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A finite element method for the buckling problem of simply supported Kirchhoff plates

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ABSTRACT

The aim of this paper is to develop a finite element method to approximate the buckling problem of *simply supported* Kirchhoff plates subjected to general plane stress tensor. We introduce an auxiliary variable $w := \Delta u$ (with u representing the displacement of the plate) to write a variational formulation of the spectral problem. We propose a conforming discretization of the problem, where the unknowns are approximated by piecewise linear and continuous finite elements. We show that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. Finally, we present some numerical experiments supporting our theoretical results.

1. Introduction

The finite element method for the approximation of eigenvalue problems is the object of great interest from both the practical and theoretical point of view. We refer to [1–3] and the references therein for the state of art in this subject area. This paper deals with the analysis of the elastic stability of plates, in particular the so-called *buckling problem*. This problem has attracted much interest since it is frequently encountered in engineering applications such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the limit of elastic stability of the plate (i.e., eigenvalues-buckling coefficients and eigenfunctions-buckling modes).

The buckling problem has been studied for years by many researchers, being the Kirchhoff–Love and the Reissner–Mindlin plate theories the most used. For the Reissner–Mindlin theory, in [4] was performed the analysis of the buckling problem of a clamped plate modeled by the Reissner–Mindlin equations. On the other hand, in [5,3,6] different formulations for the buckling problem for a thin plate subjected to clamped boundary conditions and modeled by the Kirchhoff–Love theory are considered, while [7] deals with non-conforming methods for the vibration and buckling problems of the biharmonic equation with general boundary conditions.

One of the most well-known mixed methods to deal with the source problem of thin plates modeled by the biharmonic equation is the method introduced by Ciarlet and Raviart [8]. This was thoroughly studied by many authors (see, for

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instance, [9,10], [11, Section 3(a)], [12, Section 4(a)], [13, Section III.3], [14,15]). The method was applied to the plate vibration problem in [1, Section 11.3], [16] and [3, Section 7(b)], where it was proved to converge for finite elements of degree $k \geq 2$; moreover, for this problem a procedure for accelerating the convergence of the approximation has been studied in [17,18] for finite elements of degree $k \geq 2$ and $k = 1$, respectively. A formulation of the eigenvalue problem for the Stokes equation, which turns out to be equivalent to a plate buckling problem, is also analyzed in [3, Section 7(d)], where it is proved to converge for degree $k \geq 2$, as well.

We observe that the buckling problem of a simply supported and uniformly compressed Kirchhoff plate is simpler than the case when a general plane stress tensor is applied. In fact, the solution of the problem can be related with the solution of the Laplace eigenvalue problem with homogeneous boundary conditions. We also note that the same happens for the vibration problem of a simply supported Kirchhoff plate. However, this is not true in the case when the plate is subjected to general plane stress tensor, which is the case that we will study in this work.

Conforming finite element methods for the primal formulation of the biharmonic equation involve C^1 finite elements, which are quite complicated even in two dimensions. An alternative is to use classical non-conforming finite elements as was studied in [7] for the vibration and buckling problems. The aim of this paper is to analyze a conforming discretization based on piecewise linear and continuous finite element of a variational formulation of the buckling problem of simply supported Kirchhoff plates and subjected to general plane stress tensor.

The method is based on the idea introduced by Ciarlet and Raviart [8], and consists in the introduction of an auxiliary variable $w := \Delta u$ (with u being the transverse displacement of the mean surface of the plate) to write a variational formulation of the spectral problem. To analyze the continuous problem, we introduce the so-called solution operator (whose eigenvalues are the reciprocals of the buckling coefficients) which is a compact operator. We propose a conforming discretization based on piecewise linear and continuous finite elements for the two variables. We use the so-called Babuška–Osborn abstract spectral approximation theory (see [1]) to show that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues.

The outline of the paper is as follows: we introduce in Section 2 the variational formulation of the buckling eigenvalue problem, define a solution operator and establish its spectral characterization. In Section 3, we introduce the finite element discrete formulation and describe the spectrum of a discrete solution operator. In Section 4, we prove that the numerical scheme provides a correct spectral approximation and establish optimal order approximation of the eigenfunctions. We end this section by proving that an improved order of convergence holds for the approximation of the eigenvalues. In Section 5, we report some numerical tests which confirm the theoretical order of the error and allow us to assess the performance of the proposed method. Finally, we summarize some conclusions in Section 6.

Throughout the article we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote with c and C, with or without subscripts, tildes or hats, generic constants independent of the mesh parameter h, which may take different values in different occurrences. Moreover $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support contained in Ω. Finally, we will use the following notation for any 2×2 tensor field $\boldsymbol{\tau}$, any 2D vector field $\boldsymbol{v}$, and any scalar field v:

$$\operatorname{div} \boldsymbol{v} := \partial_1 v_1 + \partial_2 v_2, \qquad \nabla v := \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix},$$

$$\mathbf{div}\, \boldsymbol{\tau} := \begin{pmatrix} \partial_1 \tau_{11} + \partial_2 \tau_{12} \\ \partial_1 \tau_{21} + \partial_2 \tau_{22} \end{pmatrix}.$$

Moreover, we denote

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. The spectral problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with polygonal boundary occupied by the mean surface of a plate, simply supported on its whole boundary Γ (see [19,20]). The plate is assumed to be homogeneous, isotropic, linearly elastic, and sufficiently thin as to be modeled by Kirchhoff–Love equations. We denote by u the transverse displacement of the mean surface of the plate.

