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A posteriori error estimates for a Virtual Element Method for the Steklov eigenvalue problem[☆]

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ABSTRACT

The paper deals with the a posteriori error analysis of a virtual element method for the Steklov eigenvalue problem. The virtual element method has the advantage of using general polygonal meshes, which allows implementing efficiently mesh refinement strategies. We introduce a residual type a posteriori error estimator and prove its reliability and global efficiency. Local efficiency estimates also hold, although in some elements they involve boundary terms that are not known to be locally negligible. We use the corresponding error estimator to drive an adaptive scheme. Finally, we report the results of a couple of numerical tests, that allow us to assess the performance of this approach.

1. Introduction

The *Virtual Element Method* (VEM), introduced in [1,2], appears as an evolution of the *Mimetic Finite Differences Method* (see [3]), VEM takes its main ideas from modern mimetic schemes, but involves the Galerkin discretization of the problem and consequently can be interpreted as a generalization of the Finite Element Method (FEM) on polygons or polyhedra meshes. In recent years, the interest in numerical methods that can make use of general polytopal meshes has undergone a significant growth in the mathematical and engineering literature; we cite [1,3–8] as a minimal sample of these works. To date, VEM has been applied successfully in a large range of problems; see for instance [1,2,9–20].

The object of this paper is to introduce and analyze an a posteriori error estimator of residual type for a VEM approximation of the Steklov eigenvalue problem, whose a priori analysis was recently performed in [18]. This problem is characterized by the presence of the eigenvalue in the boundary condition and it appears for instance in the computation of the hydroelastic vibration modes of a structure in contact with an incompressible fluid (see [21]), the analysis of the stability of mechanical oscillators immersed in a viscous media (see [22]) and the dynamics of liquids in moving containers, i.e., sloshing problems (see [23–28]).

Due to the large flexibility of the meshes to which the virtual element method is applied, mesh adaptivity becomes an appealing feature since mesh refinement strategies can be implemented very efficiently. For instance, hanging nodes can be introduced in the mesh to guarantee the mesh conformity without spreading the refined zones. In fact hanging nodes introduced by the refinement of a neighboring element are simply treated as new nodes since adjacent non matching element interfaces are perfectly acceptable. On the other hand, polygonal cells with very general shapes are admissible. Therefore,

☆ Dedicated to Ivo Babuška on the occasion of his ninetieth birthday.

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simple coarsening algorithms based on joining different element could be tried, although this issue will not be considered in the paper.

On the other hand, adaptive mesh refinement strategies based on a posteriori error indicators play a relevant role in the numerical solution of partial differential equations in a general sense. For instance, they guarantee achieving errors below a tolerance with a reasonable computer cost in presence of singular solutions. Several approaches have been considered to construct error estimators based on the residual equations (see [29–31] and the references therein). In particular, for the Steklov eigenvalue problem we mention [32–36]. On the other hand, the design and analysis of a posteriori error bounds for the VEM is a challenging task. References [37,38] are the only a posteriori error analyses for VEM currently available in the literature. In [37], a posteriori error bounds for the C^1-conforming VEM for the two-dimensional Poisson problem are proposed. In turn, in [38], a posteriori error bounds are introduced for the C^0-conforming VEM proposed in [39] for the discretization of second order linear elliptic reaction–convection–diffusion problems with non constant coefficients in two and three dimensions.

We have recently developed in [18] a virtual element method for the two dimensional Steklov eigenvalue problem. Under standard assumptions on the computational domain, we have established that the resulting scheme provides a correct approximation of the spectrum and proved optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. Having in mind the capability of VEM in the use of general polygonal meshes and its flexibility for the application of mesh adaptive strategies, we introduce and analyze an a posteriori error estimator for the virtual element approximation from [18]. Since normal fluxes of the VEM solution are not computable, they will be replaced in the estimators by a proper projection. As a consequence of this replacement, new additional terms appear in the a posteriori error estimator, which represent the virtual inconsistency of VEM. Similar terms also appear in the other existing papers about a posteriori error estimates of VEM (see [37,38]).

On the other hand, due to the fact that different eigenfunctions have in general different strengths of singularities, the optimal mesh for a particular one will not be optimal for other eigenfunctions. Because of this, we focus on computing a single eigenpair. Let us remark that in many cases (for instance in sloshing problems) only the smallest eigenvalue is sought in practice. We prove that the estimator is equivalent to the error and use the corresponding indicator to drive an adaptive scheme. Finally, let us mention that the proposed analysis combined with the results from [38] could be tried for similar three-dimensional problems.

The outline of this article is as follows: in Section 2 we recall the continuous and discrete formulations of the Steklov eigenvalue problem together with the spectral characterization and the a priori error estimates for the virtual element approximation analyzed in [18]. In Section 3, we define the a posteriori error estimator and proved its reliability and global efficiency. The proof of the latter relies in local efficiency estimates which, for some elements, involve additional boundary terms that are proved to be globally (although not locally) negligible. In Section 4, we report a set of numerical tests that allow us to assess the performance of an adaptive strategy driven by the estimator. We also make a comparison between the proposed estimator and the standard edge-residual error estimator for a finite element method. Finally, we summarize some conclusions.

Throughout the article we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

2. The Steklov eigenvalue problem and its virtual element approximation

In this section, we recall the Steklov eigenvalue problem and a VEM approximation proposed in [18]. Also, we summarize the main a priori analysis results from this reference.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\partial\Omega$. Let Γ_0 and Γ_1 be disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ with $\Gamma_0 \neq \emptyset$. We denote by n the outward unit normal vector to $\partial\Omega$.

We consider the following eigenvalue problem:

Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = \begin{cases} \lambda w & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{cases} \end{cases}$$

By testing the first equation above with $v \in H^1(\Omega)$ and integrating by parts, we arrive at the following equivalent weak formulation:

Problem 1. Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that

$$\int_\Omega \nabla w \cdot \nabla v = \lambda \int_{\Gamma_0} wv \quad \forall v \in H^1(\Omega).$$

According to [18, Theorem 2.1], we know that the solutions (λ, w) of the problem above are:

- $\lambda_0 = 0$, whose associated eigenspace is the space of constant functions in Ω;
- a sequence of positive finite-multiplicity eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ such that $\lambda_j \to \infty$.

The eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\Gamma_0)$. Therefore the eigenfunctions w_j corresponding to $\lambda_j > 0$ satisfy

$$\int_{\Gamma_0} w_j = 0. \tag{2.1}$$

We denote the bounded bilinear symmetric forms appearing in Problem 1 as follows:

$$a(w, v) := \int_{\Omega} \nabla w \cdot \nabla v, \quad w, v \in H^1(\Omega),$$
$$b(w, v) := \int_{\Gamma_0} wv, \quad w, v \in H^1(\Omega).$$

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K. We assume that for every mesh $\mathcal{T}_h$, $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ are union of edges of elements $K \in \mathcal{T}_h$. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K\in\mathcal{T}_h} h_K$.

For the analysis, we will make as in [1,18] the following assumptions.

- **A1**. Every mesh $\mathcal{T}_h$ consists of a finite number of *simple* polygons (i.e., open simply connected sets with non self intersecting polygonal boundaries).
- **A2**. There exists $\gamma > 0$ such that, for all meshes $\mathcal{T}_h$, each polygon $K \in \mathcal{T}_h$ is star-shaped with respect to a ball of radius greater than or equal to γh_K.
- **A3**. There exists $\widehat{\gamma} > 0$ such that, for all meshes $\mathcal{T}_h$, for each polygon $K \in \mathcal{T}_h$, the distance between any two of its vertices is greater than or equal to $\widehat{\gamma} h_K$.

We consider now a simple polygon K and, for $k \in \mathbb{N}$, we define

$$\mathbb{B}_k(\partial K) := \left\{ v \in C^0(\partial K) : \ v|_\ell \in \mathbb{P}_k(\ell) \text{ for all edges } \ell \subset \partial K \right\}.$$

We then consider the finite-dimensional space defined as follows:

$$V_k^K := \left\{ v \in H^1(K) : \ v|_{\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v|_K \in \mathbb{P}_{k-2}(K) \right\}, \tag{2.2}$$

where, for $k = 1$, we have used the convention that $\mathbb{P}_{-1}(K) := \{0\}$. We choose in this space the degrees of freedom introduced in [1, Section 4.1]. Finally, for every decomposition $\mathcal{T}_h$ of Ω into simple polygons K and for a fixed $k \in \mathbb{N}$, we define

$$V_h := \left\{ v \in H^1(\Omega) : \ v|_K \in V_k^K \quad \forall K \in \mathcal{T}_h \right\}.$$

In what follows, we will use standard Sobolev spaces, norms and seminorms and also the broken H^1-seminorm

$$|v|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} \|\nabla v\|_{0,K}^2,$$

which is well defined for every $v \in L^2(\Omega)$ such that $v|_K \in H^1(K)$ for each polygon $K \in \mathcal{T}_h$.

We split the bilinear form $a(\cdot, \cdot)$ as follows:

$$a(u, v) = \sum_{K\in\mathcal{T}_h} a^K(u, v), \quad u, v \in H^1(\Omega),$$

where

$$a^K(u, v) := \int_K \nabla u \cdot \nabla v, \quad u, v \in H^1(K).$$

Due to the implicit space definition, we must have into account that we would not know how to compute $a^K(\cdot, \cdot)$ for $u_h, v_h \in V_h$. Nevertheless, the final output will be a local matrix on each element K whose associated bilinear form can be exactly computed whenever one of the two entries is a polynomial of degree k. This will allow us to retain the optimal approximation properties of the space V_h.

