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NONCONFORMING MIXED FINITE ELEMENT APPROXIMATION OF A FLUID-STRUCTURE INTERACTION SPECTRAL PROBLEM

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Abstract. We aim to provide a finite element analysis for the elastoacoustic vibration problem. We use a dual-mixed variational formulation for the elasticity system and combine the lowest order Lagrange finite element in the fluid domain with the reduced symmetry element known as PEERS and introduced for linear elasticity in [1]. We show that the resulting global nonconforming scheme provides a correct spectral approximation and we prove quasi-optimal error estimates. Finally, we confirm the asymptotic rates of convergence by numerical experiments.

1. **Introduction.** We are concerned with the computation of the free vibration modes of a coupled system consisting of an elastic structure which is in contact with an internal compressible fluid. We refer to [3, 5, 15, 18] for the analysis of different formulations of this eigenvalue problem. Here, we follow [12, 16, 17] and consider a dual-mixed formulation with reduced symmetry in the solid. This leads to a symmetric saddle point problem that delivers direct finite element approximations of the stresses and that is immune to the locking phenomenon that arises in the nearly incompressible case. Recently, a Galerkin scheme based on the lowest-order Lagrange finite element in the fluid and the lowest order Arnold-Falk-Winther [2] mixed finite element in the solid has been analyzed in [17]. It was shown that such a mixed finite element Galerkin approximation is spectrally correct and provides optimal convergence error estimates for eigenvalues and eigenfunctions. Our purpose here is to show that the same order of convergence can be achieved, at a lower computational cost, when PEERS element [1] is used in the solid in association with the lowest order Lagrange finite element in the fluid. Compared with [17], the main

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technical difficulty is related with the fact that, in this case, the Galerkin scheme is nonconforming.

The paper is organized as follows. In Section 2 we recall the mixed formulation with reduced symmetry of the fluid-structure eigenvalue problem and provide a spectral description of the corresponding solution operator. In Section 3 we introduce the mixed finite element approximation of the saddle point eigenproblem and characterize the spectrum of the discrete solution operator. In Section 4 we provide the conditions under which the numerical scheme is spectrally correct and we provide abstract convergence error estimates for the eigenfunctions and the eigenvalues. In Section 5 we establish asymptotic error estimates and finally, in Section 6, we present numerical tests and confirm that the experimental rates of convergence are in accordance with the theoretical ones.

Notations. In all what follows we will denote the vectorial and tensorial counterparts of order n ($n = 2, 3$) of a given Hilbert space $\mathcal{V}$ by $\mathcal{V}^n$ and $\mathcal{V}^{n\times n}$ respectively. We use standard notation for the Hilbertian Sobolev space $\mathrm{H}^s(\Omega)$, $s \geq 0$, defined on a Lipschitz bounded domain $\Omega \subset \mathbb{R}^n$ and denote by $\|\cdot\|_{s,\Omega}$ the norms in $\mathrm{H}^s(\Omega)$, $\mathrm{H}^s(\Omega)^n$ and $\mathrm{H}^s(\Omega)^{n\times n}$.

The component-wise inner product of two matrices $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{R}^{n\times n}$ is denoted $\boldsymbol{\sigma} : \boldsymbol{\tau} := \operatorname{tr}(\boldsymbol{\sigma}^{\mathtt{t}}\boldsymbol{\tau})$, where $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^n \tau_{ii}$ and $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$ stand for the trace and the transpose of $\boldsymbol{\tau} = (\tau_{ij})$ respectively. For $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{n\times n}$ and $\boldsymbol{u} : \Omega \to \mathbb{R}^n$, we define the row-wise divergence $\operatorname{\mathbf{div}}\boldsymbol{\sigma} : \Omega \to \mathbb{R}^n$ and the row-wise gradient $\nabla\boldsymbol{u} : \Omega \to \mathbb{R}^{n\times n}$ by,

$$(\operatorname{\mathbf{div}}\boldsymbol{\sigma})_i := \sum_j \partial_j \sigma_{ij} \qquad \text{and} \qquad (\nabla\boldsymbol{u})_{ij} := \partial_j u_i.$$

We introduce for $s \geq 0$ the Hilbert space

$$\mathrm{H}^s(\operatorname{\mathbf{div}};\Omega) := \left\{\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^{n\times n} :\ \operatorname{\mathbf{div}}\boldsymbol{\tau} \in \mathrm{H}^s(\Omega)^n\right\}$$

endowed with the norm $\|\boldsymbol{\tau}\|^2_{\mathrm{H}^s(\operatorname{\mathbf{div}};\Omega)} := \|\boldsymbol{\tau}\|^2_{s,\Omega} + \|\operatorname{\mathbf{div}}\boldsymbol{\tau}\|^2_{s,\Omega}$ and we use the convention $\mathrm{H}(\operatorname{\mathbf{div}};\Omega) := \mathrm{H}^0(\operatorname{\mathbf{div}};\Omega)$.

Given two Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$ and a bounded bilinear form $c : \mathcal{V}\times\mathcal{W} \to \mathbb{R}$, we say that c satisfies the inf-sup condition for the pair $\{\mathcal{V}, \mathcal{W}\}$, whenever there exists $\beta > 0$ such that

$$\sup_{0\neq s\in\mathcal{V}} \frac{c(s,t)}{\|s\|_{\mathcal{V}}} \geq \beta\,\|t\|_{\mathcal{W}} \qquad \forall t \in \mathcal{W}.$$

Finally, $\mathbf{0}$ stands for a generic null vector or tensor and denote by C generic constants independent of the discretization parameters, which may take different values at different places.

2. The spectral problem.

We consider an elastic structure occupying a Lipschitz and polyhedral domain Ω_{S}. We assume that the structure is fixed at $\emptyset \neq \Gamma_{\mathrm{D}} \subset \partial\Omega_{\mathrm{S}}$ and free of stress on $\Gamma_{\mathrm{N}} := \partial\Omega_{\mathrm{S}} \setminus (\Gamma_{\mathrm{D}} \cup \Sigma)$. We are interested by the simplified model in which the stress tensor $\boldsymbol{\sigma}$ is related to the linearized deformation tensor $\boldsymbol{\varepsilon} := \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ through the constitutive law

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega_{\mathrm{S}},$$

where λ_{S} and μ_{S} are Lamé coefficients, $\boldsymbol{I}$ is the identity matrix of $\mathbb{R}^{n\times n}$ and $\mathcal{C} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is given by $\mathcal{C}\boldsymbol{\tau} := \lambda_{\mathrm{S}}(\operatorname{tr}\boldsymbol{\tau})\,\boldsymbol{I} + 2\mu_{\mathrm{S}}\boldsymbol{\tau}$. We will also consider the rotation $\boldsymbol{r} := \frac{1}{2}\left[\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^{\mathtt{t}}\right]$ as a further variable. The interior fluid domain is given by a Lipschitz and polyhedral domain Ω_{F} and the fluid-structure interface is represented by by Σ, see Figure 1. The boundary $\partial\Omega_{\mathrm{F}}$ of the fluid domain is the union of the interface Σ and the open boundary of the fluid Γ_0 (we don't exclude the case $\Gamma_0 = \emptyset$).