The buckling problem of a plate, which is subjected to a plane stress tensor field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{2\times 2}$, $\boldsymbol{\sigma} \neq 0$ reads as the following eigenvalue problem:

$$\begin{cases} \Delta^2 u = -\lambda \operatorname{div}(\boldsymbol{\sigma} \nabla u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where in this case λ is the critical load. To simplify the notation we have taken the Young modulus and the density of the plate, both equal to 1. The applied stress tensor field is assumed to satisfy the equilibrium equations:

$$\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{\sigma} \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{div}\, \boldsymbol{\sigma} = 0 \quad \text{in } \Omega. \tag{2.3}$$

Moreover, we assume that,

$$\boldsymbol{\sigma} \in L^\infty(\Omega)^{2\times 2}. \tag{2.4}$$

However, we do not need to assume $\boldsymbol{\sigma}$ to be positive definite. Let us remark that, in practice, $\boldsymbol{\sigma}$ is the stress distribution on the plate subjected to in-plane loads, which does not need to be positive definite (see, for instance, Section 5).

A classical variational formulation of (2.1) is obtained by testing with $v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$ and using integration by parts in Ω. Thus, we obtain the following symmetric weak formulation:

Find $(\lambda, u) \in \mathbb{R} \times \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega),\ u \neq 0$, such that

$$\int_\Omega \Delta u\, \Delta v = \lambda \int_\Omega (\boldsymbol{\sigma}\nabla u)\cdot\nabla v \quad \forall v \in \mathrm{H}_0^1(\Omega)\cap \mathrm{H}^2(\Omega). \tag{2.5}$$

It is immediate to prove that the eigenvalues of the problem above are real and positive whenever $\boldsymbol{\sigma}$ is positive definite. Since, we are not assuming that hypothesis we can prove that these eigenvalues are real (see Lemma 2.1 below).

In what follows we write another variational formulation of (2.1).

We introduce the auxiliary variable $w := \Delta u$ (see [1, Section 11.3]). Then (2.1) can be rewritten equivalently as follows:

$$\begin{cases} w = \Delta u & \text{in } \Omega,\\ \Delta w = -\lambda \mathrm{div}\, (\boldsymbol{\sigma}\nabla u) & \text{in } \Omega,\\ u = w = 0 & \text{on } \Gamma. \end{cases}$$

Therefore, by testing the system above with functions in $\mathrm{H}_0^1(\Omega)$, we arrive at the following weak formulation:

Find $(\lambda, w, u) \in \mathbb{R}\times \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega),\ u\neq 0$, such that

$$\begin{cases} \displaystyle\int_\Omega \nabla w\cdot\nabla v = -\lambda\int_\Omega (\boldsymbol{\sigma}\nabla u)\cdot\nabla v & \forall v\in \mathrm{H}_0^1(\Omega),\\ \displaystyle\int_\Omega \nabla u\cdot\nabla\tau + \int_\Omega w\tau = 0 & \forall \tau\in \mathrm{H}_0^1(\Omega). \end{cases} \tag{2.6}$$

We note that all the solutions of problem above are solutions of (2.1) in the sense of distributions. In fact, if (λ, w, u) is a solution of (2.6), then by taking as test-function in the second equation $\tau \in \mathcal{D}(\Omega)$, we obtain that $w = \Delta u \in \mathrm{H}_0^1(\Omega)$. On the other hand, taking $v \in \mathcal{D}(\Omega)$ in the first equation, then we have that $\Delta w = -\lambda \mathrm{div}\,(\boldsymbol{\sigma}\nabla u)$; therefore, (λ, u) is a solution of problem (2.1).

The goal of this paper is to propose and analyze a finite element method to solve problem (2.6). In particular, our aim is to obtain accurate approximations of the smallest (in absolute value) eigenvalues λ, which correspond to the buckling coefficients and the associated eigenfunctions or buckling modes.

Remark 2.1. In problem (2.6), the eigenvalues cannot vanish. In fact, if $\lambda = 0$, then the first equation yields $w = 0$, and, from the second one, $u = 0$. Moreover, $\int_\Omega (\boldsymbol{\sigma}\nabla u)\cdot\nabla u \neq 0$ in problem (2.6), despite the fact that $\boldsymbol{\sigma}$ is not positive definite. In fact, if $\int_\Omega (\boldsymbol{\sigma}\nabla u)\cdot\nabla u = 0$, the first and second equations of problem (2.6) imply that $w = 0$ and $\Delta u = 0$, hence, $u = 0$.

Now, we introduce a more compact notation for the spectral problem (2.6). Let $\mathbb{A} : \left(\mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega)\right) \times \left(\mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega)\right) \to \mathbb{R}$, and $\mathbb{B} : \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega) \to \mathbb{R}$, be the continuous and symmetric bilinear forms respectively defined by

$$\mathbb{A}((w,u),(\tau,v)) := \int_\Omega \nabla w\cdot\nabla v + \int_\Omega \nabla u\cdot\nabla\tau + \int_\Omega w\tau,$$

$$\mathbb{B}(u,v) := \int_\Omega (\boldsymbol{\sigma}\nabla u)\cdot\nabla v.$$

Using this notation, problem (2.6) can be written as follows:

Find $(\lambda, w, u) \in \mathbb{R}\times \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega),\ u\neq 0$, such that

$$\mathbb{A}((w,u),(\tau,v)) = -\lambda\mathbb{B}(u,v) \quad \forall(\tau,v)\in \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega). \tag{2.7}$$

Before introducing the numerical method, we define the linear operator corresponding to the source problem associated with the buckling spectral problem (2.7) and prove some properties that will be used for the subsequent convergence analysis.