With this end, for any $K \in \mathcal{T}_h$ and for any sufficiently regular function φ, we define first

$$\overline{\varphi} := \frac{1}{N_K} \sum_{i=1}^{N_K} \varphi(P_i),$$

where P_i, $1 \le i \le N_K$, are the vertices of K. Then, we define the projector $\Pi_k^K : V_k^K \longrightarrow \mathbb{P}_k(K) \subseteq V_k^K$ for each $v_h \in V_k^K$ as the solution of

$$a^K\left(\Pi_k^K v_h, q\right) = a^K(v_h, q) \quad \forall q \in \mathbb{P}_k(K),$$
$$\overline{\Pi_k^K v_h} = \overline{v_h}.$$

On the other hand, let $S^K(\cdot,\cdot)$ be any symmetric positive definite bilinear form to be chosen as to satisfy

$$c_0\, a^K(v_h,v_h) \le S^K(v_h,v_h) \le c_1\, a^K(v_h,v_h) \quad \forall v_h \in V_k^K \text{ with } \Pi_k^K v_h = 0, \tag{2.3}$$

for some positive constants c_0 and c_1 independent of K. Then, set

$$a_h(u_h,v_h) := \sum_{K\in\mathcal{T}_h} a_h^K(u_h,v_h), \quad u_h, v_h \in V_h,$$

where $a_h^K(\cdot,\cdot)$ is the bilinear form defined on $V_k^K \times V_k^K$ by

$$a_h^K(u_h,v_h) := a^K\left(\Pi_k^K u_h, \Pi_k^K v_h\right) + S^K\left(u_h - \Pi_k^K u_h, v_h - \Pi_k^K v_h\right), \quad u_h, v_h \in V_k^K.$$

Notice that the bilinear form $S^K(\cdot,\cdot)$ has to be actually computable for $u_h, v_h \in V_k^K$.

The following properties of $a_h^K(\cdot,\cdot)$ have been established in [1, Theorem 4.1].

- *k-Consistency*:

$$a_h^K(p,v_h) = a^K(p,v_h) \quad \forall p \in \mathbb{P}_k(K),\ \forall v_h \in V_k^K. \tag{2.4}$$

- *Stability*: There exist two positive constants α_* and α^*, independent of K, such that:

$$\alpha_* a^K(v_h,v_h) \le a_h^K(v_h,v_h) \le \alpha^* a^K(v_h,v_h) \quad \forall v_h \in V_k^K. \tag{2.5}$$

Now, we are in a position to write the virtual element discretization of Problem 1.

Problem 2. Find $(\lambda_h, w_h) \in \mathbb{R}\times V_h$, $w_h \neq 0$, such that

$$a_h(w_h,v_h) = \lambda_h b(w_h,v_h) \quad \forall v_h \in V_h.$$

According to [18, Theorem 3.1] we know that the solutions (λ_h, w_h) of the problem above are:

- $\lambda_{h0} = 0$, whose associated eigenspace is the space of the constant functions in Ω.
- $\{\lambda_{hj}\}_{j=1}^{N_h}$, with $N_h := \dim\left\{v_h|_{\Gamma_0}, v_h \in V_h\right\} - 1$, which are positive eigenvalues repeated according to their respective multiplicities.

Moreover, the eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\Gamma_0)$. Therefore the eigenfunctions w_{hj} corresponding to $\lambda_{hj} > 0$ satisfy

$$\int_{\Gamma_0} w_{hj} = 0. \tag{2.6}$$

Let (λ, w) be a solution to Problem 1. We assume $\lambda > 0$ is a simple eigenvalue and we normalize w so that $\|w\|_{0,\Gamma_0} = 1$. Then, for each mesh $\mathcal{T}_h$, there exists a solution (λ_h, w_h) of Problem 2 such that $\lambda_h \to \lambda$, $\|w_h\|_{0,\Gamma_0} = 1$ and $\|w - w_h\|_{1,\Omega} \to 0$ as $h \to 0$. Moreover, according to (2.1) and (2.6), we have that w and w_h belong to the space

$$V := \left\{ v \in H^1(\Omega) : \int_{\Gamma_0} v = 0 \right\}.$$

Let us remark that the following generalized Poincaré inequality holds true in this space: there exists $C > 0$ such that

$$\|v\|_{1,\Omega} \le C|v|_{1,\Omega} \quad \forall v \in V. \tag{2.7}$$

The following a priori error estimates have been proved in [18, Theorems 4.2–4.4]: there exists $C > 0$ such that for all $r \in [\frac{1}{2}, r_\Omega)$

$$\|w - w_h\|_{1,\Omega} \le Ch^{\min\{r,k\}}, \tag{2.8}$$

$$|\lambda - \lambda_h| \le Ch^{2\min\{r,k\}}, \tag{2.9}$$

$$\|w - w_h\|_{0,\Gamma_0} \le Ch^{\min\{r,1\}/2+\min\{r,k\}}, \tag{2.10}$$

where the constant $r_\Omega > \frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. Let us remark that $r_\Omega > 1$, if Ω is convex, and $r_\Omega := \frac{\pi}{\omega}$ with ω being the largest re-entrant angle of Ω, otherwise.

3. A posteriori error analysis

The aim of this section is to introduce a suitable residual-based error estimator for the VEM approximation of a single eigenpair of the Steklov eigenvalue problem corresponding to a simple eigenvalue. The estimator must be fully computable, in the sense that it must depend only on quantities available from the VEM solution. Then, we will show its equivalence with the error. For this purpose, we introduce the following definitions and notations.

For any polygon $K \in \mathcal{T}_h$, we denote by $\mathcal{E}_K$ the set of edges of K and

$$\mathcal{E} := \bigcup_{K\in\mathcal{T}_h} \mathcal{E}_K.$$

We decompose $\mathcal{E} = \mathcal{E}_\Omega \cup \mathcal{E}_{\Gamma_0} \cup \mathcal{E}_{\Gamma_1}$, where $\mathcal{E}_{\Gamma_0} := \{\ell \in \mathcal{E} : \ell \subset \Gamma_0\}$, $\mathcal{E}_{\Gamma_1} := \{\ell \in \mathcal{E} : \ell \subset \Gamma_1\}$ and $\mathcal{E}_\Omega := \mathcal{E} \setminus (\mathcal{E}_{\Gamma_0} \cup \mathcal{E}_{\Gamma_1})$. For each inner edge $\ell \in \mathcal{E}_\Omega$ and for any sufficiently smooth function v, we define the jump of its normal derivative on ℓ by

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]_\ell := \nabla(v|_K)\cdot n_K + \nabla(v|_{K'})\cdot n_{K'},$$

where K and K' are the two elements in $\mathcal{T}_h$ sharing the edge ℓ and n_K and $n_{K'}$ are the respective outer unit normal vectors.

As a consequence of the mesh regularity assumptions, we have that each polygon $K \in \mathcal{T}_h$ admits a sub-triangulation $\mathcal{T}_h^K$ obtained by joining each vertex of K with the midpoint of the ball with respect to which K is starred. Let $\widehat{\mathcal{T}}_h := \bigcup_{K\in\mathcal{T}_h} \mathcal{T}_h^K$. Since we are also assuming **A3**, $\{\widehat{\mathcal{T}}_h\}_h$ is a shape-regular family of triangulations of Ω.

We introduce bubble functions on polygons as follows (see [38]). An interior bubble function $\psi_K \in H_0^1(K)$ for a polygon K can be constructed piecewise as the sum of the cubic bubble functions for each triangle of the sub-triangulation $\mathcal{T}_h^K$ that attain the value 1 at the barycenter of each triangle. On the other hand, an edge bubble function ψ_ℓ for $\ell \in \partial K$ is a piecewise quadratic function attaining the value 1 at the barycenter of ℓ and vanishing on the triangles $T \in \widehat{\mathcal{T}}_h$ that do not contain ℓ on its boundary.

The following results which establish standard estimates for bubble functions will be useful in what follows (see [29,31]).

Lemma 3.1 (*Interior Bubble Functions*). *For any $K \in \mathcal{T}_h$, let ψ_K be the corresponding interior bubble function. Then, there exists a constant $C > 0$ independent of h_K such that*

$$C^{-1}\|q\|_{0,K}^2 \le \int_K \psi_K q^2 \le \|q\|_{0,K}^2 \quad \forall q \in \mathbb{P}_k(K),$$
$$C^{-1}\|q\|_{0,K} \le \|\psi_K q\|_{0,K} + h_K\|\nabla(\psi_K q)\|_{0,K} \le C\|q\|_{0,K} \quad \forall q \in \mathbb{P}_k(K).$$

Lemma 3.2 (*Edge Bubble Functions*). *For any $K \in \mathcal{T}_h$ and $\ell \in \mathcal{E}_K$, let ψ_ℓ be the corresponding edge bubble function. Then, there exists a constant $C > 0$ independent of h_K such that*

$$C^{-1}\|q\|_{0,\ell}^2 \le \int_\ell \psi_\ell q^2 \le \|q\|_{0,\ell}^2 \quad \forall q \in \mathbb{P}_k(\ell).$$

Moreover, for all $q \in \mathbb{P}_k(\ell)$, there exists an extension of $q \in \mathbb{P}_k(K)$ (again denoted by q) such that

$$h_K^{-1/2}\|\psi_\ell q\|_{0,K} + h_K^{1/2}\|\nabla(\psi_\ell q)\|_{0,K} \le C\|q\|_{0,\ell}.$$

Remark 3.1. A possible way of extending q from $\ell \in \mathcal{E}_K$ to K so that Lemma 3.2 holds is as follows: first we extend q to the straight line $L \supset \ell$ using the same polynomial function. Then, we extend it to the whole plain through a constant prolongation in the normal direction to L. Finally, we restrict the latter to K.

The following lemma provides an error equation which will be the starting point of our error analysis. From now on, we will denote by $e := (w - w_h) \in V$ the eigenfunction error and by

$$J_\ell := \begin{cases} \dfrac{1}{2}\left[\!\!\left[\dfrac{\partial(\Pi_k^K w_h)}{\partial n}\right]\!\!\right]_\ell, & \ell \in \mathcal{E}_\Omega, \\ \lambda_h w_h - \dfrac{\partial(\Pi_k^K w_h)}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_0}, \\ -\dfrac{\partial(\Pi_k^K w_h)}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_1}, \end{cases} \tag{3.1}$$

the edge residuals. Notice that J_ℓ are actually computable since they only involve values of w_h on Γ_0 (which are computable in terms of the boundary degrees of freedom) and $\Pi_k^K w_h \in \mathbb{P}_k(K)$ which is also computable.