The spectral structural-acoustic coupled problem described in Figure 1, with natural frequencies ω, can be formulated as follows in terms of the stress tensor and the pressure (see, for instance, [5, 18]): Find $\boldsymbol{\sigma} : \Omega_{\mathrm{S}} \to \mathbb{R}^{n\times n}$ symmetric, $\boldsymbol{r} : \Omega_{\mathrm{S}} \to \mathbb{R}^{n\times n}$ skew symmetric, $p : \Omega_{\mathrm{F}} \to \mathbb{R}$ and $\omega \in \mathbb{R}$ such that,

$$\nabla\left(\frac{1}{\rho_{\mathrm{S}}}\,\mathbf{div}\,\boldsymbol{\sigma}\right) + \omega^2\left(\mathcal{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{r}\right) = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{S}}, \tag{1}$$

$$\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{2}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{N}} \tag{3}$$

$$\Delta p + \frac{\omega^2}{c^2}p = 0 \qquad \text{in } \Omega_{\mathrm{F}}, \tag{4}$$

$$\frac{\partial p}{\partial \boldsymbol{\nu}} - \frac{\omega^2}{g}p = 0 \qquad \text{on } \Gamma_0, \tag{5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} + p\boldsymbol{\nu} = \mathbf{0} \qquad \text{on } \Sigma, \tag{6}$$

$$\frac{\partial p}{\partial \boldsymbol{\nu}} + \frac{\rho_{\mathrm{F}}}{\rho_{\mathrm{S}}}\,\mathbf{div}\,\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0 \qquad \text{on } \Sigma, \tag{7}$$

where c is the acoustic speed, g is the gravity acceleration and ρ_{F} and ρ_{S} represent the fluid and solid mass densities respectively.

Notice that the displacement can be recovered, and also post-precessed at the discrete level, from

$$\mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho_{\mathrm{S}}\boldsymbol{u} = \mathbf{0}. \tag{8}$$

We consider $\boldsymbol{\mathcal{W}} := \{\boldsymbol{\tau} \in \mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}}), \quad \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma_{\mathrm{N}}\}$, and introduce the product space $\widetilde{\boldsymbol{\mathcal{Y}}} := \boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_{\mathrm{F}})$ endowed with the Hilbertian norm

$$\|(\boldsymbol{\tau}, q)\|^2 := \|\boldsymbol{\tau}\|^2_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})} + \|q\|^2_{1,\Omega_{\mathrm{F}}}.$$

It is straightforward to prove that the subspace $\boldsymbol{\mathcal{Y}}$ given by

$$\boldsymbol{\mathcal{Y}} := \left\{(\boldsymbol{\tau}, q) \in \widetilde{\boldsymbol{\mathcal{Y}}} : \ \boldsymbol{\tau}\boldsymbol{\nu} + q\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Sigma\right\}$$

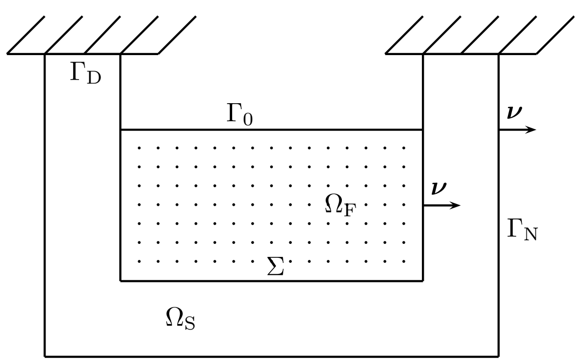

FIGURE 1. Fluid and solid domains

is closed in $\widetilde{\boldsymbol{\mathcal{Y}}}$. The rotation $\boldsymbol{r}$ will be sought in the space

$$\boldsymbol{\mathcal{Q}} := \left\{ \boldsymbol{s} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^{n\times n} : \ \boldsymbol{s}^{\mathrm{t}} = -\boldsymbol{s} \right\}.$$

For commodity we will also denote the Hilbertian product norm in $\widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$ by

$$|||((\boldsymbol{\tau}, q), \boldsymbol{s})|||^2 := \|(\boldsymbol{\tau}, q)\|^2 + \|\boldsymbol{s}\|^2_{0,\Omega_{\mathrm{S}}}.$$

The closed subspace $\boldsymbol{\mathcal{W}}_\Sigma := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} : \ \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\}$ will also be useful in the following.

For $(\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q) \in \widetilde{\boldsymbol{\mathcal{Y}}}$, $\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}$, and $\boldsymbol{v} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n$, we introduce the bounded bilinear forms

$$\begin{aligned}
a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &:= \int_{\Omega_{\mathrm{S}}} \frac{1}{\rho_{\mathrm{S}}} \mathbf{div}\, \boldsymbol{\sigma} \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}}} \nabla p \cdot \nabla q, \\
d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &:= \int_{\Omega_{\mathrm{S}}} \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} pq + \int_{\Gamma_0} \frac{1}{\rho_{\mathrm{F}} g} pq, \\
b((\boldsymbol{\tau}, q), \boldsymbol{s}) &:= \int_{\Omega_{\mathrm{S}}} \boldsymbol{\tau} : \boldsymbol{s}, \\
A((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) &:= a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)), \\
B((\boldsymbol{\tau}, q), (\boldsymbol{s}, \boldsymbol{v})) &:= b((\boldsymbol{\tau}, q), \boldsymbol{s}) + \int_{\Omega_{\mathrm{S}}} \mathbf{div}\, \boldsymbol{\tau} \cdot \boldsymbol{v}.
\end{aligned}$$

It is clear that the kernel of the bilinear form a in $\boldsymbol{\mathcal{Y}}$ is

$$\ker(a) := \{(\boldsymbol{\tau}, \xi) \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}} : \ \mathbf{div}\, \boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega_{\mathrm{S}}\},$$

where $\boldsymbol{\mathcal{Y}}_{\mathbb{R}}$ is the closed subspace of $\boldsymbol{\mathcal{Y}}$ given by

$$\boldsymbol{\mathcal{Y}}_{\mathbb{R}} := \{(\boldsymbol{\tau}, \xi) \in \boldsymbol{\mathcal{W}} \times \mathbb{R} : \ \boldsymbol{\tau}\boldsymbol{\nu} + \xi\boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\}.$$

The variational formulation of the eigenvalue problem (1)-(7) is given, in terms of $\lambda := 1 + \omega^2$ as follows (see [17] for more details): Find $\lambda \in \mathbb{R}$, $0 \neq (\boldsymbol{\sigma}, p) \in \boldsymbol{\mathcal{Y}}$, and $0 \neq \boldsymbol{r} \in \boldsymbol{\mathcal{Q}}$ such that

$$A((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) = \lambda \left[d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) \right] \tag{9}$$

$$b((\boldsymbol{\sigma}, p), \boldsymbol{s}) = \lambda\, b((\boldsymbol{\sigma}, p), \boldsymbol{s}) \tag{10}$$

for all $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}$.