First, we introduce the following bounded linear operator which is called the *solution operator*:

$$\begin{aligned} T:\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega),\\ f &\mapsto u, \end{aligned}$$

with $(w, u) \in \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega)$ being the solution of the corresponding source problem:

$$\mathbb{A}((w,u),(\tau,v)) = -\mathbb{B}(f,v) \quad \forall(\tau,v)\in \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega). \tag{2.8}$$

This problem is well posed. In fact, it can be decomposed into the following well posed problems:

- Find $w \in \mathrm{H}_0^1(\Omega)$ such that

$$\int_\Omega \nabla w \cdot \nabla v = -\int_\Omega (\boldsymbol{\sigma}\nabla f)\cdot\nabla v \quad \forall v \in \mathrm{H}_0^1(\Omega). \tag{2.9}$$

- Find $u \in \mathrm{H}_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \tau = -\int_\Omega w\tau \quad \forall \tau \in \mathrm{H}_0^1(\Omega). \tag{2.10}$$

Clearly λ is an eigenvalue of problem (2.7) if and only if $\mu := \frac{1}{\lambda}$ is a non-zero eigenvalue of T, with the same multiplicity and corresponding eigenfunctions u (recall $\lambda \neq 0$; cf. Remark 2.1).

In order to obtain the spectral characterization, we introduce the following well posed problem:

Given $f \in \mathrm{H}_0^1(\Omega)$, find $u \in \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)$ such that

$$\int_\Omega \Delta u\,\Delta v = \int_\Omega (\boldsymbol{\sigma}\nabla f)\cdot\nabla v \quad \forall v \in \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega). \tag{2.11}$$

The following result regarding the equivalence of problems (2.8) and (2.11) has been dealt with in [21]. We include a proof for the sake of completeness.

Proposition 2.1. *(w, u) is a solution of problem (2.8) if and only if u is a solution of problem (2.11).*

Proof. Let (w, u) be a solution of (2.8), equivalently, w and u are solutions of problems (2.9) and (2.10), respectively. On the one hand, from (2.10) we have that $w = \Delta u$. Therefore, $\Delta u \in \mathrm{H}_0^1(\Omega)$ and since Ω is a convex domain, we have that $u \in \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)$. On the other hand, from (2.9), first using that $w = \Delta u$, and then integration by parts we conclude that u is a solution of problem (2.11).

Now, let $u \in \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)$ be a solution of (2.11). Since Ω is a convex polygonal domain, by resorting to a well known regularity result for the biharmonic problem with its right-hand side in $\mathrm{H}_0^1(\Omega)'$, we have that $u \in \mathrm{H}^3(\Omega)$ and $w = \Delta u \in \mathrm{H}_0^1(\Omega)$. So, testing problem (2.11) with adequate functions we have that (w, u) be a solution of (2.8). Thus, the proof is complete.

As a consequence, we have the following spectral characterization result.

Lemma 2.1. *The spectrum of T satisfies $\mathrm{sp}(T) = \{0\}\cup\{\mu_n : n \in \mathbb{N}\}$, where $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of real eigenvalues which converges to 0. The multiplicity of each non-zero eigenvalue is finite and its ascent is 1.*

Proof. By virtue of the equivalence between problems (2.8) and (2.11), T is also a bounded linear operator from $\mathrm{H}_0^1(\Omega)$ into $\mathrm{H}^2(\Omega)$. Hence, because of the compact inclusion $\mathrm{H}^2(\Omega) \hookrightarrow \mathrm{H}_0^1(\Omega)$ and the spectral characterization of compact operators, we have that $\mathrm{sp}(T) = \{0\}\cup\{\mu_n : n \in \mathbb{N}\}$, with $\{\mu_n\}_{n\in\mathbb{N}}$ a sequence of finite-multiplicity eigenvalues which converges to 0.

Moreover, it is simple to prove by using (2.2) that $T|_{\mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)} : \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega) \to \mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)$ is self-adjoint with respect to the inner product $(u, v) \mapsto \int_\Omega \Delta u\,\Delta v$. Therefore, since $\mathrm{sp}(T) = \{0\}\cup\mathrm{sp}(T|_{\mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)})$, we conclude that the non-zero eigenvalues of T are real and have ascent 1. Thus, we end the proof.

The following is an additional regularity result for the solution of problem (2.8).

Lemma 2.2. *There exists $C > 0$ such that, for all $f \in \mathrm{H}_0^1(\Omega)$, the solution (w, u) of problem (2.8) satisfies $u \in \mathrm{H}^2(\Omega)$, and*

$$\|w\|_{1,\Omega} + \|u\|_{2,\Omega} \le C\,\|f\|_{1,\Omega}\,.$$

Proof. The estimate for w (which does not involve any additional regularity) follows directly from (2.9) and (2.4). The estimate for u follows from the classical regularity result for the Laplace equation on convex domains with right-hand side $w \in H^1(\Omega)$ (cf. [22]). Thus, we conclude the proof.

Remark 2.2. The lemma above does not fix any further regularity for the solution w of problem (2.9). Indeed, no additional regularity can be expected for arbitrary $f \in \mathrm{H}_0^1(\Omega)$. For instance, from (2.9), if $\boldsymbol{\sigma} = \mathbf{I}$, then $w \equiv f$.