Lemma 3.3. *For any $v \in H^1(\Omega)$, we have the following identity:*

$$a(e,v) = \lambda b(w,v) - \lambda_h b(w_h,v) - \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, v) + \sum_{K\in\mathcal{T}_h}\left[\int_K \Delta(\Pi_k^K w_h)v + \sum_{\ell\in\mathcal{E}_K}\int_\ell J_\ell v\right].$$

Proof. Using that (λ, w) is a solution of Problem 1, adding and subtracting $\Pi_k^K w_h$ and integrating by parts, we obtain

$$\begin{aligned}
a(e, v) &= \lambda b(w, v) - a(w_h, v) \\
&= \lambda b(w, v) - \sum_{K\in\mathcal{T}_h} \left[a^K(w_h - \Pi_k^K w_h, v) + a^K(\Pi_k^K w_h, v)\right] \\
&= \lambda b(w, v) - \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, v) - \sum_{K\in\mathcal{T}_h} \left[-\int_K \Delta(\Pi_k^K w_h)\, v + \int_{\partial K} \frac{\partial(\Pi_k^K w_h)}{\partial n}\, v\right] \\
&= \lambda b(w, v) - \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, v) \\
&\quad + \sum_{K\in\mathcal{T}_h} \left[\int_K \Delta(\Pi_k^K w_h)\, v - \sum_{\ell\in\mathcal{E}_K\cap(\mathcal{E}_{\Gamma_0}\cup\mathcal{E}_{\Gamma_1})} \int_\ell \frac{\partial(\Pi_k^K w_h)}{\partial n}\, v + \frac{1}{2}\sum_{\ell\in\mathcal{E}_K\cap\mathcal{E}_\Omega} \int_\ell \left[\!\left[\frac{\partial(\Pi_k^K w_h)}{\partial n}\right]\!\right]_\ell v\right].
\end{aligned}$$

Finally, the proof follows by adding and subtracting the term $\lambda_h b(w_h, v)$. $\square$

For each $K \in \mathcal{T}_h$, we introduce the local consistency term θ_K, the volumetric residual R_K and the local error indicator η_K as follows:

$$\begin{aligned}
\theta_K^2 &:= a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h), \\
R_K^2 &:= h_K^2 \|\Delta(\Pi_k^K w_h)\|_{0,K}^2, \\
\eta_K^2 &:= \theta_K^2 + R_K^2 + \sum_{\ell\in\mathcal{E}_K} h_K \|J_\ell\|_{0,\ell}^2.
\end{aligned}$$

We also introduce the global error estimator by

$$\eta^2 := \sum_{K\in\mathcal{T}_h} \eta_K^2.$$

Remark 3.2. The indicators η_K include the terms θ_K which do not appear in standard finite element estimators. This term, which represent the virtual inconsistency of the method, has been introduced in [37,38] for a posteriori error estimates of other VEM. Let us emphasize that it can be directly computed in terms of the bilinear form $S^K(\cdot,\cdot)$. In fact,

$$\theta_K^2 = a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) = S^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h).$$

3.1. Reliability of the a posteriori error estimator

Our next goal is to prove upper bounds for different error terms. These bounds will not be actually computable, since they will involve the quantity $\|w - w_h\|_{0,\Gamma_0}$. However, later on, this quantity will be proved to be asymptotically negligible, so that the following three lemmas can be seen as intermediary steps to obtain a fully computable a posteriori error estimate.

Lemma 3.4. *There exists a constant $C > 0$ independent of h such that*

$$|w - w_h|_{1,\Omega} \le C\left(\eta + \frac{\lambda + \lambda_h}{2}\|w - w_h\|_{0,\Gamma_0}\right).$$

Proof. Since $e = w - w_h \in V \subset H^1(\Omega)$, there exists $e_I \in V_h$ satisfying (see [18, Proposition 4.2])

$$\|e - e_I\|_{0,K} + h_K|e - e_I|_{1,K} \le C h_K \|e\|_{1,K}. \tag{3.2}$$

Then, we have that

$$\begin{aligned}
|w - w_h|_{1,\Omega}^2 &= a(w - w_h, e) \\
&= a(w - w_h, e - e_I) + a(w, e_I) - a_h(w_h, e_I) + a_h(w_h, e_I) - a(w_h, e_I) \\
&= \underbrace{\lambda b(w, e) - \lambda_h b(w_h, e)}_{T_1} + \underbrace{\sum_{K\in\mathcal{T}_h}\left[\int_K \Delta(\Pi_k^K w_h)(e - e_I) + \sum_{\ell\in\mathcal{E}_K}\int_\ell J_\ell(e - e_I)\right]}_{T_2} \\
&\quad \underbrace{- \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, e - e_I)}_{T_3} + \underbrace{a_h(w_h, e_I) - a(w_h, e_I)}_{T_4},
\end{aligned} \tag{3.3}$$

the last equality thanks to Lemma 3.3. Next, we bound each term T_i separately.

For T_1, we use the definition of $b(\cdot,\cdot)$, the fact that $\|w\|_{0,\Gamma_0} = \|w_h\|_{0,\Gamma_0} = 1$, a trace theorem and (2.7) to write

$$T_1 = \lambda + \lambda_h - (\lambda + \lambda_h)\int_{\Gamma_0} w w_h = \frac{\lambda + \lambda_h}{2}\|e\|_{0,\Gamma_0}^2 \leq C\frac{\lambda + \lambda_h}{2}\|e\|_{0,\Gamma_0}|e|_{1,\Omega}. \tag{3.4}$$

For T_2, first, we use a local trace inequality (see [14, Lemma 14]) and (3.2) to write for each $\ell \in \mathcal{E}_K$ and $K \in \mathcal{T}_h$

$$\|e - e_I\|_{0,\ell} \leq C\left(h_K^{-1/2}\|e - e_I\|_{0,K} + h_K^{1/2}|e - e_I|_{1,K}\right) \leq C h_K^{1/2}\|e\|_{1,K}.$$

Hence, using (3.2) again, we have

$$\begin{aligned} T_2 &\leq C\sum_{K\in\mathcal{T}_h}\left[\|\Delta(\Pi_k^K w_h)\|_{0,K}\|e - e_I\|_{0,K} + \sum_{\ell\in\mathcal{E}_K}\|J_\ell\|_{0,\ell}\|e - e_I\|_{0,\ell}\right] \\ &\leq C\sum_{K\in\mathcal{T}_h}\left[h_K\|\Delta(\Pi_k^K w_h)\|_{0,K}\|e\|_{1,K} + \sum_{\ell\in\mathcal{E}_K} h_K^{1/2}\|J_\ell\|_{0,\ell}\|e\|_{1,K}\right] \\ &\leq C\left\{\sum_{K\in\mathcal{T}_h}\left[h_K^2\|\Delta(\Pi_k^K w_h)\|_{0,K}^2 + \sum_{\ell\in\mathcal{E}_K} h_K\|J_\ell\|_{0,\ell}^2\right]\right\}^{1/2}|e|_{1,\Omega}, \end{aligned} \tag{3.5}$$

where for the last estimate we have used (2.7).

To bound T_3, we use the *stability* property (2.5) and (3.2) to write

$$T_3 \leq C\sum_{K\in\mathcal{T}_h} a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2}\|e\|_{1,K} \leq C\left(\sum_{K\in\mathcal{T}_h}\theta_K^2\right)^{1/2}|e|_{1,\Omega}, \tag{3.6}$$

where for the last estimate we have used Remark 3.2 and (2.7) again.

Finally, to bound T_4, we add and subtract $\Pi_k^K w_h$ on each $K \in \mathcal{T}_h$ and use the *k-consistency* property (2.4):

$$\begin{aligned} T_4 &= \sum_{K\in\mathcal{T}_h}\left[a_h^K(w_h - \Pi_k^K w_h, e_I) - a^K(w_h - \Pi_k^K w_h, e_I)\right] \\ &\leq \sum_{K\in\mathcal{T}_h} a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2} a_h^K(e_I, e_I)^{1/2} + \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2} a^K(e_I, e_I)^{1/2} \\ &\leq C\sum_{K\in\mathcal{T}_h} a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2}|e_I|_{1,K} \\ &\leq C\left(\sum_{K\in\mathcal{T}_h}\theta_K^2\right)^{1/2}|e|_{1,\Omega}, \end{aligned} \tag{3.7}$$

where we have used the *stability* property (2.5), (3.2) and (2.7) for the last two inequalities.

Thus, the result follows from (3.3)–(3.7). □

Although the virtual approximate eigenfunction is w_h, this function is not known in practice. Instead of w_h, what can be used as an approximation of the eigenfunction is $\Pi_k w_h$, where Π_k is defined for $v_h \in V_h$ by

$$(\Pi_k v_h)|_K := \Pi_k^K v_h \quad \forall K \in \mathcal{T}_h.$$

Notice that $\Pi_k w_h$ is actually computable. The following result shows that an estimate similar to that of Lemma 3.4 holds true for $\Pi_k w_h$.

Lemma 3.5. *There exists a constant $C > 0$ independent of h such that*

$$|w - \Pi_k w_h|_{1,h} \leq C\left(\eta + \frac{\lambda + \lambda_h}{2}\|w - w_h\|_{0,\Gamma_0}\right).$$

Proof. For each polygon $K \in \mathcal{T}_h$, we have that

$$|w - \Pi_k^K w_h|_{1,K} \leq |w - w_h|_{1,K} + |w_h - \Pi_k^K w_h|_{1,K}.$$

Then, summing over all polygons we obtain

$$|w - \Pi_k w_h|_{1,h} \leq C\left(\sum_{K\in\mathcal{T}_h}|w - w_h|_{1,K}^2 + \sum_{K\in\mathcal{T}_h}|w_h - \Pi_k^K w_h|_{1,K}^2\right)^{1/2}.$$

Now, using (2.3) together with Remark 3.2, we have that

$$|w_h - \Pi_k^K w_h|_{1,K}^2 \leq \frac{1}{c_0} S^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) = \frac{1}{c_0}\theta_K^2 \leq \frac{1}{c_0}\eta_K^2.$$

Thus, the result follows from Lemma 3.4. □

In what follows, we prove also an upper bound for the eigenvalue approximation.