The solution operator corresponding to this eigenvalue problem is

$$\begin{aligned}
\widetilde{\boldsymbol{T}} : \ \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}} &\longrightarrow \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}, \\
((\boldsymbol{F}, f), \boldsymbol{g}) &\longmapsto \widetilde{\boldsymbol{T}}((\boldsymbol{F}, f), \boldsymbol{g}) := ((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*),
\end{aligned}$$

where $((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ solves the source problem:

$$A((\boldsymbol{\sigma}^*, p^*), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}^*) = d((\boldsymbol{F}, f), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{g}) \tag{11}$$

$$b((\boldsymbol{\sigma}^*, p^*), \boldsymbol{s}) = b((\boldsymbol{F}, f), \boldsymbol{s}) \tag{12}$$

for all $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{s} \in \boldsymbol{\mathcal{Q}}$.

Theorem 2.1. *The linear operator $\widetilde{\boldsymbol{T}}$ is well defined and bounded. Moreover, the norm of this operator remains bounded in the nearly incompressible case (i.e., when $\lambda_{\mathrm{S}} \to \infty$).*

Proof. As a consequence of [17, Lemma 2.2], it is straightforward that b satisfies the inf-sup condition for the pair $\{\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Q}}\}$. Moreover, [17, Lemma 2.1] proves that $A(\cdot,\cdot)$ is elliptic on $\ker(b)$ with an ellipticity constant independent of λ_{S}. The result is then an application of the Babuška-Brezzi theory. ☐

Notice that $(\lambda, (\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \mathbb{R} \times \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ solves problem (9)-(10) if and only if $\Big(\mu := \frac{1}{\lambda}, ((\boldsymbol{\sigma}, p), \boldsymbol{r})\Big)$, is an eigenpair of $\boldsymbol{T} := \widetilde{\boldsymbol{T}}|_{\boldsymbol{\mathcal{Y}}\times\boldsymbol{\mathcal{Q}}}$, i.e., if and only if $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \neq \mathbf{0}$ and

$$\boldsymbol{T}((\boldsymbol{\sigma}, p), \boldsymbol{r}) = \frac{1}{\lambda}\,((\boldsymbol{\sigma}, p), \boldsymbol{r})\,.$$

Moreover, it is clear that $\mu = 1$ is an eigenvalue of $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ with associated eigenspace $\ker(a) \times \boldsymbol{\mathcal{Q}}$.

Let us now rewrite the equations of problem (9)-(10) as follows: Find $\lambda \in \mathbb{R}$ and $0 \neq ((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ such that,

$$\mathbb{A}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) = \lambda\, \mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) \qquad \forall((\boldsymbol{\tau}, q), \boldsymbol{s}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}},$$

where $\mathbb{A}$ and $\mathbb{B}$ are the bounded bilinear forms in $\widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$ defined by

$$\mathbb{A}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) := A((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) + b((\boldsymbol{\sigma}, p), \boldsymbol{s}),$$

$$\mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) := d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) + b((\boldsymbol{\sigma}, p), \boldsymbol{s}).$$

To continue with the spectral description of $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ we introduce the orthogonal subspace to $\ker(a) \times \boldsymbol{\mathcal{Q}}$ in $\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ with respect to the bilinear form $\mathbb{B}$,

$$[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}} := \{((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} :\ \mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) = 0 \quad \forall((\boldsymbol{\tau}, q), \boldsymbol{s}) \in \ker(a) \times \boldsymbol{\mathcal{Q}}\}\,.$$

Lemma 2.2. *The subspace* $[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ *is invariant for* $\boldsymbol{T}$*, i.e.,*

$$\boldsymbol{T}([\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}) \subset [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}. \tag{13}$$

Moreover, we have the direct and stable decomposition

$$\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} = [\ker(a) \times \boldsymbol{\mathcal{Q}}] \oplus [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}. \tag{14}$$

Proof. We refer to [16, Proposition A.1] for the proof of (13) and proceed as follows to obtain the splitting of a given $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ according to (14). We consider the problem: find $((\boldsymbol{\sigma}_0, p_0), \boldsymbol{r}_0) \in \ker(a) \times \boldsymbol{\mathcal{Q}}$ solution of

$$d((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}_0) = d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}) \quad \forall(\boldsymbol{\tau}, q) \in \ker(a),$$
$$b((\boldsymbol{\sigma}_0, p_0), \boldsymbol{s}) = b((\boldsymbol{\sigma}, p), \boldsymbol{s}) \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}.$$

The first equation shows that the linear form $(\boldsymbol{\tau}, q) \mapsto d((\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}, p_0 - p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}_0 - \boldsymbol{r})$ belongs to the polar of $\ker(a)$ in $\boldsymbol{\mathcal{Y}}_{\mathbb{R}}$. Hence, the well-known inf-sup condition

$$\sup_{(\boldsymbol{\tau},q)\in\boldsymbol{\mathcal{Y}}_{\mathbb{R}}} \frac{\displaystyle\int_{\Omega_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\tau} \cdot \boldsymbol{v}}{\|(\boldsymbol{\tau}, q)\|} \geq \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{W}}_{\Sigma}} \frac{\displaystyle\int_{\Omega_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\tau} \cdot \boldsymbol{v}}{\|\boldsymbol{\tau}\|_{\mathrm{H}(\mathbf{div};\Omega_{\mathrm{S}})}} \geq \beta\, \|\boldsymbol{v}\|_{0,\Omega_{\mathrm{S}}} \quad \forall \boldsymbol{v} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n,$$

proves the existence of $\boldsymbol{u}_0 \in \mathrm{L}^2(\Omega_{\mathrm{S}})^n$ such that

$$-\int_{\Omega_{\mathrm{S}}} \mathbf{div}\,\boldsymbol{\tau} \cdot \boldsymbol{u}_0 = d((\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}, p_0 - p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}_0 - \boldsymbol{r}) \qquad \forall(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}}.$$

Therefore, $((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{r}_0, \boldsymbol{u}_0)) \in [\boldsymbol{\mathcal{Y}}_{\mathbb{R}} \times (\boldsymbol{\mathcal{Q}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n)]$ satisfies

$$d((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{\tau}, q)) + B((\boldsymbol{\tau}, q), (\boldsymbol{r}_0, \boldsymbol{u}_0)) = d((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b((\boldsymbol{\tau}, q), \boldsymbol{r}), \tag{15}$$

$$B((\boldsymbol{\sigma}_0, p_0), (\boldsymbol{s}, \boldsymbol{v})) = b((\boldsymbol{\sigma}, p), \boldsymbol{s}), \tag{16}$$

for all $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}}$ and $(\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n$. The saddle point problem (15)-(16) is well-posed (see [17, Section 3]) and $((\boldsymbol{\sigma} - \boldsymbol{\sigma}_0, p - p_0), \boldsymbol{r} - \boldsymbol{r}_0)$ belongs to $[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ by construction. The decomposition (14) follows then from

$$((\boldsymbol{\sigma}, p), \boldsymbol{r}) = ((\boldsymbol{\sigma}_0, p_0), \boldsymbol{r}_0) + ((\boldsymbol{\sigma} - \boldsymbol{\sigma}_0, p - p_0), \boldsymbol{r} - \boldsymbol{r}_0).$$

□

It is clear now that the solution of the continuous eigenvalue problem (9)-(10) relies on the spectral description of

$$\boldsymbol{T}|_{[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}} : \ [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}} \to [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}.$$

To this end, we need to provide a characterization of the unique projection $\boldsymbol{P}$: $\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ with range $[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ and kernel $\ker(a) \times \boldsymbol{\mathcal{Q}}$ associated to the splitting (14).

