spectral convergence result analogous to Theorem 4.1 holds for $\widetilde{T}_h$ and $\widetilde{T}$. Moreover, we are in a position to establish the following estimate.

Theorem 4.4. *Let w_h be an eigenfunction of T_h associated with the eigenvalue $\mu_h^{(i)}$, $1 \leq i \leq m$, with $\|w_h\|_{0,\Gamma_0} = 1$. Then, there exists an eigenfunction w of T associated with μ such that, for all $r \in [\frac{1}{2}, r_\Omega)$, there exists $C > 0$ such that*

$$\|w - w_h\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}}.$$

Proof. Thanks to Lemma 4.5, Theorem 7.1 from Ref. 6 yields spectral convergence of $\widetilde{T}_h$ to $\widetilde{T}$. In particular, because of the relation between the eigenfunctions of T

and T_h with those of $\widetilde{T}$ and $\widetilde{T}_h$, respectively, we have that $w_h|_{\Gamma_0} \in \widetilde{\mathcal{E}}_h$ and there exists $w \in \mathcal{E}$ such that

$$\|w - w_h\|_{0,\Gamma_0} \leq C \sup_{\widetilde{f} \in \widetilde{\mathcal{E}}: \|\widetilde{f}\|_{0,\Gamma_0}=1} \|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\|_{0,\Gamma_0}. \tag{4.16}$$

On the other hand, because of Lemma 4.3, for all $\widetilde{f} \in \widetilde{\mathcal{E}}$, if $f \in \mathcal{E}$ is such that $\widetilde{f} = f|_{\Gamma_0}$, then

$$\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\|_{0,\Gamma_0} = \|(T - T_h)f\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}}\|f\|_{1,\Omega}.$$

Now, for $f \in \mathcal{E},\ Tf = \mu f$. Hence, $\|f\|_{1,\Omega} = \frac{1}{\mu}\|Tf\|_{1,\Omega} \leq C\|f\|_{0,\Gamma_0}$ (cf. Lemma 4.4). Thus, substituting this expression into the previous inequality, we have that

$$\|(\widetilde{T} - \widetilde{T}_h)\widetilde{f}\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}}\|\widetilde{f}\|_{0,\Gamma_0},$$

which together with (4.16) allow us to conclude the proof. □

Remark 4.1. The result above is actually an improved error estimate in $L^2(\Gamma_0)$-norm as compared with the obvious one $\|w - w_h\|_{0,\Gamma_0} \leq Ch^{\min\{r,k\}}$ which follows from Theorems 4.1 and 4.2 and the trace theorem.

5. Numerical Results

We report in this section some numerical examples which have allowed us to assess the theoretical results proved above. With this aim, we have implemented in a MATLAB code a lowest-order VEM ($k = 1$) on arbitrary polygonal meshes, by following the ideas proposed in Ref. 9.

To complete the choice of the VEM, we had to fix the bilinear forms $S^K(\cdot,\cdot)$ satisfying (3.4) to be used. To do this, we have proceeded as in Ref. 7: for each polygon K with vertices $P_1, \ldots, P_{N_K}$, we have used

$$S^K(u,v) := \sigma_K \sum_{i=1}^{N_K} u(P_i)v(P_i), \quad u,v \in V_1^K, \tag{5.1}$$

where σ_K is the so-called *stability constant* that can be chosen freely as far as it satisfies

$$0 < \sigma_* \leq \sigma_K \leq \sigma^*, \tag{5.2}$$

with the two constants σ_* and σ^* independent of h and the particular element K.

As stated in Sec. 4.6 of Ref. 7, under assumption **A0.3**, this choice of $S^K(\cdot,\cdot)$ satisfies (3.4).

5.1. *Test 1: Sloshing in a square domain*

In this test, we have taken $\Omega := (0,1)^2$, with Γ_0 and Γ_1 as shown in Fig. 1.

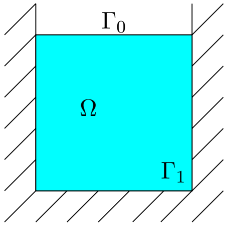

Fig. 1. Sloshing in a square domain.