Lemma 3.6. *There exists a constant $C > 0$ independent of h such that*

$$|\lambda - \lambda_h| \leq C\left(\eta + \frac{\lambda + \lambda_h}{2}\|w - w_h\|_{0,\Gamma_0}\right)^2.$$

Proof. From the symmetry of the bilinear forms together with the facts that $a(w, v) = \lambda b(w, v)$ for all $v \in H^1(\Omega)$, $a_h(w_h, v_h) = \lambda_h b(w_h, v_h)$ for all $v_h \in V_h$ and $b(w_h, w_h) = 1$, we have

$$\begin{aligned} |\lambda - \lambda_h| &= \frac{|a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) + a_h(w_h, w_h) - a(w_h, w_h)|}{b(w_h, w_h)} \\ &\leq C\left[|w - w_h|_{1,\Omega}^2 + \|w - w_h\|_{0,\Gamma_0}^2 + |a_h(w_h, w_h) - a(w_h, w_h)|\right] \\ &\leq C\left[|w - w_h|_{1,\Omega}^2 + |a_h(w_h, w_h) - a(w_h, w_h)|\right], \end{aligned} \tag{3.8}$$

where we have also used a trace theorem and (2.7). We now bound the last term on the right-hand side above using the definition of $a_h(\cdot,\cdot)$ and (2.3):

$$\begin{aligned} &|a_h(w_h, w_h) - a(w_h, w_h)| \\ &= \left|\sum_{K\in\mathcal{T}_h}\left[a^K(\Pi_k^K w_h, \Pi_k^K w_h) + S^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right)\right] - \sum_{K\in\mathcal{T}_h} a^K(w_h, w_h)\right| \\ &\leq \left|\sum_{K\in\mathcal{T}_h}\left[a^K\left(\Pi_k^K w_h, \Pi_k^K w_h\right) - a^K(w_h, w_h)\right]\right| + \sum_{K\in\mathcal{T}_h} c_1\, a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right) \\ &= \sum_{K\in\mathcal{T}_h}(1 + c_1)\, a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right) \\ &\leq (1 + c_1)\sum_{K\in\mathcal{T}_h}\left(|w_h - w|_{1,K}^2 + \left|w - \Pi_k^K w_h\right|_{1,K}^2\right). \end{aligned}$$

Finally, from the above estimate and (3.8) we obtain

$$|\lambda - \lambda_h| \leq C\left(|w - w_h|_{1,\Omega}^2 + |w - \Pi_k w_h|_{1,h}^2\right). \tag{3.9}$$

Hence, we conclude the proof thanks to Lemmas 3.4 and 3.5. □

As claimed above, the upper bounds of the last three lemmas are not computable since they involve the error term $\|w - w_h\|_{0,\Gamma_0}$. Our next goal is to prove that this term is asymptotically negligible in these estimates. With this aim, we will improve the estimate (2.10) by proving that

$$\|w - w_h\|_{0,\Gamma_0} \leq Ch^{\min\{r,1\}/2}\left(|w - w_h|_{1,\Omega} + |w - \Pi_k w_h|_{1,h}\right). \tag{3.10}$$

This proof is based on the arguments used in Section 4 from [18]. To avoid repeating them step by step, in what follows we will only report the changes that have to be made in order to prove (3.10).

We define in $H^1(\Omega)$ the bilinear form $\widehat{a}(\cdot,\cdot) := a(\cdot,\cdot) + b(\cdot,\cdot)$, which is elliptic [18, Lemma 2.1]. Let $u \in H^1(\Omega)$ be the solution of

$$\widehat{a}(u, v) = b(w, v) \quad \forall v \in H^1(\Omega).$$

Since $a(w, v) = \lambda b(w, v)$ we have that $u = w/(\lambda + 1)$. We also define in V_h the bilinear form $\widehat{a}_h(\cdot,\cdot) := a_h(\cdot,\cdot) + b(\cdot,\cdot)$, which is elliptic uniformly in h [18, Lemma 3.1]. Let $u_h \in V_h$ be the solution of

$$\widehat{a}_h(u_h, v_h) = b(w, v_h) \quad \forall v_h \in V_h. \tag{3.11}$$

The arguments in the proof of Lemma 4.3 from [18] can be easily modified to prove that

$$\|u - u_h\|_{0,\Gamma_0} \leq Ch^{\min\{r,1\}/2}\left(|u - u_h|_{1,\Omega} + |u - \Pi_k u_h|_{1,h}\right).$$

Then, using this estimate in the proof of Theorem 4.4 from [18] yields

$$\|w - w_h\|_{0,\Gamma_0} \le Ch^{\min\{r,1\}/2} \left(|u - u_h|_{1,\Omega} + |u - \Pi_k u_h|_{1,h} \right). \tag{3.12}$$

Now, since as stated above $u = w/(\lambda + 1)$, we have that

$$|u - u_h|_{1,\Omega} \le \frac{|w - w_h|_{1,\Omega}}{|\lambda + 1|} + \left| \frac{1}{\lambda + 1} - \frac{1}{\lambda_h + 1} \right| |w_h|_{1,\Omega} + \left| \frac{w_h}{\lambda_h + 1} - u_h \right|_{1,\Omega}. \tag{3.13}$$

For the second term on the right hand side above, we use (3.9) to write

$$\left| \frac{1}{\lambda + 1} - \frac{1}{\lambda_h + 1} \right| = \frac{|\lambda - \lambda_h|}{|\lambda + 1|\,|\lambda_h + 1|} \le C \left(|w - w_h|^2_{1,\Omega} + |w - \Pi_k w_h|^2_{1,h} \right). \tag{3.14}$$

To estimate the third term we recall first that

$$\widehat{a}_h(w_h, v_h) = (\lambda_h + 1) b(w_h, v_h) \quad \forall v_h \in V_h.$$

Then, subtracting this equation divided by $\lambda_h + 1$ from (3.11) we have that

$$\widehat{a}_h \left(u_h - \frac{w_h}{\lambda_h + 1}, v_h \right) = b(w - w_h, v_h) \quad \forall v_h \in V_h.$$

Hence, from the uniform ellipticity of $\widehat{a}_h(\cdot, \cdot)$ in V_h, we obtain

$$\left\| u_h - \frac{w_h}{\lambda_h + 1} \right\|^2_{1,\Omega} \le C \|w - w_h\|_{0,\Gamma_0} \left\| u_h - \frac{w_h}{\lambda_h + 1} \right\|_{0,\Gamma_0} \le C \|w - w_h\|_{0,\Gamma_0} \left\| u_h - \frac{w_h}{\lambda_h + 1} \right\|_{1,\Omega}.$$

Therefore

$$\left\| u_h - \frac{w_h}{\lambda_h + 1} \right\|_{1,\Omega} \le C \|w - w_h\|_{0,\Gamma_0} \le C \|w - w_h\|_{1,\Omega} \le C |w - w_h|_{1,\Omega}, \tag{3.15}$$

the last inequality because of Poincaré inequality (2.7). Then, substituting (3.14) and (3.15) into (3.13) we obtain

$$|u - u_h|_{1,\Omega} \le C \left(|w - w_h|_{1,\Omega} + |w - \Pi_k w_h|_{1,h} \right). \tag{3.16}$$

For the other term on the right hand side of (3.12) we have

$$|u - \Pi_k u_h|_{1,h} \le |u - u_h|_{1,\Omega} + |u_h - \Pi_k u_h|_{1,h}, \tag{3.17}$$

whereas

$$\begin{aligned} |u_h - \Pi_k u_h|_{1,h} &\le \left| u_h - \frac{w_h}{\lambda_h + 1} \right|_{1,\Omega} + \frac{|w_h - \Pi_k w_h|_{1,h}}{\lambda_h + 1} + \left| \Pi_k \left(\frac{w_h}{\lambda_h + 1} - u_h \right) \right|_{1,h} \\ &\le 2 \left| u_h - \frac{w_h}{\lambda_h + 1} \right|_{1,\Omega} + \frac{|w - w_h|_{1,\Omega}}{\lambda_h + 1} + \frac{|w - \Pi_k w_h|_{1,h}}{\lambda_h + 1} \\ &\le C \left(|w - w_h|_{1,\Omega} + |w - \Pi_k w_h|_{1,h} \right), \end{aligned}$$

where we have used (3.15) for the last inequality. Substituting this and estimate (3.16) into (3.17) we obtain

$$|u - \Pi_k u_h|_{1,h} \le C \left(|w - w_h|_{1,\Omega} + |w - \Pi_k w_h|_{1,h} \right).$$

Finally, substituting the above estimate and (3.16) into (3.12), we conclude the proof of the following result.

Lemma 3.7. *There exists $C > 0$ independent of h such that*

$$\|w - w_h\|_{0,\Gamma_0} \le Ch^{\min\{r,1\}/2} \left(|w - w_h|_{1,\Omega} + |w - \Pi_k w_h|_{1,h} \right).$$

Using this result, now it is easy to prove that the term $\|w - w_h\|_{0,\Gamma_0}$ in Lemmas 3.4–3.6 is asymptotically negligible. In fact, we have the following result.

Theorem 3.1. *There exist positive constants C and h_0 such that, for all $h < h_0$, there holds*

$$|w - w_h|_{1,\Omega} + |w - \Pi_k w_h|_{1,h} \le C\eta; \tag{3.18}$$

$$|\lambda - \lambda_h| \le C\eta^2. \tag{3.19}$$

Proof. From Lemmas 3.4, 3.5 and 3.7 we have

$$|w-w_h|_{1,\Omega}+|w-\Pi_k w_h|_{1,h}\leq C\left(\eta+h^{\min\{r,1\}/2}\left(|w-w_h|_{1,\Omega}+|w-\Pi_k w_h|_{1,h}\right)\right).$$

Hence, it is straightforward to check that there exists $h_0>0$ such that for all $h<h_0$ (3.18) holds true.