In what follows, $\bar{q} := q - \frac{1}{|\Omega_{\mathrm{F}}|} \int_{\Omega_{\mathrm{F}}} q$ stands for the zero mean value component of functions $q \in \mathrm{L}^2(\Omega_{\mathrm{F}})$. For a given $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ we denote $((\widetilde{\boldsymbol{\sigma}}, \tilde{p}), \widetilde{\boldsymbol{r}}) := \boldsymbol{P}((\boldsymbol{\sigma}, p), \boldsymbol{r})$. By definition of the projection $\boldsymbol{P}$, we should have that $((\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}, \tilde{p} - \bar{p}), \widetilde{\boldsymbol{r}} - \boldsymbol{r}) \in \ker(a) \times \boldsymbol{\mathcal{Q}}$ and $((\widetilde{\boldsymbol{\sigma}}, \tilde{p}), \widetilde{\boldsymbol{r}}) \in [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$. In other words,

$$\mathbf{div}\, \widetilde{\boldsymbol{\sigma}} = \mathbf{div}\, \boldsymbol{\sigma} \qquad \text{and} \qquad \tilde{p} = \bar{p} + \tilde{c}, \quad \text{with} \qquad \tilde{c} \in \mathbb{R} \tag{17}$$

$$d((\widetilde{\boldsymbol{\sigma}}, \widetilde{p}), (\boldsymbol{\tau}, \xi)) + b((\boldsymbol{\tau}, \xi), \widetilde{\boldsymbol{r}}) = 0 \qquad \forall (\boldsymbol{\tau}, \xi) \in \ker(a), \tag{18}$$

$$b((\widetilde{\boldsymbol{\sigma}}, \widetilde{p}), \boldsymbol{s}) = 0 \qquad \forall (\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}}. \tag{19}$$

It is convenient to incorporate the divergence restriction on $\widetilde{\boldsymbol{\sigma}}$ by means of a Lagrange multiplier and use a shift argument to deal properly with the affine transmission condition relating $\widetilde{\boldsymbol{\sigma}}$ and $\tilde{p}$ on Σ. For this purpose, given $q \in \mathrm{H}^1(\Omega_{\mathrm{F}})$, we let $\widehat{\boldsymbol{u}} \in \mathrm{H}^1(\Omega_{\mathrm{S}})^n$ and $\widehat{\boldsymbol{\sigma}} \in \mathrm{H}(\mathbf{div}; \Omega_{\mathrm{S}})$ be the solution of the following linear elasticity problem:

$$\begin{aligned} -\mathbf{div}\, \widehat{\boldsymbol{\sigma}} &= \mathbf{0} && \text{in } \Omega_{\mathrm{S}}, \\ \widehat{\boldsymbol{\sigma}} &= \mathcal{C}\boldsymbol{\varepsilon}(\widehat{\boldsymbol{u}}) && \text{in } \Omega_{\mathrm{S}}, \\ \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu} &= q\boldsymbol{\nu} && \text{on } \Sigma, \\ \widehat{\boldsymbol{u}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{D}}, \\ \widehat{\boldsymbol{\sigma}}\boldsymbol{\nu} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

and we define the bounded linear operator $\boldsymbol{E} : \ \mathrm{H}^1(\Omega_{\mathrm{F}}) \longrightarrow \boldsymbol{\mathcal{W}}$ given by $\boldsymbol{E}q := -\widehat{\boldsymbol{\sigma}}$. We notice that $\boldsymbol{E}$ provides a symmetric divergence-free extension of a given pressure field q to the solid domain. Classical regularity results for the elasticity equations in polyhedral (polygonal) domains (cf. [10, 13]) ensure the existence of $t_{\mathrm{S}} \in (0, 1]$, which depends on the geometry of Ω_{S} and the Lamé coefficients, such that $\boldsymbol{E}q \in \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n \times n}$ and

$$\|\boldsymbol{E}q\|_{t_{\mathrm{S}}, \Omega_{\mathrm{S}}} \le C \|q\|_{1, \Omega_{\mathrm{F}}} \qquad \forall q \in \mathrm{H}^1(\Omega_{\mathrm{F}}). \tag{20}$$

We consider $\widehat{\boldsymbol{E}}q := (\boldsymbol{E}q, q) \in \boldsymbol{\mathcal{Y}}$ and introduce the operator

$$\widetilde{\boldsymbol{P}} : \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}} \ \to \ \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$$

$$((\boldsymbol{\sigma}, p), \boldsymbol{r}) \quad \mapsto \quad \widetilde{\boldsymbol{P}}((\boldsymbol{\sigma}, p), \boldsymbol{r}) := ((\widetilde{\boldsymbol{\sigma}}_0, \widetilde{c}) + \widehat{\boldsymbol{E}}\bar{p}, \widetilde{\boldsymbol{r}})$$

where $(\widetilde{\boldsymbol{\sigma}}_0, \widetilde{c}) \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}}$ and $(\widetilde{\boldsymbol{r}}, \widetilde{\boldsymbol{u}}) \in \boldsymbol{\mathcal{Q}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n$ satisfy

$$d((\widetilde{\boldsymbol{\sigma}}_0, \widetilde{c}), (\boldsymbol{\tau}, \xi)) + B((\boldsymbol{\tau}, \xi), (\widetilde{\boldsymbol{r}}, \widetilde{\boldsymbol{u}})) = -d(\widehat{\boldsymbol{E}}\bar{p}, (\boldsymbol{\tau}, \xi)) \tag{21}$$

$$B((\widetilde{\boldsymbol{\sigma}}_0, \widetilde{c}), (\boldsymbol{s}, \boldsymbol{v})) = \int_{\Omega_{\mathrm{S}}} \mathbf{div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v} \tag{22}$$

for all $(\boldsymbol{\tau}, \xi) \in \boldsymbol{\mathcal{Y}}_{\mathbb{R}}$ and $(\boldsymbol{s}, \boldsymbol{v}) \in \boldsymbol{\mathcal{Q}} \times \mathrm{L}^2(\Omega_{\mathrm{S}})^n$. The arguments given for the well-posedness of (15)-(16) are valid for the saddle point problem (21)-(22). Moreover, it is clear from (17)–(19) that $\boldsymbol{P} = \widetilde{\boldsymbol{P}}|_{\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}}$.

The following regularity results obtained in Lemma 3.1 and Proposition 4.1 of [17] are essential for the forthcoming analysis.