This problem corresponds to the computation of the sloshing modes of a two-dimensional fluid contained in Ω with a horizontal free surface Γ_0. The analytical solutions of this problem are

$$\lambda_n = n\pi \tanh(n\pi), \quad w_n(x,y) = \cos(n\pi x)\sinh(n\pi y), \quad n \in \mathbb{N}.$$

We have taken $\sigma_K = 1$ in (5.1). We have used three different families of meshes (see Fig. 2):

- $\mathcal{T}_h^1$: triangular meshes, considering the middle point of each edge as a new degree of freedom;
- $\mathcal{T}_h^2$: trapezoidal meshes which consist of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertexes $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{2}{3})$, and $(0,\frac{1}{3})$;
- $\mathcal{T}_h^3$: meshes built from $\mathcal{T}_h^1$ with the edge midpoint moved randomly; note that these meshes contain non-convex elements.

The refinement parameter N used to label each mesh is the number of elements on each edge.

We report in Table 1 the lowest eigenvalues λ_{hi} computed with this method. The table also includes the estimated orders of convergence. The exact eigenvalues are also reported in the last column to allow for comparison.

It can be seen from Table 1 that the computed eigenvalues converge to the exact ones with an optimal quadratic order as predicted by the theory.

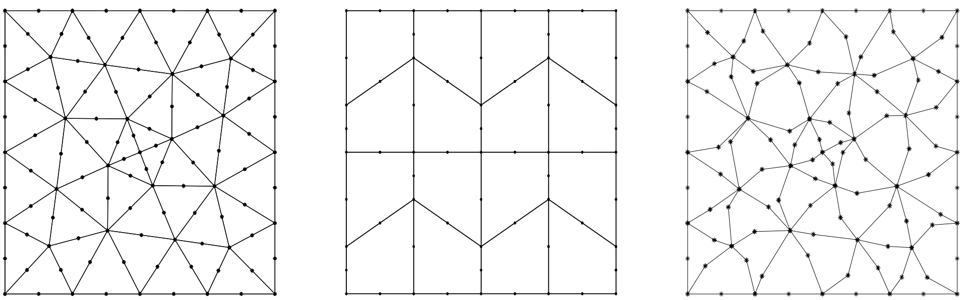

Fig. 2. Test 1. Sample meshes: $\mathcal{T}_h^1$ (left), $\mathcal{T}_h^2$ (middle) and $\mathcal{T}_h^3$ (right) for $N = 4$.

Table 1. Test 1. Computed lowest eigenvalues λ_{hi}, $1 \leq i \leq 3$, on different meshes.

$\mathcal{T}_h$	λ_{hi}	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	λ_i
$\mathcal{T}_h^1$	λ_{h1}	3.1330	3.1306	3.1301	3.1299	2.03	3.1299
	λ_{h2}	6.3095	6.2894	6.2846	6.2835	2.07	6.2831
	λ_{h3}	9.5183	9.4459	9.4298	9.4260	2.09	9.4248
$\mathcal{T}_h^2$	λ_{h1}	3.1424	3.1331	3.1307	3.1301	1.98	3.1299
	λ_{h2}	6.3765	6.3095	6.2900	6.2849	1.92	6.2831
	λ_{h3}	9.6929	9.5092	9.4475	9.4306	1.85	9.4248
$\mathcal{T}_h^3$	λ_{h1}	3.1331	3.1308	3.1301	3.1299	2.03	3.1299
	λ_{h2}	6.3105	6.2896	6.2847	6.2835	2.05	6.2831
	λ_{h3}	9.5193	9.4470	9.4300	9.4261	2.06	9.4248

Table 2. Test 1. Errors $\|w - w_h\|_{0,\Gamma_0}$ of the vibration mode for the lowest eigenvalue λ_{h1} on different meshes.

$\mathcal{T}_h$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	Order
$\mathcal{T}_h^1$	3.633e − 3	8.715e − 4	2.265e − 4	5.567e − 5	2.00
$\mathcal{T}_h^2$	2.507e − 2	5.939e − 3	1.445e − 3	3.558e − 4	2.05
$\mathcal{T}_h^3$	4.559e − 3	9.943e − 4	2.576e − 4	6.592e − 5	2.03

We report in Table 2 the $L^2(\Gamma_0)$-errors of the eigenfunctions corresponding to the lowest eigenvalue for each family of meshes and different refinement levels. We also include in this table the estimated orders of convergence.