On the other hand, from Lemma 3.7 and (3.18) we have that for all $h<h_0$

$$\|w-w_h\|_{0,\Gamma_0}\leq Ch^{\min\{r,1\}/2}\eta.$$

Then, for h small enough, (3.19) follows from Lemma 3.6 and the above estimate. $\square$

3.2. Efficiency of the a posteriori error estimator

We will show in this section that the local error indicators η_K are efficient in the sense that they point out correctly which polygons should be refined.

With this end, first, we prove an upper estimate for the volumetric residual term R_K.

Lemma 3.8. *There exists a constant $C>0$ independent of h_K such that*

$$R_K\leq C\left(|w-w_h|_{1,K}+\left|w-\Pi_k^K w_h\right|_{1,K}\right).$$

Proof. For any $K\in\mathcal{T}_h$, let ψ_K be the corresponding interior bubble function. We define $v:=\psi_K\Delta(\Pi_k^K w_h)$. Since v vanishes on the boundary of K, it may be extended by zero to the whole domain Ω. This extension, again denoted by v, belongs to $H^1(\Omega)$ and from Lemma 3.3 we have

$$a^K(e,v)=-a^K\left(w_h-\Pi_k^K w_h,\psi_K\Delta(\Pi_k^K w_h)\right)+\int_K\Delta(\Pi_k^K w_h)\psi_K\Delta(\Pi_k^K w_h).$$

Since $\Delta(\Pi_k^K w_h)\in\mathbb{P}_{k-2}(K)$, using Lemma 3.1 and the above equality we obtain

$$\begin{aligned}
C^{-1}\|\Delta(\Pi_k^K w_h)\|_{0,K}^2&\leq\int_K\psi_K\Delta(\Pi_k^K w_h)^2\\
&=a^K\left(e,\psi_K\Delta(\Pi_k^K w_h)\right)+a^K\left(w_h-\Pi_k^K w_h,\psi_K\Delta(\Pi_k^K w_h)\right)\\
&\leq C\left(|e|_{1,K}+\left|w_h-\Pi_k^K w_h\right|_{1,K}\right)\left|\psi_K\Delta(\Pi_k^K w_h)\right|_{1,K}\\
&\leq Ch_K^{-1}\left(|e|_{1,K}+\left|w-\Pi_k^K w_h\right|_{1,K}\right)\left\|\Delta(\Pi_k^K w_h)\right\|_{0,K},
\end{aligned}\tag{3.20}$$

where, for the last inequality, we have used again Lemma 3.1 and triangular inequality. Multiplying the above inequality by h_K allows us to conclude the proof. $\square$

Next goal is to obtain an upper estimate for the local consistency term θ_K.

Lemma 3.9. *There exists $C>0$ independent of h_K such that*

$$\theta_K\leq C\left(|w-w_h|_{1,K}+|w-\Pi_k^K w_h|_{1,K}\right).$$

Proof. From the definition of θ_K together with Remark 3.2 and estimate (2.3) we have

$$\theta_K\leq C|w_h-\Pi_k^K w_h|_{1,K}\leq C\left(|w_h-w|_{1,K}+|w-\Pi_k^K w_h|_{1,K}\right).$$

The proof is complete. $\square$

The following lemma provides an upper estimate for the jump terms of the local error indicator.

Lemma 3.10. *There exists a constant $C>0$ independent of h_K such that*

$$h_K^{1/2}\|J_\ell\|_{0,\ell}\leq C\left(|w-w_h|_{1,K}+\left|w-\Pi_k^K w_h\right|_{1,K}\right)\quad\forall\ell\in\mathcal{E}_K\cap\mathcal{E}_{\Gamma_1},\tag{3.21}$$

$$h_K^{1/2}\|J_\ell\|_{0,\ell}\leq C\left(|w-w_h|_{1,K}+\left|w-\Pi_k^K w_h\right|_{1,K}+h_K^{1/2}\|\lambda w-\lambda_h w_h\|_{0,\ell}\right)\quad\forall\ell\in\mathcal{E}_K\cap\mathcal{E}_{\Gamma_0},\tag{3.22}$$

$$h_K^{1/2}\|J_\ell\|_{0,\ell}\leq C\sum_{K'\in\omega_\ell}\left(|w-w_h|_{1,K'}+|w-\Pi_k^{K'}w_h|_{1,K'}\right)\quad\forall\ell\in\mathcal{E}_K\cap\mathcal{E}_\Omega,\tag{3.23}$$

where $\omega_\ell:=\{K'\in\mathcal{T}_h:\ell\in\mathcal{E}_{K'}\}$.

Proof. First, for $\ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_1}$, we extend $J_\ell \in \mathbb{P}_{k-1}(\ell)$ to the element K as in Remark 3.1. Let ψ_ℓ be the corresponding edge bubble function. We define $v := J_\ell\psi_\ell$. Then, v may be extended by zero to the whole domain Ω. This extension, again denoted by v, belongs to $H^1(\Omega)$ and from Lemma 3.3 we have that

$$a^K(e,v) = -a^K(w_h - \Pi_k^K w_h, J_\ell\psi_\ell) + \int_K \Delta\left(\Pi_k^K w_h\right) J_\ell\psi_\ell + \int_\ell J_\ell^2\psi_\ell.$$

For $J_\ell \in \mathbb{P}_{k-1}(\ell)$, from Lemma 3.2 and the above equality we obtain

$$\begin{aligned}
C^{-1}\|J_\ell\|_{0,\ell}^2 &\le \int_\ell J_\ell^2\psi_\ell \le C\left[\left(|e|_{1,K} + |w_h - \Pi_k^K w_h|_{1,K}\right)|\psi_\ell J_\ell|_{1,K} + \left\|\Delta(\Pi_k^K w_h)\right\|_{0,K}\|J_\ell\psi_\ell\|_{0,K}\right]\\
&\le C\left[\left(|e|_{1,K} + |w_h - \Pi_k^K w_h|_{1,K}\right)h_K^{-1/2}\|J_\ell\|_{0,\ell} + h_K^{-1}\left(\left|w - \Pi_k^K w_h\right|_{1,K} + |e|_{1,K}\right)h_K^{1/2}\|J_\ell\|_{0,\ell}\right]\\
&\le Ch_K^{-1/2}\|J_\ell\|_{0,\ell}\left(|e|_{1,K} + \left|w - \Pi_k^K w_h\right|_{1,K}\right),
\end{aligned}$$

where we have used again Lemma 3.2 together with estimate (3.20). Multiplying by $h_K^{1/2}$ the above inequality allows us to conclude (3.21).

Secondly, for $\ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_0}$, we extend $v := J_\ell\psi_\ell$ to $H^1(\Omega)$ as in the previous case and use Lemma 3.3 to write

$$a^K(e,v) = \lambda\int_\ell wJ_\ell\psi_\ell - \lambda_h\int_\ell w_hJ_\ell\psi_\ell - a^K\left(w_h - \Pi_k^K w_h, J_\ell\psi_\ell\right) + \int_K \Delta\left(\Pi_k^K w_h\right)J_\ell\psi_\ell + \int_\ell J_\ell^2\psi_\ell.$$

Then, since $J_\ell \in \mathbb{P}_k(\ell)$, repeating the previous arguments we obtain

$$\left|\int_\ell J_\ell^2\psi_\ell\right| \le C\left[\left|\lambda_h\int_\ell w_hJ_\ell\psi_\ell - \lambda\int_\ell wJ_\ell\psi_\ell\right| + h_K^{-1/2}\|J_\ell\|_{0,\ell}\left(\left|w - \Pi_k^K w_h\right|_{1,K} + |e|_{1,K}\right)\right].$$

Hence, using Lemma 3.2 and a local trace inequality we arrive at

$$\begin{aligned}
\|J_\ell\|_{0,\ell}^2 &\le C\left[\|\lambda w - \lambda_h w_h\|_{0,\ell}\|\psi_\ell J_\ell\|_{0,\ell} + h_K^{-1/2}\left(\left|w - \Pi_k^K w_h\right|_{1,K} + |e|_{1,K}\right)\|J_\ell\|_{0,\ell}\right]\\
&\le Ch_K^{-1/2}\|J_\ell\|_{0,\ell}\left(\left|w - \Pi_k^K w_h\right|_{1,K} + |e|_{1,K} + h_K^{1/2}\|\lambda w - \lambda_h w_h\|_{0,\ell}\right),
\end{aligned}$$

where we have used Lemma 3.2 again. Multiplying by $h_K^{1/2}$ the above inequality yields (3.22).

Finally, for $\ell \in \mathcal{E}_K \cap \mathcal{E}_\Omega$, we extend $v := J_\ell\psi_\ell$ to $H^1(\Omega)$ as above again and use Lemma 3.3 to write

$$a(e,v) = -\sum_{K'\in\omega_\ell} a^{K'}(w_h - \Pi_k^{K'} w_h, J_\ell\psi_\ell) + \sum_{K'\in\omega_\ell}\int_{K'}\Delta\left(\Pi_k^{K'} w_h\right)J_\ell\psi_\ell + \sum_{K'\in\omega_\ell}\int_\ell J_\ell^2\psi_\ell.$$

Then, proceeding analogously to the previous case we obtain

$$\|J_\ell\|_{0,\ell}^2 \le Ch_K^{-1/2}\|J_\ell\|_{0,\ell}\left[\sum_{K'\in\omega_\ell}\left(|e|_{1,K'} + |w - \Pi_k^{K'} w_h|_{1,K'}\right)\right].$$

Thus, the proof is complete. □

Now, we are in a position to prove an upper bound for the local error indicators η_K.

Theorem 3.2. *There exists $C > 0$ such that*

$$\eta_K^2 \le C\left[\sum_{K'\in\omega_K}\left(|w - \Pi_k^{K'} w_h|_{1,K'}^2 + |w - w_h|_{1,K'}^2 + \sum_{\ell\in\mathcal{E}_K\cap\mathcal{E}_{\Gamma_0}} h_K\|\lambda w - \lambda_h w_h\|_{0,\ell}^2\right)\right],$$

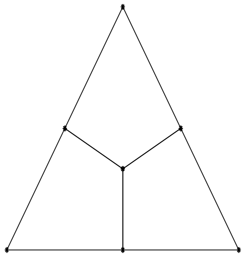
where $\omega_K := \{K' \in \mathcal{T}_h : K' \text{ and } K \text{ share an edge}\}$.