3. The discrete problem

We will study in this section, the numerical approximation of the eigenvalue problem (2.7). With this aim, let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of triangulations of the polygonal domain Ω by triangles T with mesh size h. In what follows, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^2$, $\mathcal{P}_k(S)$ denotes the space of polynomials defined in S of total degree less than or equal to k.

We consider the space of piecewise linear continuous finite elements:

$$\mathrm{H}_h := \left\{ v_h \in \mathrm{H}_0^1(\Omega) : \; v_h|_T \in \mathcal{P}_1(T) \; \forall T \in \mathcal{T}_h \right\}.$$

Now, we are in a position to write the finite element discretization of problem (2.7).

Find $(\lambda_h, w_h, u_h) \in \mathbb{R} \times \mathrm{H}_h \times \mathrm{H}_h, \; u_h \neq 0$, such that

$$\mathbb{A}((w_h, u_h), (\tau_h, v_h)) = -\lambda_h \mathbb{B}(u_h, v_h) \quad \forall (\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{H}_h. \tag{3.12}$$

As in the continuous case, we introduce for the analysis the discrete *solution operator*:

$$\begin{array}{rccl} T_h : & \mathrm{H}_0^1(\Omega) & \to & \mathrm{H}_0^1(\Omega), \\ & f & \mapsto & u_h, \end{array}$$

with $(w_h, u_h) \in \mathrm{H}_h \times \mathrm{H}_h$ being the solution of the following discrete problem:

$$\mathbb{A}((w_h, u_h), (\tau_h, v_h)) = -\mathbb{B}(f, v_h) \quad \forall (\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{H}_h. \tag{3.13}$$

This problem decomposes into a sequence of two well posed problems, which are the respective discretizations of problems (2.9) and (2.10):

- Find $w_h \in \mathrm{H}_h$ such that

$$\int_\Omega \nabla w_h \cdot \nabla v_h = -\int_\Omega (\boldsymbol{\sigma} \nabla f) \cdot \nabla v_h \quad \forall v_h \in \mathrm{H}_h. \tag{3.14}$$

- Find $u_h \in \mathrm{H}_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla \tau_h = -\int_\Omega w_h \tau_h \quad \forall \tau_h \in \mathrm{H}_h. \tag{3.15}$$

Also, as in the continuous case, λ_h is an eigenvalue of problem (3.12) if and only if $\mu_h := \frac{1}{\lambda_h}$ is a non-zero eigenvalue of T_h, with the same multiplicity and corresponding eigenfunctions u_h.

Remark 3.1. The same arguments leading to Remark 2.1 allow us to show that any solution of problem (3.12) satisfies $\lambda_h \neq 0$.

The matrix form of the discrete spectral problem (3.12) reads as follows:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{t}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w}_h \\ \mathbf{u}_h \end{pmatrix} = \lambda_h \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{E} \end{pmatrix} \begin{pmatrix} \mathbf{w}_h \\ \mathbf{u}_h \end{pmatrix}, \tag{3.16}$$

where $\mathbf{w}_h$, and $\mathbf{u}_h$ denote the vectors whose entries are the components of w_h, and u_h, respectively, in particular given bases of the discrete space H_h.

In this generalized eigenvalue problem, matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$ are symmetric, whereas $\mathbf{A}$, and $\mathbf{B}$ are also positive definite.

Now, we are in a position to prove the following characterization of the discrete spectral problem (3.12):

Proposition 3.1. *Let* $\mathrm{Z}_h := \{u_h \in \mathrm{H}_h : \; \mathbb{B}(u_h, v_h) = 0 \; \forall v_h \in \mathrm{H}_h\}$*. Then, problem* (3.12) *has exactly* $\dim \mathrm{H}_h - \dim \mathrm{Z}_h$ *eigenvalues, repeated accordingly to their respective multiplicities. All of them are real and non-zero.*

Proof. Since $\mathbf{A}$ is positive definite and consequently non-singular, $\mathbf{w}_h$ can be eliminated in the generalized eigenvalue problem (3.16) as follows:

$$\mathbf{w}_h = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u}_h \Longrightarrow \mathbf{E}\mathbf{u}_h = \mu_h \left(\mathbf{B}^{\mathrm{t}}\mathbf{A}^{-1}\mathbf{B} \right) \mathbf{u}_h,$$

with $\mu_h := \frac{1}{\lambda_h}$ (recall $\lambda_h \neq 0$; cf. Remark 3.1).

Now, since also $\mathbf{B}$ is non-singular, $\mathbf{B}^{\mathrm{t}}\mathbf{A}^{-1}\mathbf{B}$ is symmetric and positive definite and, $\mathbf{E}$ being symmetric too, the generalized eigenvalue problem $\mathbf{E}\mathbf{u}_h = \mu_h(\mathbf{B}^{\mathrm{t}}\mathbf{A}^{-1}\mathbf{B})\mathbf{u}_h$ is well posed and all its eigenvalues are real. Therefore, the number of eigenvalues of problem (3.16) (which is the matrix form of problem (3.12)) equals the number of non-zero eigenvalues of this problem, namely, $\dim \mathrm{H}_h - \dim(\mathrm{Ker}(\mathbf{E}))$. Thus, we conclude the lemma by noting that $\mathbf{E}\mathbf{u}_h = \mathbf{0}$ if and only if $u_h \in \mathrm{Z}_h$.

As an immediate consequence of the proof of this proposition, note that problem (3.12) always has real non-zero eigenvalues, as long as $\mathbf{E} \neq \mathbf{0}$.