We observe from this table a clear quadratic order of convergence. Let us remark that this is the optimal order attainable with the virtual elements used, which is actually larger than the order $\mathcal{O}(h^{3/2})$ predicted by the theory.

Figure 3 shows the eigenfunctions on Γ_0 corresponding to the three lowest eigenvalues. Let us remark that, in the sloshing problem, this corresponds to the shape of the fluid free surface ($\partial_n w = \lambda w$) for each sloshing mode.

5.2. *Test 2: Effect of the stability*

The aim of this test is to analyze the influence of the *stability constant* σ_K in (5.1) on the computed spectrum. We will show that the introduction of the stability terms

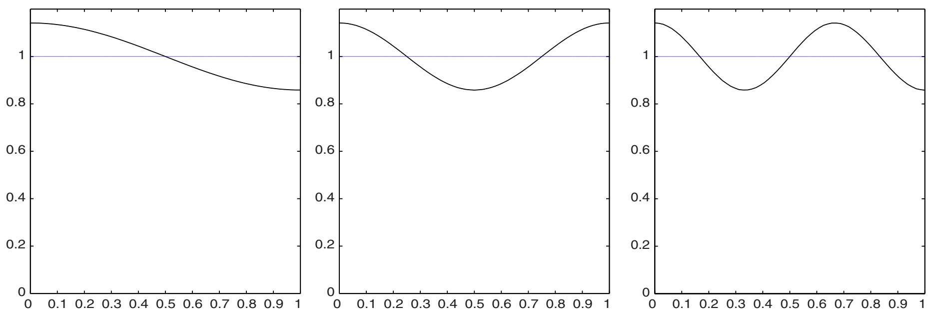

Fig. 3. Test 1. Sloshing modes: w_{h1} (left), w_{h2} (middle) and w_{h3} (right) computed with $N = 256$.

S_K in (3.5) leads to spurious eigenvalues. We will also show that these spurious eigenvalues can be driven by appropriately choosing the stability constant σ_K. If the same value of σ_K is chosen for all $K \in \mathcal{T}_h$ (as in the previous test), roughly speaking, the spurious eigenvalues will be proportional to this stability constant. This can be seen from Table 3, where we report the lowest eigenvalues computed by the method with varying values of σ_K on a fixed mesh $\mathcal{T}_h^1$ with refinement level $N = 8$ (see Fig. 2, left).

The table also includes on the last column the three lowest exact eigenvalues. The computed eigenvalues into boxes correspond to approximations of these physical eigenvalues, whereas the rest correspond to spurious spectrum.

For $\sigma_K \geq 1$ we observe that the lowest computed eigenvalues are correct approximations of the physical ones, whereas the largest are spurious and behave roughly speaking proportional to σ_K as claimed above. For values of $\sigma_K < 1$, the spurious eigenvalues appear interspersed among the correct ones, which makes it hard to distinguish spurious and physical eigenvalues. For very small values of σ_K, the spurious eigenvalues become even smaller than the physical ones, as can be seen on the first column of Table 3 for $\sigma_K = 1/64$.

The above analysis suggests to use a sufficiently large σ_K in order to avoid the correct spectrum to be polluted. This phenomenon seems to contradict the theoretical analysis: spurious eigenvalues should not appear, when there is convergence in norm as was shown to happen in our case (see Lemma 4.2). However, this assertion is of an asymptotic nature: spurious eigenvalues will not appear interspersed among the correct spectrum for h small enough. In fact, this is what happens in our case as is shown in Table 4. In this table we report the eigenvalues computed only with the smallest σ_K of the previous experiment, but with increasingly refined meshes.

Table 3. Test 2. Computed lowest eigenvalues for $\sigma_K = 4^{-k}$ with $-3 \leq k \leq 3$.