Proof. It follows immediately from Lemmas 3.8–3.10. □

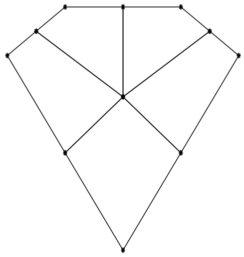
According to the above theorem, for those elements K whose edges do not lie on Γ_0, the error indicators η_K^2 provide lower bounds of the error terms $\sum_{K'\in\omega_K}\left(|w - \Pi_k^{K'} w_h|_{1,K'}^2 + |w - w_h|_{1,K'}^2\right)$ in the neighborhood ω_K of K. For those elements K with an edge on Γ_0, the term $h_K\|\lambda w - \lambda_h w_h\|_{0,\ell}^2$ also appears in the estimate. Let us remark that it is reasonable to expect this terms to be asymptotically negligible. In fact, this is the case at least for the global estimator $\eta^2 = \sum_{K\in\mathcal{T}_h}\eta_K^2$ as is shown in the following result.

Corollary 3.1. *There exists a constant $C > 0$ such that*

$$\eta^2 \le C\left(|w - w_h|_{1,\Omega}^2 + |w - \Pi_k w_h|_{1,h}^2\right).$$



(a) Triangle K refined into 3 quadrilaterals.



(b) Pentagon K refined into 5 quadrilaterals.

Fig. 1. Example of refined elements for VEM strategy.

Proof. From Theorem 3.2 we have that

$$\eta^2 \leq C\left(|w - w_h|^2_{1,\Omega} + |w - \Pi_k w_h|^2_{1,h} + h\|\lambda w - \lambda_h w_h\|^2_{0,\Gamma_0}\right).$$

The last term on the right hand side above is bounded as follows:

$$\|\lambda w - \lambda_h w_h\|^2_{0,\Gamma_0} \leq 2\lambda^2\|w - w_h\|^2_{0,\Gamma_0} + 2|\lambda - \lambda_h|^2,$$

where we have used that $\|w_h\|_{0,\Gamma_0} = 1$. Now, by using a trace inequality and Poincaré inequality (2.7) we have

$$\|w - w_h\|_{0,\Gamma_0} \leq C|w - w_h|_{1,\Omega}.$$

On the other hand, using the estimate (3.9), we have

$$|\lambda - \lambda_h|^2 \leq (|\lambda| + |\lambda_h|)|\lambda - \lambda_h| \leq C\left(|w - w_h|^2_{1,\Omega} + |w - \Pi_k w_h|^2_{1,h}\right).$$

Therefore,

$$\eta^2 \leq C\left(|w - w_h|^2_{1,\Omega} + |w - \Pi_k w_h|^2_{1,h}\right)$$

and we conclude the proof. □

4. Numerical results

In this section, we will investigate the behavior of an adaptive scheme driven by the error indicator in two numerical tests that differ in the shape of the computational domain Ω and, hence, in the regularity of the exact solution. With this aim, we have implemented in a MATLAB code a lowest-order VEM ($k = 1$) on arbitrary polygonal meshes following the ideas proposed in [2].

To complete the choice of the VEM, we had to choose the bilinear forms $S^K(\cdot,\cdot)$ satisfying (2.3). In this respect, we proceeded as in [1, Section 4.6]: for each polygon K with vertices $P_1, \ldots, P_{N_K}$, we used

$$S^K(u, v) := \sum_{r=1}^{N_K} u(P_r)v(P_r), \quad u, v \in V_1^K.$$

We have used the MATLAB command `eigs` to compute the smallest eigenvalue λ_h and the corresponding eigenfunction w_h. Then, we have computed $\|w_h\|_{0,\Gamma_0}$ to normalize w_h. Notice that this norm is actually computable, since it only involves values of w_h on edges of the mesh.

One of the goals of our numerical tests is to compare the performance of our VEM code with that of a standard classical finite element method (FEM). Let us remark that, for $k = 1$ and meshes of triangles, VEM reduces to FEM. This fact allowed us to use the VEM code for most of the FEM computations. Actually, both codes only differ in the refinement stage.

In fact, in all our tests we have initiated the adaptive processes with a coarse triangular mesh, but we have used different algorithms to refine the meshes for VEM and FEM. The refinement for FEM was based on the so-called *blue–green-closure* strategy (see [40]), for which all the subsequent meshes consist of triangles. Instead, for VEM, we have used the procedure to refine the meshes described in [37]. It consists of splitting each element into n quadrilaterals (n being the number of edges of the polygon) by connecting the barycenter of the element with the midpoint of each edge as shown in Fig. 1 (see [37] for more details). Notice that although this process is initiated with a mesh of triangles, the successively created meshes will contain other kind of convex polygons, as it can be seen in Figs. 3 and 7.

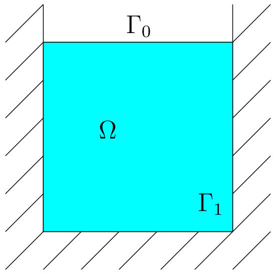


Fig. 2. Test 1. Sloshing in a square domain.

In both procedures (VEM and FEM) we have used the classical strategy of marking to refine those elements K which satisfy

$$\eta_K \geq 0.5 \max_{K' \in \mathcal{T}_h} \{\eta_{K'}\}.$$

Let us remark that we have also tried Dörfler's marking strategy [41], but no significant change appears in the results.

Since we have chosen $k = 1$, according to the definition of the local virtual element space V_1^K (cf. (2.2)), the term $R_K^2 := h_K^2 \|\Delta w_h\|_{0,K}^2$ vanishes. Thus, the quantities that we have actually computed as error indicators for our VEM code are the following:

$$\eta_K^2 = \theta_K^2 + \sum_{\ell \in \mathcal{E}_K} h_K \|J_\ell\|_{0,\ell}^2 \quad \forall K \in \mathcal{T}_h.$$

As claimed above, we have used the same VEM code on triangular meshes for the FEM procedure. In such a case, the term $\theta_K^2 := a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)$ vanishes too, since $V_1^K = \mathbb{P}_1(K)$ and hence Π_k^K is the identity. By the same reason, the projection Π_k^K also disappears in the definition (3.1) of J_ℓ. Therefore, for triangular meshes, not only VEM reduces to FEM, but also the error indicator becomes a classical well-known edge-residual error estimator (see [33]):

$$\eta_K^2 = \sum_{\ell \in \mathcal{E}_K} h_K \|J_\ell\|_{0,\ell}^2 \quad \text{with}\, J_\ell = \begin{cases} \dfrac{1}{2}\left[\!\!\left[\dfrac{\partial w_h}{\partial n}\right]\!\!\right]_\ell, & \ell \in \mathcal{E}_\Omega, \\ \lambda_h w_h - \dfrac{\partial w_h}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_0}, \\ -\dfrac{\partial w_h}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_1}. \end{cases}$$

In what follows, we report the results of a couple of tests. In both cases, we will restrict our attention to the approximation of the eigenvalues. Let us recall that according to Corollary 3.6, the global error estimator η^2 provides an upper bound of the error of the computed eigenvalue.

4.1. Test 1: sloshing in a square domain

We have chosen for this test a problem with known analytical solution. It corresponds to the computation of the sloshing modes of a two-dimensional fluid contained in the domain $\Omega := (0, 1)^2$ with a horizontal free surface Γ_0 as shown in Fig. 2. The solutions of this problem are

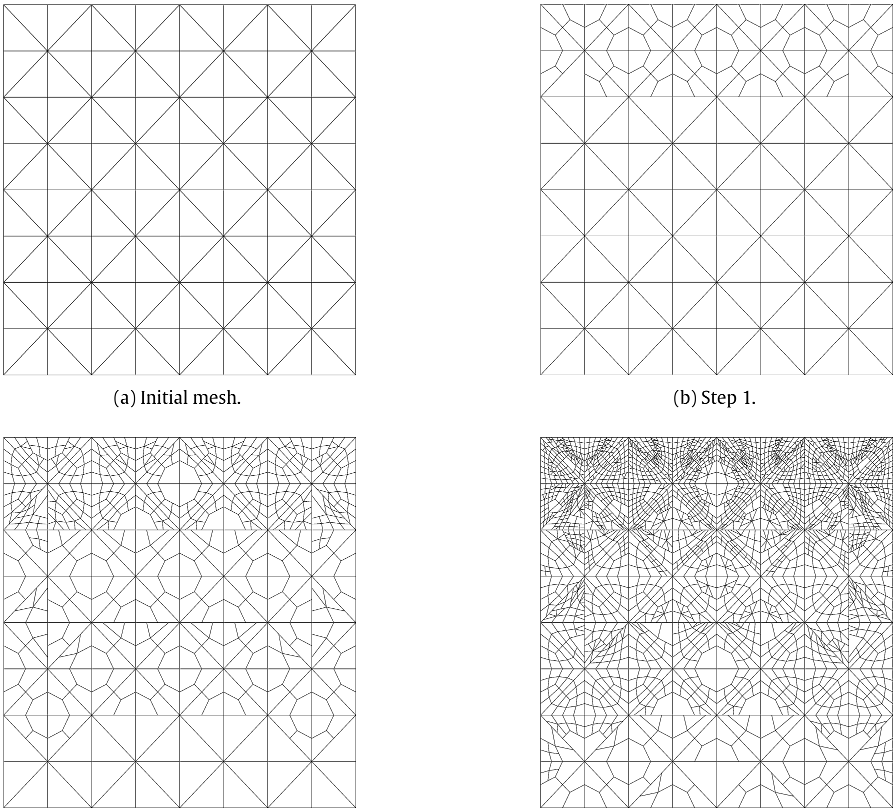
$$\lambda_j = j\pi \tanh(j\pi), \qquad w_j(x, y) = \cos(j\pi x) \sinh(j\pi y), \quad j \in \mathbb{N}.$$

Figs. 3 and 4 show the adaptively refined meshes obtained with VEM and FEM procedures, respectively.