Remark 3.2. For all the solutions (λ_h, w_h, u_h) of problem (3.12), there holds $\int_\Omega (\boldsymbol{\sigma} \nabla u_h) \cdot \nabla u_h \neq 0$, despite the fact that the stress tensor $\boldsymbol{\sigma}$ is not necessarily positive definite. In fact, as shown in the proof of Proposition 3.1,

$$\int_\Omega (\boldsymbol{\sigma} \nabla u_h) \cdot \nabla u_h = \mathbb{B}(u_h, u_h) = \mathbf{u}_h^{\mathrm{t}} \mathbf{E} \mathbf{u}_h = \frac{1}{\lambda_h} \mathbf{u}_h^{\mathrm{t}} \left(\mathbf{B}^{\mathrm{t}}\mathbf{A}^{-1}\mathbf{B} \right) \mathbf{u}_h \neq 0.$$

4. Spectral approximation and error estimates

To prove that T_h provides a correct spectral approximation of T, we will resort to the classical theory for compact operators (see [1]), which is based on the convergence in norm of T_h to T as h goes to zero.

The following lemma yields the uniform convergence of T_h to T as $h \to 0$.

Lemma 4.1. *There exists $C > 0$ such that, for all $f \in \mathrm{H}_0^1(\Omega)$,*

$$\|(T - T_h)f\|_{1,\Omega} \leq Ch\,\|f\|_{1,\Omega}\,.$$

Proof. Given $f \in \mathrm{H}_0^1(\Omega)$, let (w, u) and (w_h, u_h) be the solutions of problems (2.8) and (3.13), respectively, so that $u = Tf$ and $u_h = T_h f$. From (2.10) and (3.15), and the first Strang Lemma (cf. [23]), we have

$$\|u - u_h\|_{1,\Omega} \leq C \left(\inf_{v_h \in \mathrm{H}_h} \|u - v_h\|_{1,\Omega} + \sup_{v_h \in \mathrm{H}_h} \frac{\int_\Omega (w - w_h) v_h}{\|v_h\|_{1,\Omega}} \right). \tag{4.17}$$

To estimate the first term on the right-hand side above, we use standard approximation results and the regularity of u proved in Lemma 2.2:

$$\inf_{v_h \in \mathrm{H}_h} \|u - v_h\|_{1,\Omega} \leq Ch\,\|u\|_{2,\Omega} \leq Ch\,\|f\|_{1,\Omega}\,. \tag{4.18}$$

For the second term, we use the Cauchy–Schwarz inequality to obtain

$$\sup_{v_h \in \mathrm{H}_h} \frac{\int_\Omega (w - w_h) v_h}{\|v_h\|_{1,\Omega}} \leq \|w - w_h\|_{0,\Omega}\,. \tag{4.19}$$

Now, we resort to a duality argument to estimate $\|w - w_h\|_{0,\Omega}$, since no additional regularity holds for w (cf. Remark 2.2). We consider the following well posed problem:

Find $\mathrm{w} \in \mathrm{H}_0^1(\Omega)$ such that

$$\int_\Omega \nabla \mathrm{w} \cdot \nabla \tau = \int_\Omega (w - w_h)\,\tau \quad \forall \tau \in \mathrm{H}_0^1(\Omega). \tag{4.20}$$

By virtue of standard regularity results for the Laplace equation on convex domains, we have that

$$\|\mathrm{w}\|_{2,\Omega} \leq C\,\|w - w_h\|_{0,\Omega}\,.$$

Let $\mathrm{w}_h^I \in \mathrm{H}_h$ be the Lagrange interpolant of w. Taking $\tau = w - w_h$ in (4.20) and using (2.9), (3.14), and standard approximation results, we have

$$\begin{aligned} \|w - w_h\|_{0,\Omega}^2 &= \int_\Omega \nabla \mathrm{w} \cdot \nabla (w - w_h) = \int_\Omega \nabla (\mathrm{w} - \mathrm{w}_h^I) \cdot \nabla (w - w_h) \\ &\leq Ch\,\|\mathrm{w}\|_{2,\Omega}\,\|\nabla (w - w_h)\|_{0,\Omega} \\ &\leq Ch\,\|w - w_h\|_{0,\Omega}\,\|\nabla (w - w_h)\|_{0,\Omega}\,. \end{aligned}$$

Therefore, from (2.9) and (3.14), again, and (2.4),

$$\|w - w_h\|_{0,\Omega} \leq Ch \left(\|\nabla w\|_{0,\Omega} + \|\nabla w_h\|_{0,\Omega} \right) \leq Ch\,\|f\|_{1,\Omega}\,. \tag{4.21}$$

Thus, the lemma follows from (4.17), (4.18), (4.19), and (4.21).

As a direct consequence of Lemma 4.1, T_h converges in norm to T as h goes to zero. Hence, standard results of spectral approximation (see, for instance, [24]) show that isolated parts of $\mathrm{sp}(T)$ are approximated by isolated parts of $\mathrm{sp}(T_h)$. More precisely, let $\mu \neq 0$ be an eigenvalue of T with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. There exist m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which converge to μ. Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces.

We recall the definition of the *gap* $\widehat{\delta}$ between two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathrm{H}_0^1(\Omega)$:

$$\widehat{\delta}(\mathcal{X}, \mathcal{Y}) := \max\left\{\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\right\}, \quad \text{where } \delta(\mathcal{X}, \mathcal{Y}) := \sup_{x \in \mathcal{X}: \|x\|_{1,\Omega} = 1} \left(\inf_{y \in \mathcal{Y}} \|x - y\|_{1,\Omega} \right).$$

The following error estimates for the approximation of eigenvalues and eigenfunctions hold true.