$\sigma_K = 1/64$	$\sigma_K = 1/16$	$\sigma_K = 1/4$	$\sigma_K = 1$	$\sigma_K = 4$	$\sigma_K = 16$	$\sigma_K = 64$	λ_i
1.517	[3.078]	[3.101]	[3.142]	[3.175]	[3.189]	[3.193]	3.1299
1.531	5.563	[6.065]	[6.393]	[6.668]	[6.784]	[6.819]	6.2831
1.587	5.646	[8.716]	[9.788]	[10.755]	[11.181]	[11.309]	9.4248
1.693	5.824	11.020	13.487	15.948	17.105	17.460	
1.715	5.903	12.878	17.494	22.298	24.551	25.235	
2.171	6.368	14.527	22.314	30.464	34.046	35.093	
2.180	6.537	15.731	27.513	39.415	43.831	45.055	
2.196	7.806	16.752	33.436	44.896	49.191	50.407	
2.214	8.134	18.433	41.706	113.899	405.657	1573.904	
[3.071]	8.251	19.318	49.196	141.263	501.449	1941.283	
[5.834]	8.409	21.881	61.343	183.830	655.050	2534.774	
[8.037]	8.607	23.403	72.371	231.747	844.138	3285.114	
9.645	10.182	27.705	85.040	282.392	1051.315	4119.669	
10.747	11.494	28.317	92.654	324.962	1237.860	4883.585	
11.080	12.064	29.445	99.697	358.294	1377.508	5448.886	
11.712	12.800	29.840	101.609	369.180	1426.727	5652.202	

Table 4. Test 2. Computed lowest eigenvalues for $\sigma_K = 1/64$.

$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	λ_i
0.747	1.517	2.929	3.126	3.129	3.1299
0.801	1.530	2.975	5.854	6.276	6.2831
0.874	1.587	3.033	5.901	9.399	9.4248
1.096	1.693	3.044	5.931	11.703	
1.098	1.715	3.056	5.981	11.752	
2.898	2.171	3.114	5.987	11.815	
4.760	2.180	3.150	5.990	11.857	
5.425	2.196	3.260	6.046	11.890	
	2.214	3.367	6.109	11.897	
	3.071	3.394	6.140	11.915	
	5.834	3.407	6.220	11.952	
	8.037	3.482	6.252	11.976	
	9.645	3.569	6.404	11.991	
	10.747	3.626	6.438	12.033	
	11.080	4.367	6.446	12.179	
	11.712	4.382	6.469	12.185	
		4.389	6.516	12.199	
		4.401	6.618	12.312	
		6.158	6.827	12.463	

The table shows that the spurious eigenvalues are also roughly speaking proportional to $1/h$, so that they blow up as the mesh is refined. For instance, for the most refined mesh reported in Table 4 ($N = 64$), the three lowest correct eigenvalues are not polluted by the spurious ones.

This analysis suggests, that the user of VEM for spectral problems, has to be aware of the risk of spurious eigenvalue pollution. The way of minimizing this risk is to take a reasonably large σ_K (which will depend, in real problems, on the value of the physical constants) and sufficiently refined meshes. Moreover, the spurious character of an eigenvalue can be easily checked from its dependence on σ_K and the mesh.

5.3. *Test 3: Circular domain*

In this test, we have taken as domain the unit circle $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with $\Gamma_0 = \partial\Omega$ and $\Gamma_1 = \emptyset$.

It is easy to check that any homogeneous harmonic polynomial of degree n satisfies $\partial_n w = nw$ on $\partial\Omega$. Therefore, for all $n \in \mathbb{N}$, $\lambda = n$ is an eigenvalue of this problem and the corresponding eigenspace is the set of homogeneous harmonic polynomials of degree n, whose dimension is 2.

We have taken $\sigma_K = 1$ in (5.1). We have used polygonal meshes created with PolyMesher,[30] as that shown in Fig. 4. The refinement parameter N used to label each mesh is now the number of elements intersecting the boundary.

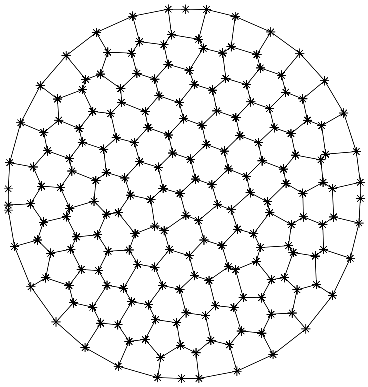

Fig. 4. Test 3. Sample polygonal mesh for $N = 29$.

Table 5. Test 3. Computed lowest eigenvalues λ_{hi}, $1 \leq i \leq 4$.