Lemma 2.3. *There exists $C > 0$ such that, for all $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$,*

$$\|\widetilde{\boldsymbol{\sigma}}\|_{t_{\mathrm{S}}, \Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{u}}\|_{1+t_{\mathrm{S}}, \Omega_{\mathrm{S}}} + \|\widetilde{\boldsymbol{r}}\|_{t_{\mathrm{S}}, \Omega_{\mathrm{S}}} + \|\widetilde{p}\|_{1, \Omega_{\mathrm{F}}} \leq C \left(\|\mathbf{div}\, \boldsymbol{\sigma}\|_{0, \Omega_{\mathrm{S}}} + \|p\|_{1, \Omega_{\mathrm{F}}} \right),$$

where $((\widetilde{\boldsymbol{\sigma}}, \widetilde{p}), (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{r}})) \in \left[\boldsymbol{\mathcal{Y}}_{\mathbb{R}} + \widehat{\boldsymbol{E}}p\right] \times \left[\mathrm{L}^2(\Omega_{\mathrm{S}})^n \times \boldsymbol{\mathcal{Q}}\right]$ is the solution to (21)–(22). Consequently, $\widetilde{\boldsymbol{P}}(\widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}) \subset \left[\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n} \times \mathrm{H}^1(\Omega_{\mathrm{F}})\right] \times \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$.

Lemma 2.4. *If $((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*) = \widetilde{\boldsymbol{T}}((\boldsymbol{\sigma}, p), \boldsymbol{r})$, with $((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$, then $\mathbf{div}\, \boldsymbol{\sigma}^* \in \mathrm{H}^1(\Omega_{\mathrm{S}})^n$ and there exists $t_{\mathrm{F}} \in (0, 1]$ such that $p^* \in \mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})$. Moreover, there exists a constant $C > 0$ such that*

$$\|\mathbf{div}\, \boldsymbol{\sigma}^*\|_{1, \Omega_{\mathrm{S}}} + \|p^*\|_{1+t_{\mathrm{F}}, \Omega_{\mathrm{F}}} \leq C\, |\!|\!|((\boldsymbol{\sigma}, p), \boldsymbol{r})|\!|\!|\,.$$

As a first consequence of Lemmas 2.3 and 2.4 and the fact that $\boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) = [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp}$ is $\boldsymbol{T}$-invariant we have that

$$\boldsymbol{T} \circ \boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) \subset \boldsymbol{P}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) \cap \boldsymbol{T}(\boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}) \hookrightarrow \left[\mathrm{H}^{t_{\mathrm{S}}}(\mathbf{div}; \Omega_{\mathrm{S}}) \times \mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})\right] \times \mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n} \tag{23}$$

and the compactness of $\boldsymbol{T} : [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}} \to [\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$ follows. This permits us to announce the following spectral characterization of $\widetilde{\boldsymbol{T}}$.

Theorem 2.5. *The spectrum of $\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}} \to \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$ decomposes as follows: $\mathrm{sp}(\widetilde{\boldsymbol{T}}) = \{0, 1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, where:*

i) $\mu = 1$ is an infinite-multiplicity eigenvalue of $\widetilde{\boldsymbol{T}}$ and its associated eigenspace is $\ker(a) \times \boldsymbol{\mathcal{Q}}$;

ii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0, 1)$ is a sequence of finite-multiplicity eigenvalues of $\widetilde{\boldsymbol{T}}$ which converge to 0 and the corresponding eigenspaces lie on $[\ker(a) \times \boldsymbol{\mathcal{Q}}]^{\perp_{\mathbb{B}}}$; moreover, the ascent of each of these eigenvalues is 1;

iii) $\mu = 0$ is an infinite-multiplicity eigenvalue of $\widetilde{\boldsymbol{T}}$ and its associated eigenspace is

$$\ker \widetilde{\boldsymbol{T}} := \Big\{((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}} : \ \mathbb{B}(((\boldsymbol{\sigma}, p), \boldsymbol{r}), ((\boldsymbol{\tau}, q), \boldsymbol{s})) = 0 \quad \forall ((\boldsymbol{\tau}, q), \boldsymbol{s}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}} \Big\}\,.$$

Proof. See [16, Proposition A.2] and [17, Theorem 4.3] for more details. ∎

3. **The discrete eigenproblem.** Let $\{\mathcal{T}_h(\Omega_S)\}_{h>0}$ and $\{\mathcal{T}_h(\Omega_F)\}_{h>0}$ be shape-regular families of triangulations of the polyhedral (polygonal) regions $\bar{\Omega}_S$ and $\bar{\Omega}_F$, respectively, by tetrahedrons (triangles) T of diameter h_T, with mesh size $h := \max\{h_T : T \in \mathcal{T}_h(\Omega_S) \cup \mathcal{T}_h(\Omega_F)\}$. For the sake of simplicity, in the forthcoming analysis we assume that $\mathcal{T}_h(\Omega_S)$ and $\mathcal{T}_h(\Omega_F)$ induce on Σ a coincident triangulation denoted Σ_h. In what follows, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^n$, $\mathcal{P}_k(S)$ denotes the space of polynomial functions defined in S of total degree $\leq k$.

We consider the first order Raviart-Thomas finite element

$$\mathcal{RT}_0(T) := \{a\boldsymbol{x} + \boldsymbol{b}, \quad a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^n\}$$

and denote by b_T the usual bubble function on $T \in \mathcal{T}_h(\Omega_S)$. We introduce

$$\boldsymbol{\mathcal{W}}_h := \big\{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}} : \ (\boldsymbol{\tau}_{h,i}|_T)^{\mathtt{t}} \in \mathcal{RT}_0(T) \oplus \mathbf{curl}(\mathcal{P}_0(T)^{2n-3} b_T) \\ \forall i \in \{1, \cdots, n\}, \quad \forall T \in \mathcal{T}_h(\Omega_S)\big\},$$

and

$$\mathcal{V}_h := \left\{q_h \in \mathrm{H}^1(\Omega_F) : \ q_h|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h(\Omega_F)\right\},$$

where $\boldsymbol{\tau}_{h,i}$ stands for the i-th row of $\boldsymbol{\tau}_h$. It is well-known that

$$\boldsymbol{\Phi}_S^h := \{\boldsymbol{\tau}_h \cdot \boldsymbol{\nu} \, : \, \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h\} \subset \mathcal{P}_0(\Sigma_h)^n,$$

where $\mathcal{P}_0(\Sigma_h) := \{\phi_h : \Sigma \to \mathbb{R} : \phi_h|_F \in \mathbb{P}_0(F) \quad \forall F \in \Sigma_h\}$. We denote by ϱ_h the $L^2(\Sigma)$-orthogonal projection onto $\mathcal{P}_0(\Sigma_h)$ and introduce the finite element subspaces

$$\boldsymbol{\mathcal{Y}}_h := \{(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{W}}_h \times \mathcal{V}_h : \ \boldsymbol{\tau}_h \boldsymbol{\nu} + \varrho_h(q_h)\boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\},$$
$$\boldsymbol{\mathcal{Q}}_h := \left\{\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}} \cap \mathcal{C}^0(\bar{\Omega}_S)^{n\times n} : \ \boldsymbol{s}_h|_T \in \mathcal{P}_1(T)^{n\times n} \ \forall T \in \mathcal{T}_h(\Omega_S)\right\}.$$

We point out that $\boldsymbol{\mathcal{Y}}_h \subset \widetilde{\boldsymbol{\mathcal{Y}}}$ is not a subspace of $\boldsymbol{\mathcal{Y}}$. In addition, for the analysis below we will also use the space

$$\boldsymbol{\mathcal{U}}_h := \left\{\boldsymbol{v}_h \in \mathrm{L}^2(\Omega_S)^n : \ \boldsymbol{v}_h|_T \in \mathcal{P}_0(T)^n \ \forall T \in \mathcal{T}_h(\Omega_S)\right\}.$$

Notice that $\boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h$ is the lowest-order mixed finite element of the PEERS family introduced for linear elasticity by Arnold, Brezzi and Douglas (see [1]). In particular we have the inf-sup condition [1, 8]: There exists $\beta^* > 0$, independent of h, such that

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h\cap\boldsymbol{\mathcal{W}}_\Sigma} \frac{B((\boldsymbol{\tau}_h, q_h), (\boldsymbol{s}_h, \boldsymbol{v}_h))}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}(\mathbf{div};\Omega_S)}} \geq \beta^* \left(\|\boldsymbol{v}_h\|_{0,\Omega_S} + \|\boldsymbol{s}_h\|_{0,\Omega_S}\right), \tag{24}$$

for all $(\boldsymbol{s}_h, \boldsymbol{v}_h) \in \boldsymbol{\mathcal{Q}}_h \times \boldsymbol{\mathcal{U}}_h$.