Theorem 4.2. *There exists a strictly positive constant C such that*

$$\hat{\delta}(\mathcal{E}, \mathcal{E}_h) \le Ch,$$
$$\left|\mu - \mu_h^{(i)}\right| \le Ch, \quad i = 1, \dots, m.$$

Proof. As a consequence of Lemma 4.1, T_h converges in norm to T as h goes to zero. Then, the proof follows as a direct consequence of Theorems 7.1 and 7.3 from [1].

The error estimates for the eigenvalues $\mu \neq 0$ of T yield analogous estimates for the eigenvalues $\lambda = \frac{1}{\mu}$ of problem (2.7). However, the order of convergence $\mathcal{O}(h)$ in Theorem 4.2 is not optimal for μ. Our next goal is to improve this order.

With this purpose, let us denote $\lambda_h := 1/\mu_h^{(i)}$, with $\mu_h^{(i)}$ being any particular eigenvalue of T_h converging to μ. Let u_h, and w_h be such that (λ_h, w_h, u_h) is a solution of problem (3.12) with $\|u_h\|_{1,\Omega} = 1$. According to Theorem 4.2, there exists a solution (λ, w, u) of problem (2.7) with $\|u\|_{1,\Omega} = 1$ such that

$$\|u - u_h\|_{1,\Omega} \le Ch. \tag{4.22}$$

The following lemma, which will be used to prove an improved order of convergence for the corresponding eigenvalues, shows estimates for $\|w - w_h\|_{1,\Omega}$.

Lemma 4.2. *There exists $C > 0$ such that*

$$\|w - w_h\|_{1,\Omega} \le Ch.$$

Proof. First, note that (w, u) is the solution of problem (2.8) with $f = \lambda u$. Hence, from Lemma 2.2, $u \in \mathrm{H}^2(\Omega)$ with $\|u\|_{2,\Omega} \le C\lambda \|u\|_{1,\Omega}$. Hence, w is the solution of (2.9), then

$$\begin{cases} -\Delta w = \operatorname{div}(\boldsymbol{\sigma} \nabla(\lambda u)) \in L^2(\Omega), \\ w = 0 \quad \text{on } \Gamma. \end{cases}$$

Therefore, from (2.2)–(2.4), we have that $w \in \mathrm{H}^2(\Omega)$, and

$$\|w\|_{2,\Omega} \le C \|u\|_{2,\Omega} \le C\lambda \|u\|_{1,\Omega}.$$

On the other hand, (w_h, u_h) is the solution of problem (3.13) with $f = \lambda_h u_h$. Thus, from the equivalence between this problem and problems (3.14) and (3.15), w_h is the solution of (3.14) with $f = \lambda_h u_h$. Hence, from the first Strang Lemma again,

$$\|w - w_h\|_{1,\Omega} \le C \left(\inf_{\tau_h \in \mathrm{H}_h} \|w - \tau_h\|_{1,\Omega} + \sup_{\tau_h \in \mathrm{H}_h} \frac{\int_\Omega [\boldsymbol{\sigma}(\lambda \nabla u - \lambda_h \nabla u_h)] \cdot \nabla \tau_h}{\|\tau_h\|_{1,\Omega}} \right).$$

To estimate the first term on the right-hand side above, we use standard approximation results:

$$\inf_{\tau_h \in \mathrm{H}_h} \|w - \tau_h\|_{1,\Omega} \le Ch \|w\|_{2,\Omega} \le Ch \|u\|_{1,\Omega}.$$

For the second term, we use the Cauchy–Schwarz inequality, (2.4), (4.22), and Theorem 4.2:

$$\begin{aligned} \sup_{\tau_h \in \mathrm{H}_h} \frac{\int_\Omega [\boldsymbol{\sigma}(\lambda \nabla u - \lambda_h \nabla u_h)] \cdot \nabla \tau_h}{\|\tau_h\|_{1,\Omega}} &\le C \|\lambda \nabla u - \lambda_h \nabla u_h\|_{0,\Omega} \\ &\le C |\lambda| \|u - u_h\|_{1,\Omega} + |\lambda - \lambda_h| \|u_h\|_{1,\Omega} \\ &\le Ch. \end{aligned}$$

Thus, we conclude the proof.

Now, we are in a position to prove the double order of convergence for the eigenvalues.

Theorem 4.5. *There exists a strictly positive constant C such that*

$$|\lambda - \lambda_h| \le Ch^2.$$

Proof. We adapt to our case a standard argument (cf. [1, Lemma 9.1]). Let $\mathbf{p} := (w, u)$ and $\mathbf{p}_h := (w_h, u_h)$ be as in the proof of Lemma 4.2. Because of (2.7) and (3.12), and the symmetry of the bilinear forms, we have,

$$\begin{aligned} \mathbb{A}(\mathbf{p} - \mathbf{p}_h, \mathbf{p} - \mathbf{p}_h) &= \mathbb{A}(\mathbf{p}, \mathbf{p}) - 2\mathbb{A}(\mathbf{p}, \mathbf{p}_h) + \mathbb{A}(\mathbf{p}_h, \mathbf{p}_h) \\ &= -\lambda \mathbb{B}(u, u) + 2\lambda \mathbb{B}(u, u_h) - \lambda_h \mathbb{B}(u_h, u_h), \end{aligned}$$