Since the eigenfunctions of this problem are smooth, according to (2.8) we have that $|\lambda - \lambda_h| = \mathcal{O}(h^2)$. Therefore, in case of uniformly refined meshes, $|\lambda - \lambda_h| = \mathcal{O}\left(N^{-1}\right)$, where N denotes the number of degrees of freedom which is the optimal convergence rate that can be attained.

Fig. 5 shows the error curves for the computed lowest eigenvalue on uniformly refined meshes and adaptively refined meshes with FEM and VEM schemes. The plot also includes a line of slope -1, which correspond to the optimal convergence rate of the method $\mathcal{O}\left(N^{-1}\right)$.

It can be seen from Fig. 5 that the three refinement schemes lead to the correct convergence rate. Moreover, the performance of adaptive VEM is a bit better than that of adaptive FEM, while the latter is also better than that of uniform FEM. Moreover, in spite of the fact that the eigenfunction is smooth, both adaptive processes lead to meshes more refined in the vicinity of Γ_0 and the error on these meshes is smaller than that on uniform meshes as can be seen from Fig. 5.



(a) Initial mesh.

(b) Step 1.

(c) Step 3.

(d) Step 6.

Fig. 3. Test 1. Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1, 3 and 6.

Table 1
Test 1. Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM.

N	λ_{h1}	$\|\lambda_1 - \lambda_{h1}\|$	θ^2	J^2	η^2	$\frac{\|\lambda_1-\lambda_{h1}\|}{\eta^2}$
81	3.2499	0.1200	0	0.8245	0.8245	0.1456
167	3.1644	0.0345	0.0111	0.2469	0.2580	0.1339
313	3.1450	0.0151	0.0117	0.1108	0.1225	0.1234
745	3.1355	0.0056	0.0054	0.0427	0.0481	0.1171
1 540	3.1327	0.0028	0.0033	0.0216	0.0249	0.1113
3 392	3.1311	0.0013	0.0015	0.0102	0.0117	0.1069
5 806	3.1307	0.0008	0.0009	0.0064	0.0073	0.1069
11 973	3.1303	0.0004	0.0005	0.0032	0.0037	0.1075

We report in Table 1, the errors $|\lambda_1 - \lambda_{h1}|$ and the estimators η^2 at each step of the adaptive VEM scheme. We include in the table the terms $\theta^2 := \sum_{K\in\mathcal{T}_h}\theta_K^2$ which arise from the inconsistency of VEM and $J^2 := \sum_{K\in\mathcal{T}_h}\left(\sum_{\ell\in\mathcal{E}_K} h_K\|J_\ell\|_{0,\ell}^2\right)$ which arise from the edge residuals. We also report in the table the effectivity indexes $|\lambda_1 - \lambda_{h1}|/\eta^2$.

It can be seen from Table 1 that the effectivity indexes are bounded above and below far from zero and that the inconsistency and edge residual terms are roughly speaking of the same order, none of them being asymptotically negligible.

4.2. Test 2

The aim of this test is to assess the performance of the adaptive scheme when solving a problem with a singular solution. In this test Ω consists of a unit square from which it is subtracted an equilateral triangle as shown in Fig. 6. In this case Ω has a reentrant angle $\omega = \frac{5\pi}{3}$. Therefore, the Sobolev exponent is $r_\Omega := \frac{\pi}{\omega} = 3/5$, so that the eigenfunctions will belong to $H^{1+r}(\Omega)$ for all $r < 3/5$, but in general not to $H^{1+3/5}(\Omega)$. Therefore, according to (2.8), using quasi-uniform meshes, the convergence rate for the eigenvalues should be $|\lambda - \lambda_h| \approx \mathcal{O}\left(h^{6/5}\right) \approx \mathcal{O}\left(N^{-3/5}\right)$. An efficient adaptive scheme should lead to refine the meshes in such a way that the optimal order $|\lambda - \lambda_h| = \mathcal{O}\left(N^{-1}\right)$ could be recovered.

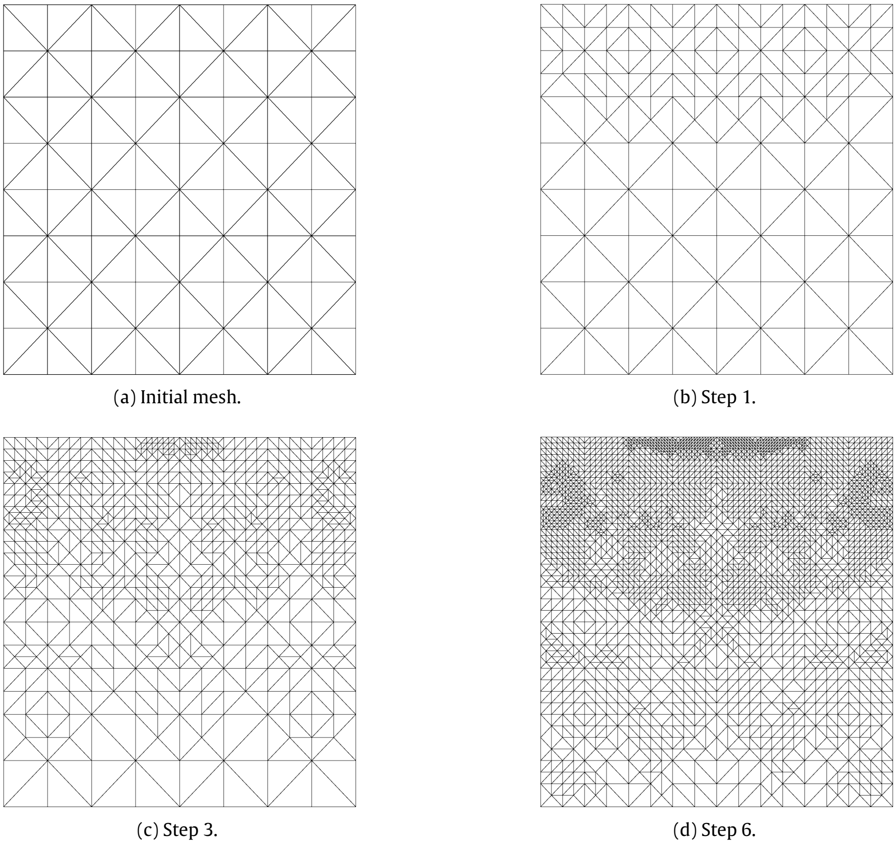


Fig. 4. Test 1. Adaptively refined meshes obtained with FEM scheme at refinement steps 0, 1, 3 and 6.

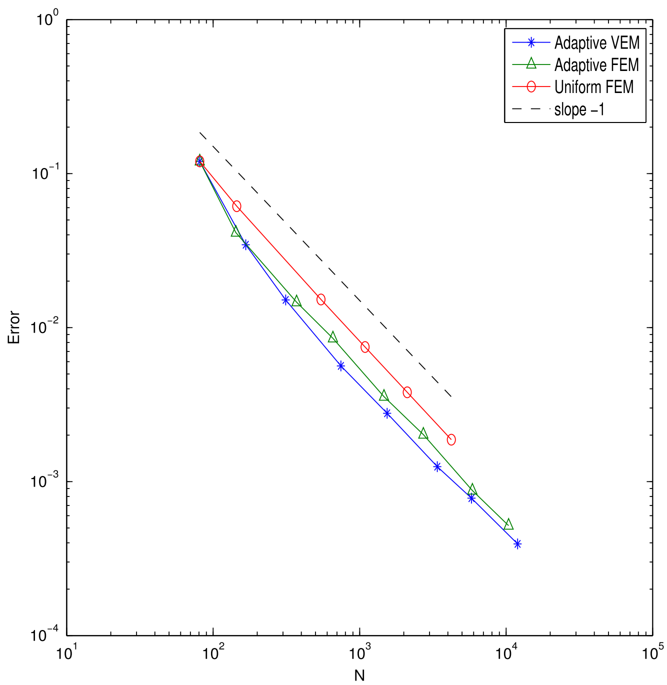


Fig. 5. Test 1. Error curves of $|\lambda_1 - \lambda_{h1}|$ for uniformly refined meshes (''Uniform FEM''), adaptively refined meshes with FEM (''Adaptive FEM'') and adaptively refined meshes with VEM (''Adaptive VEM'').

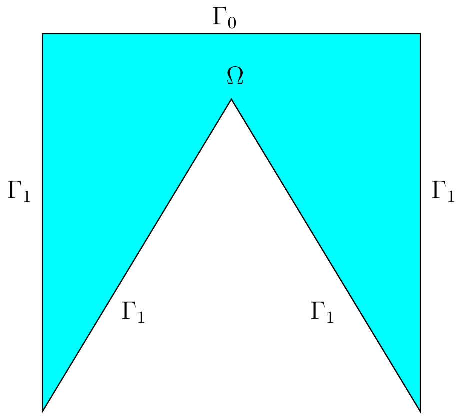


Fig. 6. Test 2. Domain Ω.