The discrete counterpart of problem (9)-(10) reads as follows: Find $\lambda_h \in \mathbb{R}$, $0 \neq (\boldsymbol{\sigma}_h, p_h) \in \boldsymbol{\mathcal{Y}}_h$, and $0 \neq \boldsymbol{r}_h \in \boldsymbol{\mathcal{Q}}_h$ such that

$$A((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{r}_h) = \lambda_h \left[d((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{r}_h)\right], \tag{25}$$

$$b((\boldsymbol{\sigma}_h, p_h), \boldsymbol{s}_h) = \lambda_h \, b((\boldsymbol{\sigma}_h, p_h), \boldsymbol{s}_h) \tag{26}$$

for all $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ and $\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$.

The discrete version of the operator $\widetilde{\boldsymbol{T}}$ is then given by

$$\widetilde{\boldsymbol{T}}_h : \ \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}} \longrightarrow \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}},$$
$$((\boldsymbol{F}, f), \boldsymbol{g}) \longmapsto \boldsymbol{T}_h((\boldsymbol{F}, f), \boldsymbol{g}) := ((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{r}_h^*),$$

where $((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{r}_h^*) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ solves the discrete source problem,

$$A((\boldsymbol{\sigma}_h^*, p_h^*), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{r}_h^*) = d((\boldsymbol{F}, f), (\boldsymbol{\tau}_h, q_h)) + b((\boldsymbol{\tau}_h, q_h), \boldsymbol{g}), \tag{27}$$

$$b((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{s}_h) = b((\boldsymbol{F}, f), \boldsymbol{s}_h) \tag{28}$$

for all $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ and $\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$. We can use the classical Babuška-Brezzi theory to prove that $\widetilde{\boldsymbol{T}}_h$ is well defined and bounded uniformly with respect to h. Indeed, we already know from [17, Lemma 2.1] that A is elliptic on the whole $\boldsymbol{\mathcal{W}} \times \mathrm{H}^1(\Omega_{\mathrm{F}})$ (and in particular on $\boldsymbol{\mathcal{Y}}_h$), whereas the discrete inf-sup condition

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h} \frac{b((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)}{\|(\boldsymbol{\tau}_h, q_h)\|} \geq \beta \|\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$$

follows immediately from (24), as shown in [17, Lemma 2.2].

Lemma 3.1. *Let* $((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*) := \widetilde{\boldsymbol{T}}((\boldsymbol{F}, f), \boldsymbol{g}) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$ *and* $((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{r}_h^*) := \widetilde{\boldsymbol{T}}_h((\boldsymbol{F}, f), \boldsymbol{g}) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ *be the solutions of* (11)-(12) *and* (27)-(28) *respectively. The following identity holds true,*

$$\mathbb{A}\Big(((\boldsymbol{\sigma}^* - \boldsymbol{\sigma}_h^*, p^* - p_h^*), \boldsymbol{r}^* - \boldsymbol{r}_h^*), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)\Big) = \int_\Sigma \frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\sigma}^* \cdot (q_h \boldsymbol{\nu} + \boldsymbol{\tau}_h \boldsymbol{\nu}) \tag{29}$$

for all $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ *and* $\boldsymbol{s}_h \in \boldsymbol{\mathcal{Q}}_h$.

Proof. We have from (27)-(28) that

$$\mathbb{A}\Big(((\boldsymbol{\sigma}_h^*, p_h^*), \boldsymbol{r}_h^*), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)\Big) = \mathbb{B}\Big(((\boldsymbol{F}, f), \boldsymbol{g}), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)\Big) \tag{30}$$

for all $((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$. On the other hand, testing (11) with $(\boldsymbol{\tau}, 0), (\mathbf{0}, q) \in \mathcal{D}(\Omega_{\mathrm{S}})^{n\times n} \times \mathcal{D}(\Omega_{\mathrm{F}}) \subset \boldsymbol{\mathcal{Y}}$ yields

$$\begin{aligned} \mathcal{C}^{-1}\boldsymbol{\sigma}^* - \nabla\left(\frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\sigma}^*\right) + \boldsymbol{r}^* = \mathcal{C}^{-1}\boldsymbol{F} + \boldsymbol{g} \qquad & \text{in } \Omega_{\mathrm{S}}, \\ -c^2 \Delta p^* + p^* = f \qquad & \text{in } \Omega_{\mathrm{F}}. \end{aligned}$$

Applying an integration by parts formula to (11) and using the last two equations we deduce that

$$\mathbb{A}\Big(((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)\Big) = \mathbb{B}\Big(((\boldsymbol{F}, f), \boldsymbol{g}), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)\Big) + \int_\Sigma \left(\frac{1}{\rho_{\mathrm{F}}} \frac{\partial p^*}{\partial \boldsymbol{\nu}} q_h - \frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\sigma}^* \cdot \boldsymbol{\tau}_h \boldsymbol{\nu}\right). \tag{31}$$

Testing now (11) with an appropriate $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ we can show that

$$\frac{\partial p^*}{\partial \boldsymbol{\nu}} = -\frac{\rho_{\mathrm{F}}}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\sigma}^* \cdot \boldsymbol{\nu} \quad \text{on } \Sigma.$$

Substituting the last identity in (31) and taking into account (30) we deduce (29). □

Since $\widetilde{\boldsymbol{T}}_h(\widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}) \subset \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$, and $\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h \subset \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$ we are allowed to consider

$$\boldsymbol{T}_h := \widetilde{\boldsymbol{T}}_h|_{\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h} : \ \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h \longrightarrow \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$$

and, as in the continuous case, we have that $(\lambda_h, (\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \mathbb{R} \times \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ solves problem (25)-(26) if and only if $(\mu_h := \frac{1}{\lambda_h}, ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h))$ is an eigenpair of $\boldsymbol{T}_h$, i.e., if and only if $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \neq \mathbf{0}$ and

$$\boldsymbol{T}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) = \frac{1}{\lambda_h} ((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) .$$