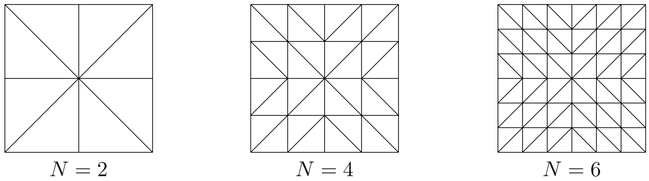

Fig. 1. Square plate: uniform meshes.

whereas

$$\lambda\mathbb{B}(u-u_h,u-u_h)=\lambda\mathbb{B}(u,u)-2\lambda\mathbb{B}(u,u_h)+\lambda\mathbb{B}(u_h,u_h).$$

Therefore, since $\mathbb{B}(u_h,u_h)\neq 0$ (cf. Remark 3.1),

$$\lambda-\lambda_h=\frac{\mathbb{A}(\mathbf{p}-\mathbf{p}_h,\mathbf{p}-\mathbf{p}_h)+\lambda\mathbb{B}(u-u_h,u-u_h)}{\mathbb{B}(u_h,u_h)}.$$

Moreover, from (4.22), $\mathbb{B}(u_h,u_h)\xrightarrow{h}\mathbb{B}(u,u)\neq 0$ (cf. Remark 2.1). Hence,

$$\begin{aligned}|\lambda-\lambda_h| &\leq C\left(|\mathbb{A}(\mathbf{p}-\mathbf{p}_h,\mathbf{p}-\mathbf{p}_h)|+|\lambda|\,|\mathbb{B}(u-u_h,u-u_h)|\right)\\ &\leq C\left(\|\mathbf{p}-\mathbf{p}_h\|^2_{\mathrm{H}^1_0(\Omega)\times\mathrm{H}^1_0(\Omega)}+\|u-u_h\|^2_{1,\Omega}\right)\\ &\leq C\left(\|w-w_h\|^2_{1,\Omega}+\|u-u_h\|^2_{1,\Omega}\right)\\ &\leq Ch^2,\end{aligned}$$

the last inequality because of (4.22) and Lemma 4.2. Thus, we conclude the proof.

5. Numerical results

We report in this section some numerical experiments which confirm the theoretical results proved above. The numerical methods have been implemented in a MATLAB code.

We have taken as an example of a convex domain the unit square $\Omega := (0,1)\times(0,1)$. We have used uniform meshes as those shown in Fig. 1. The refinement parameter N used to label each mesh is the number of elements on each edge of the plate.

5.1. Test 1: uniformly compressed square plate

The aim of this first test is to validate the computer code by solving a problem with known analytical solution.

As we stated in the introduction, the eigenvalues of this problem can be related with the eigenvalues of the Laplace eigenvalue problem with homogeneous boundary conditions In fact, we consider the following eigenvalue problem:

$$\begin{cases}\Delta u=-\lambda u & \text{in } \Omega,\\ u=0 & \text{on } \Gamma.\end{cases} \tag{5.1}$$

We have that the solution of problem (5.1) satisfies our problem when $\boldsymbol{\sigma}=\mathbf{I}$ (which corresponds to a uniformly compressed plate). More precisely, we get

$$\begin{cases}\Delta^2 u=-\lambda\Delta u & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \Gamma.\end{cases} \tag{5.2}$$

We also note that the solution of problem (5.1) also satisfies the following problem associated with the vibration problem of a simply supported Kirchhoff plate:

$$\begin{cases}\Delta^2 u=\lambda^2 u & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \Gamma.\end{cases} \tag{5.3}$$

The exact eigenvalues and eigenfunctions for the last problem are known (see [17,1]).

We report in Table 1 the lowest buckling coefficient computed with the method analyzed in this paper. The table includes computed orders of convergence and the last column shows the exact buckling coefficient.

It can be seen from Table 1 that the computed buckling coefficients converge to the exact ones with an optimal quadratic order. We also point out that the symmetry of the mesh permits to preserve the double multiplicity of the second eigenvalue at discrete level.

Fig. 2 shows the transverse displacements of the principal buckling mode (i.e., the eigenfunction corresponding to the lowest buckling coefficient of the buckling problem) computed with the method analyzed in this paper.

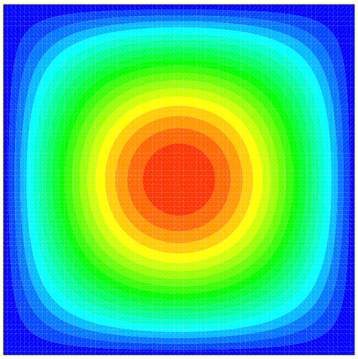

Fig. 2. Uniformly compressed square plate; principal buckling mode.

Table 1
Lowest buckling coefficients of a uniformly compressed simply supported square plate computed on uniform meshes with the method analyzed in this paper.

	$N = 10$	$N = 20$	$N = 40$	$N = 80$	Order	Exact
λ_1	20.0840	19.8273	19.7614	19.7448	1.97	19.7392
$\lambda_2 = \lambda_3$	52.0510	50.0190	49.5155	49.3899	2.01	49.3480
λ_4	86.8888	80.9137	79.4444	79.0786	2.02	78.9568

Table 2
Non-dimensional buckling intensity k_1 of a square plate subjected to linearly varying in-plane load in one direction.