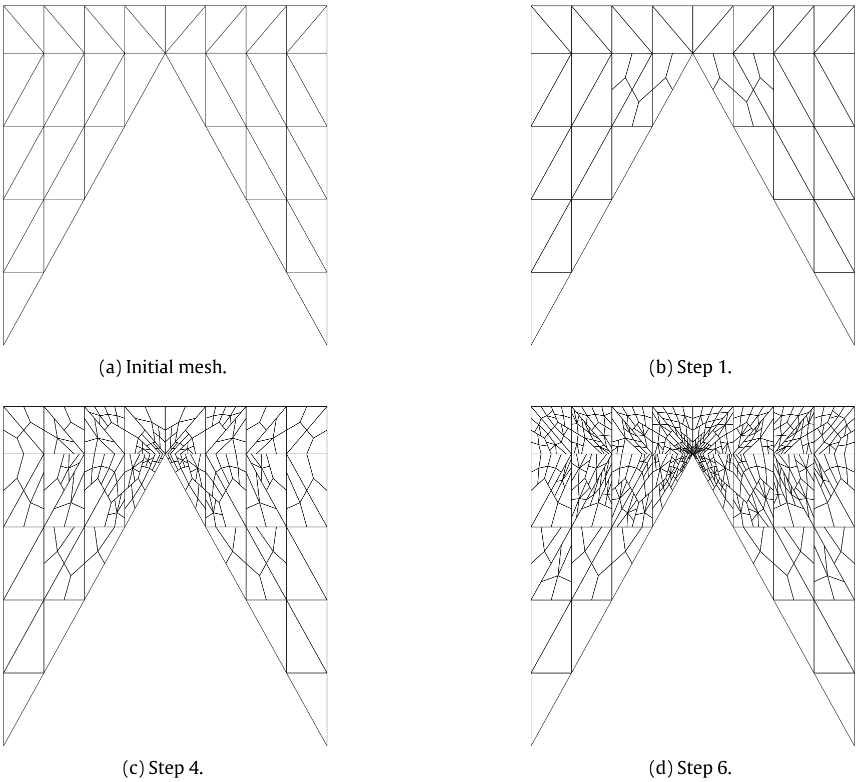


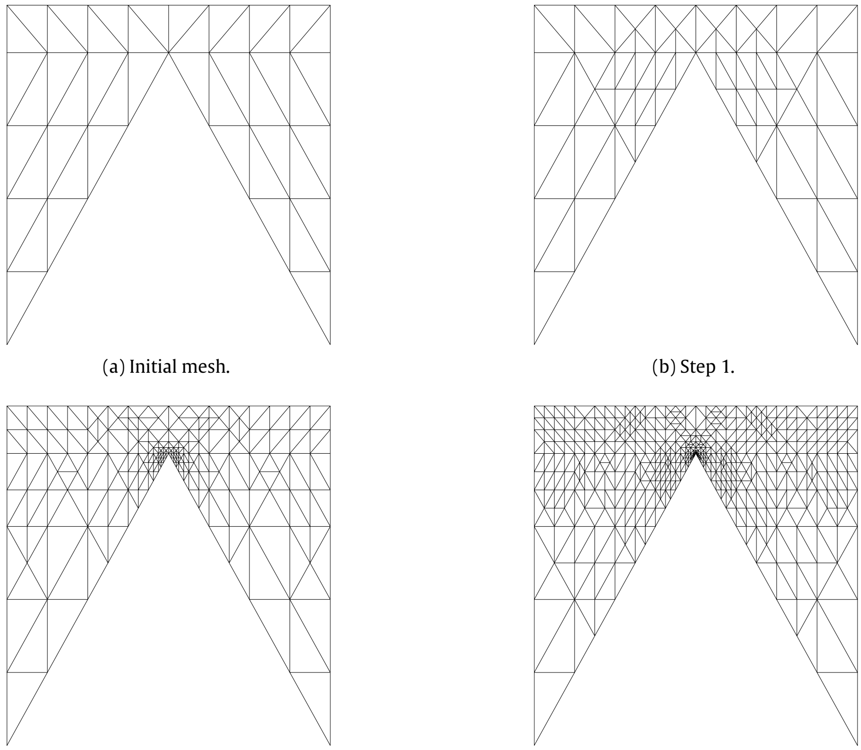
Fig. 7. Test 2. Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1, 4 and 6.

Figs. 7 and 8 show the adaptively refined meshes obtained with the VEM and FEM adaptive schemes, respectively.

In order to compute the errors $|\lambda_1 - \lambda_{h1}|$, due to the lack of an exact eigenvalue, we have used an approximation based on a least squares fitting of the computed values obtained with extremely refined meshes. Thus, we have obtained the value $\lambda_1 = 1.9288$, which has at least four correct significant digits.

We report in Table 2 the lowest eigenvalue λ_{h1} computed with each of the three schemes. Each table includes the estimated convergence rate and the errors $|\lambda_1 - \lambda_{h1}|$.

It can be seen from Table 2, that the uniform refinement leads to a convergence rate close to that predicted by the theory, $\mathcal{O}\left(N^{-3/5}\right)$, while the adaptive VEM and FEM schemes allow us to recover the optimal order of convergence $\mathcal{O}\left(N^{-1}\right)$. This



(a) Initial mesh. (b) Step 1. (c) Step 4. (d) Step 6.

Fig. 8. Test 2. Adaptively refined meshes obtained with FEM scheme at refinement steps 0, 1, 4 and 6.

Table 2
Test 2. Eigenvalue λ_{h1} computed with different schemes: uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").

Uniform FEM			Adaptive VEM			Adaptive FEM		
N	λ_{h1}	$\|\lambda_1-\lambda_{h1}\|$	N	λ_{h1}	$\|\lambda_1-\lambda_{h1}\|$	N	λ_{h1}	$\|\lambda_1-\lambda_{h1}\|$
38	2.3083	0.3795	38	2.3083	0.3795	38	2.3083	0.3795
123	2.0686	0.1398	58	2.0721	0.1433	60	2.1067	0.1779
437	1.9828	0.0540	106	1.9960	0.0672	85	2.0362	0.1074
1 641	1.9505	0.0217	229	1.9592	0.0304	148	1.9810	0.0522
6 353	1.9377	0.0089	350	1.9467	0.0179	185	1.9678	0.0390
14 137	1.9341	0.0053	666	1.9384	0.0096	280	1.9530	0.0242
24 993	1.9325	0.0037	909	1.9354	0.0066	458	1.9427	0.0139
38 291	1.9316	0.0028	1 340	1.9329	0.0041	646	1.9382	0.0094
55 921	1.9310	0.0022	2 141	1.9315	0.0027	895	1.9356	0.0068
75 993	1.9306	0.0018	3 438	1.9306	0.0018	1 593	1.9325	0.0037
99 137	1.9303	0.0015	5 172	1.9300	0.0012	2 122	1.9315	0.0027
125 353	1.9301	0.0013	8 014	1.9296	0.0008	3 178	1.9306	0.0018
154 641	1.9299	0.0011	12 365	1.9293	0.0005	5 341	1.9298	0.0010
187 001	1.9298	0.0010	19 153	1.9291	0.0003	7 522	1.9295	0.0007
222 433	1.9297	0.0009	29 403	1.9290	0.0002	11 124	1.9292	0.0004
λ_1	1.9288	$\mathcal{O}\left(N^{-0.68}\right)$	λ_1	1.9288	$\mathcal{O}\left(N^{-1.10}\right)$	λ_1	1.9288	$\mathcal{O}\left(N^{-1.16}\right)$

can be clearly seen from Fig. 9, where the three error curves are reported. The plot also includes lines of slopes -1 and $-3/5$, which correspond to the expected convergence rates of each scheme. It can also be seen from Table 2 that, although adaptive VEM leads to meshes with a larger number of degrees of freedom than adaptive FEM, the error of the former is significantly smaller than that of the latter. In consequence, as can be seen from Fig. 9, the performances of VEM and FEM strategies are almost the same.

Finally, we report in Table 3 the same information as in Table 1 for this test. Similar conclusions as in the previous test follow from this table.

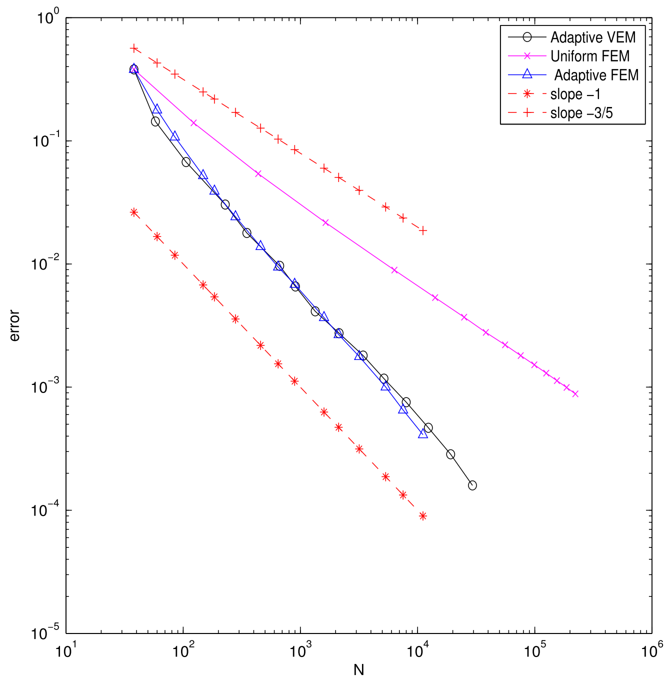


Fig. 9. Test 2. Error curves of $|\lambda_1 - \lambda_{h1}|$ for uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").

Table 3
Test 2. Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM.

N	λ_{h1}	$\|\lambda_1 - \lambda_{h1}\|$	θ^2	J^2	η^2	$\frac{\|\lambda_1-\lambda_{h1}\|}{\eta^2}$
38	2.3083	0.3795	0	2.3181	2.3181	0.1637
58	2.0721	0.1433	0.0379	0.8231	0.8609	0.1664
106	1.9960	0.0672	0.0368	0.4188	0.4556	0.1475
229	1.9592	0.0304	0.0216	0.1942	0.2158	0.1408
350	1.9467	0.0179	0.0164	0.1359	0.1522	0.1173
666	1.9384	0.0096	0.0094	0.0749	0.0844	0.1143
909	1.9354	0.0066	0.0068	0.0556	0.0624	0.1052
1340	1.9329	0.0041	0.0047	0.0408	0.0454	0.0907
2141	1.9315	0.0027	0.0032	0.0275	0.0308	0.0891
3438	1.9306	0.0018	0.0022	0.0178	0.0199	0.0904

Conclusions

We have derived an a posteriori error indicator for a VEM solution of the Steklov eigenvalue problem. We have proved that it is efficient and reliable. For lowest order elements on triangular meshes, VEM coincides with FEM and the a posteriori error indicator also coincides with a classical one. However, VEM allows using general polygonal meshes including hanging nodes, which is particularly interesting when designing an adaptive scheme. We have implemented one such scheme driven by the proposed error indicators. We have assessed its performance by means of a couple of tests which allow us to confirm that the adaptive scheme yields optimal order of convergence for regular as well as singular solutions.

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