To describe the spectrum of this operator, we proceed as in the continuous case and consider

$$\ker_h(a) := \{(\boldsymbol{\tau}_h, \xi) \in \boldsymbol{\mathcal{Y}}_{h,\mathbb{R}} : \ \mathbf{div}\, \boldsymbol{\tau}_h = \mathbf{0} \text{ in } \Omega_{\mathrm{S}}\} ,$$

where $\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}$ is the subspace of $\boldsymbol{\mathcal{Y}}_{\mathbb{R}}$ defined by

$$\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}} := \{(\boldsymbol{\tau}_h, \xi) \in \boldsymbol{\mathcal{W}}_h \times \mathbb{R} : \ \boldsymbol{\tau}_h \boldsymbol{\nu} + \xi \boldsymbol{\nu} = \mathbf{0} \text{ on } \Sigma\} .$$

Clearly, $\boldsymbol{T}_h : [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \to [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]$ reduces to the identity, which means that here again $\mu_h = 1$ is an eigenvalue of $\boldsymbol{T}_h$ with associated eigenspace $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$. Let us also consider

$$\begin{aligned}[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}} := \{&((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h : \\ &\mathbb{B}(((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h), ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)) = 0 \quad \forall ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \ker_h(a) \times \boldsymbol{\mathcal{Q}}_h\} .\end{aligned}$$

We have the following discrete analogue to Lemma 2.2.

Lemma 3.2. *The subspace $[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$ is invariant for $\boldsymbol{T}_h$. Moreover, we have the following direct and uniformly stable decomposition*

$$\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h = [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h] \oplus [\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}. \tag{32}$$

Proof. Taking into account the inf-sup condition (24), the proof is similar to the one given for Lemma 2.2. □

We denote by $\boldsymbol{P}_h : \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h \to \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$ the unique projection with range $[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$ and kernel $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$ associated to the discrete direct splitting (32). Given $((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h) \in \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$, $((\widetilde{\boldsymbol{\sigma}}_h, \tilde{p}_h), \widetilde{\boldsymbol{r}}_h) := \boldsymbol{P}_h((\boldsymbol{\sigma}_h, p_h), \boldsymbol{r}_h)$ is uniquely characterized by,

$$\mathbf{div}\, \widetilde{\boldsymbol{\sigma}}_h = \mathbf{div}\, \boldsymbol{\sigma}_h \qquad \text{and} \qquad \tilde{p}_h = \bar{p}_h + \tilde{c}_h, \quad \text{with} \quad \tilde{c}_h \in \mathbb{R}, \tag{33}$$

$$d((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), (\boldsymbol{\tau}, \xi)) + b((\boldsymbol{\tau}, \xi), \widetilde{\boldsymbol{r}}) = 0 \qquad \forall (\boldsymbol{\tau}, \xi) \in \ker_h(a), \tag{34}$$

$$b((\widetilde{\boldsymbol{\sigma}}_h, \widetilde{p}_h), \boldsymbol{s}) = 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{\mathcal{Q}}_h. \tag{35}$$

We are now in a position to provide a characterization of the spectrum of $\boldsymbol{T}_h$ and, hence, of the solutions to problem (25)-(26).

Theorem 3.3. *The spectrum of $\boldsymbol{T}_h$ consists of $M := \dim(\boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h)$ eigenvalues, repeated accordingly to their respective multiplicities. The spectrum decomposes as follows: $\mathrm{sp}(\boldsymbol{T}_h) = \{1\} \cup \{\mu_{hk}\}_{k=1}^{K}$. Moreover,*

i) the eigenspace associated to $\mu_h = 1$ is $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$;
ii) $\mu_{hk} \in (0,1)$, $k = 1, \ldots, K := M - \dim(\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h)$, are non-defective eigenvalues, repeated accordingly to their respective multiplicities, with associated eigenspaces lying on $[\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h]^{\perp_{\mathbb{B}}}$;
iii) $\mu_h = 0$ is not an eigenvalue of $\boldsymbol{T}_h$.

Proof. See [17, Theorem 6.7] for more details. □

4. **Abstract convergence analysis.** For the sake of brevity, we will denote in this section $\mathbb{X} := \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Q}}$, $\widetilde{\mathbb{X}} := \widetilde{\boldsymbol{\mathcal{Y}}} \times \boldsymbol{\mathcal{Q}}$ and $\mathbb{X}_h := \boldsymbol{\mathcal{Y}}_h \times \boldsymbol{\mathcal{Q}}_h$. Moreover, when no confusion can arise, we will use indistinctly $\boldsymbol{x}$, $\boldsymbol{y}$, etc. to denote elements in $\mathbb{X}$ and, analogously, $\boldsymbol{x}_h$, $\boldsymbol{y}_h$, etc. for those in $\mathbb{X}_h$. Finally, we will use $\|\cdot\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}$ to denote the norm of an operator restricted to the discrete subspace $\mathbb{X}_h$; namely, if $\boldsymbol{S} : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$, then

$$\|\boldsymbol{S}\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} := \sup_{\boldsymbol{0}\neq\boldsymbol{x}_h\in\mathbb{X}_h} \frac{|||\boldsymbol{S}\boldsymbol{x}_h|||}{|||\boldsymbol{x}_h|||}.$$

For $\boldsymbol{x} \in \widetilde{\mathbb{X}}$ and $\mathbb{E}$ and $\mathbb{F}$ closed subspaces of $\widetilde{\mathbb{X}}$, we set $\delta(\boldsymbol{x},\mathbb{E}) := \inf_{\boldsymbol{y}\in\mathbb{E}} |||\boldsymbol{x}-\boldsymbol{y}|||$, $\delta(\mathbb{E},\mathbb{F}) := \sup_{\boldsymbol{y}\in\mathbb{E}:\, |||\boldsymbol{y}|||=1} \delta(\boldsymbol{y},\mathbb{F})$, and $\widehat{\delta}(\mathbb{E},\mathbb{F}) := \max\{\delta(\mathbb{E},\mathbb{F}),\delta(\mathbb{F},\mathbb{E})\}$, the latter being the so called *gap* between subspaces $\mathbb{E}$ and $\mathbb{F}$.

Proposition 1. *There exists $C > 0$, independent of h, such that for all $\boldsymbol{x} := ((\boldsymbol{\sigma},p),\boldsymbol{r}) \in \widetilde{\mathbb{X}}$,*

$$\|\widetilde{\boldsymbol{T}}\boldsymbol{x} - \widetilde{\boldsymbol{T}}_h\boldsymbol{x}\| \leq C\Big(\delta(\widetilde{\boldsymbol{T}}\boldsymbol{x},\mathbb{X}_h) + \Upsilon_h(\widetilde{\boldsymbol{T}}\boldsymbol{x})\Big), \tag{36}$$

where the consistency error is given by

$$\Upsilon_h(\widetilde{\boldsymbol{T}}\boldsymbol{x}) := \sup_{0\neq\boldsymbol{y}_h:=((\boldsymbol{\tau}_h,q_h),\boldsymbol{s}_h)\in\mathbb{X}_h} \frac{\int_\Sigma \frac{1}{\rho_{\mathrm{S}}}\,\mathbf{div}\,\boldsymbol{\sigma}^*\cdot(q_h\boldsymbol{\nu}+\boldsymbol{\tau}_h\boldsymbol{\nu})}{|||\boldsymbol{y}_h|||}, \tag{37}$$

with $\widetilde{\boldsymbol{T}}\boldsymbol{x} := ((\boldsymbol{\sigma}^*,p^*),\boldsymbol{r}^*)$.