α	$N = 10$	$N = 20$	$N = 40$	$N = 80$	Order	Extrapolated	[25]	[26]	[27]
2	27.7169	26.0680	25.6628	25.5619	2.02	25.5292	25.6	–	25.53
4/3	11.2233	11.0656	11.0253	11.0152	1.97	11.0116	11.0	–	11.01
1	7.9525	7.8478	7.8210	7.8142	1.97	7.8119	7.8	7.8099	7.81
4/5	6.7119	6.6249	6.6026	6.5969	1.97	6.5950	6.6	–	6.60
2/3	6.0684	5.9902	5.9701	5.9651	1.97	5.9633	5.8	–	5.96

5.2. Test 2: square plate under combined bending and compression in one direction

In order to compare our results for the buckling problem, with those in [25–27], a non-dimensional buckling intensity is defined as:

$$k_i := \frac{\lambda_h^i L}{\pi^2},$$

where L is the plate side length.

For this test, we have computed the non-dimensional buckling intensity of the same plate as in the previous example ($L = 1$), subjected to linearly varying in-plane load in one direction. This corresponds to a plane stress field given by

$$\boldsymbol{\sigma} := \begin{pmatrix} \left(1 - \alpha\frac{y}{L}\right) & 0 \\ 0 & 0 \end{pmatrix}.$$

For $\alpha = 2$, it is the case of pure in-plane bending. For $0 < \alpha < 2$, the linearly varying load represents an eccentric bending which can be regarded as a combination of pure bending and uniform compression (see [25–27]).

We report in Table 2 the non-dimensional buckling intensity and we compare our results with those obtained in [25,26] for a Kirchhoff plate, and in [27] for a thin plate modeled by the Reissner–Mindlin theory. It is well known that the non-dimensional buckling intensity k_i is the limit to the non-dimensional buckling intensity of an identical Reissner–Mindlin simply supported plate when the thickness goes to zero (see [4]). The table includes computed orders of convergence and extrapolated more accurate values of each eigenvalue obtained by means of a least-squares fitting. Furthermore, the last three columns show the results from [25–27].

It can be seen from Table 2 that the results obtained with our method present an excellent agreement with those in [25–27] for all linearly varying loading cases, we also have that the eigenvalue approximation order is quadratic.

Figs. 3 and 4 show the transverse displacements of the principal buckling mode computed with the method analyzed in this paper and subjected to linearly varying in-plane load in one direction, considering $\alpha = 2$ and $\alpha = 1$, respectively.

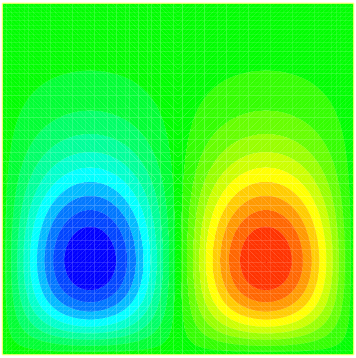

Fig. 3. Principal buckling mode corresponding to $\alpha = 2$.

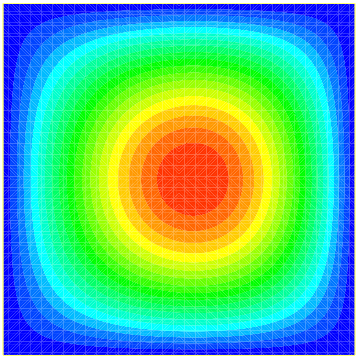

Fig. 4. Principal buckling mode corresponding to $\alpha = 1$.

Table 3
Lowest non-dimensional buckling intensity of a shear loaded simply supported square plate computed on uniform meshes with the method analyzed in this paper.

	$N = 20$	$N = 30$	$N = 40$	$N = 50$	Order	Extrapolated	[25]
k_1	9.6023	9.4496	9.3952	9.3699	1.96	9.3236	9.34

5.3. *Test 3: shear loaded square plate*

For this test we have computed the non-dimensional buckling intensity of the same plate as in the previous example, subjected to a uniform shear load. This corresponds to a plane stress field

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\boldsymbol{\sigma}$ is not positive definite in this case.

We report in Table 3 the lowest non-dimensional buckling intensity and we compare our results with those obtained in [25]. The table includes computed orders of convergence and extrapolated more accurate values of each eigenvalue obtained by means of a least-squares fitting. Furthermore, the last column shows the results from [25].

Once more, the method converges with optimal quadratic order.

Fig. 5 shows the transverse displacements of the principal buckling mode for the shear loaded square plate computed with the method analyzed in this paper.

6. Conclusions

We have introduced a finite element method for the buckling problem of a simply supported Kirchhoff polygonal plate subjected to general plane stress tensor. The method is based on the introduction of an auxiliary variable $w := \Delta u$, where u represents the transverse displacements. The formulation was discretized considering standard piecewise linear finite elements for the two variables. We have proved that the method yields an $\mathcal{O}(h)$ approximation to the transverse displacements u of buckling modes and the auxiliary variable w. Moreover, the method yields $\mathcal{O}(h^2)$ approximation to

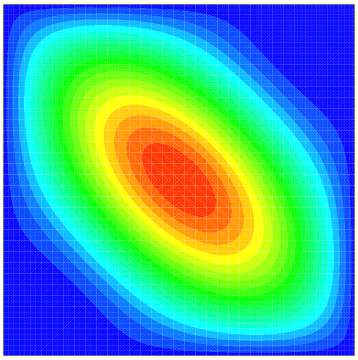

Fig. 5. Shear loaded square plate; principal buckling mode.

the buckling coefficients, too. Finally, we reported numerical results that confirm the numerical analysis of the proposed method.

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