λ_{hi}	$N = 8$	$N = 30$	$N = 104$	$N = 342$	Order	λ_i
λ_{h1}	0.9509	0.9960	0.9997	1.0000	1.97	1
λ_{h2}	0.9762	0.9971	0.9997	1.0000	1.81	1
λ_{h3}	1.9528	1.9957	1.9997	2.0000	1.91	2
λ_{h4}	2.0601	2.0002	1.9998	2.0000	1.89	2

We report in Table 5 the four lowest eigenvalues λ_{hi} computed with this method. The table also includes the estimated orders of convergence. The last column shows the exact eigenvalues.

Once more, a quadratic order of convergence can be clearly appreciated from Table 5.

Finally, Fig. 5 shows a plot of the eigenfunctions computed with the finest mesh.

In order to compute the $L^2(\Gamma_0)$-errors, some special care had to be taken because of the double multiplicity of each eigenvalue. We focused on the eigenvalue $\lambda_3 = \lambda_4 = 2$, whose corresponding eigenspace is spanned by the eigenfunctions $w_3(x, y) = xy$ and $w_4(x, y) = x^2 - y^2$. As can be seem from Fig. 5 (down), the computed eigenfunctions are not necessarily w_3 or w_4, but linear combination of these two. To isolate one particular eigenfunction, we took advantage of the symmetry of the domain and solved the problem in the quarter $x, y > 0$ of the unit circle. Thus, to compute $w_3(x, y) = xy$, which vanishes on $x = 0$ and $y = 0$, we imposed these values as homogeneous Dirichlet data of the problem. Proceeding in this way, we could ensure that the computed eigenfunction was actually an approximation of w_3.

Another difficulty of this test is that the curved domain is approximated by a polygonal one, so that the boundary values of these computed eigenfunctions are defined on the polygonal domain, whereas those of the exact eigenfunctions are given on the curved domain. To avoid these drawbacks, we projected the latter onto the polygonal domain.

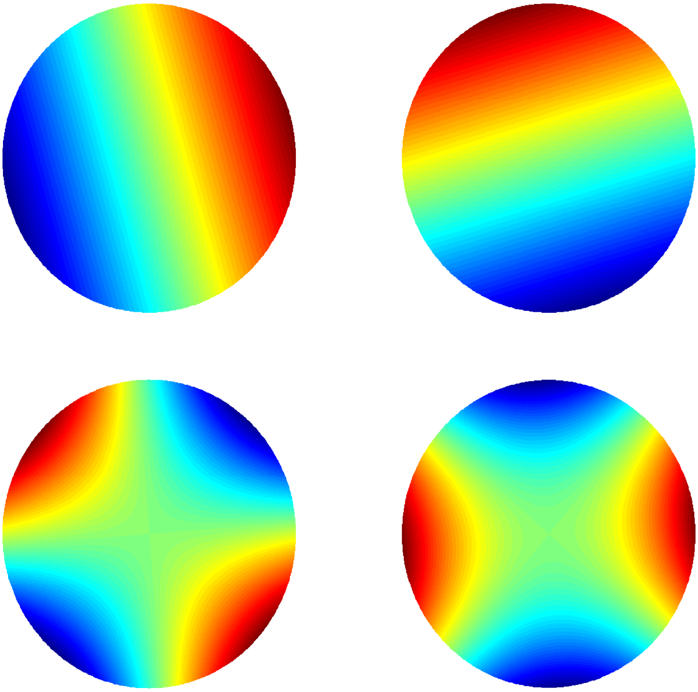

Fig. 5. Test 3. Eigenfunctions: w_{h1} (upper left), w_{h2} (upper right), w_{h3} (lower left) and w_{h4} (lower right) computed with a very refined mesh ($N = 342$).

Table 6. Test 3. $L^2(\Gamma_0)$-errors of the eigenfunction $w_3(x, y) = xy$ on different polygonal meshes.

$N = 9$	$N = 37$	$N = 117$	$N = 379$	Order
3.485e − 3	4.354e − 4	2.970e − 5	1.672e − 6	2.06

We report in Table 6 the $L^2(\Gamma_0)$-errors computed as described above on the curved boundary of the quarter of circle by using again polygonal meshes obtained with PolyMesher, similar to that shown in Fig. 4, but for the quarter of circle. We also include in this table the computed order of convergence which, once more, is clearly quadratic.

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