Proof. We deduce from the well-posedness of problem (27)-(28) that the operator

$$\mathbb{X}_h \ni \boldsymbol{y}_h \longmapsto \mathbb{A}\big(\boldsymbol{y}_h,\cdot\big) : \mathbb{X}_h \to \mathbb{R}$$

has a uniformly bounded inverse. It follows that (see [9]), there exists $\gamma > 0$ independent of h such that

$$\sup_{0\neq\boldsymbol{y}_h\in\mathbb{X}_h} \frac{\mathbb{A}\big(\boldsymbol{x}_h,\boldsymbol{y}_h\big)}{|||\boldsymbol{y}_h|||} \geq \gamma\,|||\boldsymbol{x}_h|||\,, \qquad \forall\boldsymbol{x}_h\in\mathbb{X}_h. \tag{38}$$

Let $\boldsymbol{x}^* := \widetilde{\boldsymbol{T}}\boldsymbol{x} \in \mathbb{X}$ and $\boldsymbol{x}_h^* := \widetilde{\boldsymbol{T}}_h\boldsymbol{x} \in \mathbb{X}_h$ be the solutions of (11)-(12) and (27)-(28) respectively with data $\boldsymbol{x} = ((\boldsymbol{F},f),\boldsymbol{g})$. The triangle inequality and (38) show that, for all $\tilde{\boldsymbol{y}}_h \in \mathbb{X}_h$,

$$|||\boldsymbol{x}^*-\boldsymbol{x}_h^*||| \leq |||\boldsymbol{x}^*-\tilde{\boldsymbol{y}}_h||| + |||\boldsymbol{x}_h^*-\tilde{\boldsymbol{y}}_h||| \leq |||\boldsymbol{x}^*-\tilde{\boldsymbol{y}}_h||| + \frac{1}{\gamma}\sup_{\boldsymbol{y}_h\in\mathbb{X}_h}\frac{\mathbb{A}(\boldsymbol{x}_h^*-\tilde{\boldsymbol{y}}_h,\boldsymbol{y}_h)}{|||\boldsymbol{y}_h|||}.$$

Denoting by $\|\mathbb{A}\|$ the norm of the bilinear form $\mathbb{A}$ and using again the triangle inequality we deduce that

$$|||\boldsymbol{x}^*-\boldsymbol{x}_h^*||| \leq \big(1+\frac{\|\mathbb{A}\|}{\gamma}\big)\,|||\boldsymbol{x}^*-\tilde{\boldsymbol{y}}_h||| + \frac{1}{\gamma}\sup_{\boldsymbol{y}_h\in\mathbb{X}_h}\frac{\mathbb{A}(\boldsymbol{x}^*-\boldsymbol{x}_h^*,\boldsymbol{y}_h)}{|||\boldsymbol{y}_h|||} \qquad \forall\tilde{\boldsymbol{y}}_h\in\mathbb{X}_h$$

and the result follows from (29). ∎

Lemma 4.1. *There exists $C > 0$, independent of h, such that*

$$\Big\|\widetilde{\boldsymbol{T}}-\boldsymbol{T}_h\Big\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} \leq C\Big(\Big\|\widetilde{\boldsymbol{P}}-\boldsymbol{P}_h\Big\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} + \delta(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\mathbb{X}_h),\mathbb{X}_h) + \sup_{\boldsymbol{x}_h\in\mathbb{X}_h}\frac{\Upsilon_h(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}\boldsymbol{x}_h)}{|||\boldsymbol{x}_h|||}\Big).$$

Proof. Given $\boldsymbol{x}_h \in \mathbb{X}_h$, we have that

$$(\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h)\boldsymbol{x}_h = (\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h)\boldsymbol{P}_h\boldsymbol{x}_h + (\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h)(\boldsymbol{I} - \boldsymbol{P}_h)\boldsymbol{x}_h = (\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h)\boldsymbol{P}_h\boldsymbol{x}_h,$$

where the last equality is because both $\widetilde{\boldsymbol{T}}$ and $\boldsymbol{T}_h$ become the identity when restricted to $\ker_h(a) \times \boldsymbol{\mathcal{Q}}_h$. On the other hand,

$$(\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h)\boldsymbol{P}_h\boldsymbol{x}_h = (\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{T}}_h)(\boldsymbol{P}_h - \widetilde{\boldsymbol{P}})\boldsymbol{x}_h + (\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{T}}_h)(\widetilde{\boldsymbol{P}}\boldsymbol{x}_h)$$

yields the estimate

$$\left|\!\left|\!\left|(\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{T}}_h)\boldsymbol{P}_h\boldsymbol{x}_h\right|\!\right|\!\right| \leq \left(\left(\left\|\widetilde{\boldsymbol{T}}\right\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})} + \|\widetilde{\boldsymbol{T}}_h\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})} \right) \left|\!\left|\!\left|(\boldsymbol{P}_h - \widetilde{\boldsymbol{P}})\boldsymbol{x}_h\right|\!\right|\!\right| + \left|\!\left|\!\left|(\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{T}}_h) \circ \widetilde{\boldsymbol{P}}\boldsymbol{x}_h\right|\!\right|\!\right| \right) \tag{39}$$

and the result follows from the uniform boundedness of $\widetilde{\boldsymbol{T}}_h$ (as established in Proposition 1) and from (36). □

Given an eigenvalue $\mu \notin \{0,1\}$ of $\widetilde{\boldsymbol{T}}$, we let D_γ be an open disk in the complex plane with boundary γ, such that μ is the only eigenvalue of $\widetilde{\boldsymbol{T}}$ lying in D_γ and $\gamma \cap \mathrm{sp}(\widetilde{\boldsymbol{T}}) = \emptyset$. We recall that the spectral projector $\boldsymbol{G} := \frac{1}{2\pi i}\int_\gamma \left(z\boldsymbol{I} - \widetilde{\boldsymbol{T}}\right)^{-1} dz : \widetilde{\mathbb{X}} \longrightarrow \widetilde{\mathbb{X}}$ is well-defined and bounded. It is shown in [11, Lemma 1] that, if $\lim_{h\to 0}\left\|\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} = 0$, then (for h sufficiently small) the discrete projection $\boldsymbol{G}_h := \frac{1}{2\pi i}\int_\gamma \left(z\boldsymbol{I} - \widetilde{\boldsymbol{T}}_h\right)^{-1} dz : \mathbb{X}_h \longrightarrow \mathbb{X}_h$ is also well-defined and uniformly bounded. Moreover, (cf. [11, Lemma 2]) there exists $C > 0$ independent of h such that

$$\|\boldsymbol{G} - \boldsymbol{G}_h\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} \leq C\left\|\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}. \tag{40}$$

Theorem 4.2. *Assume that*

$$\lim_{h\to 0}\left(\left\|\widetilde{\boldsymbol{P}} - \boldsymbol{P}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} + \delta(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}}),\mathbb{X}_h) + \sup_{\boldsymbol{x}_h\in\mathbb{X}_h}\frac{\Upsilon_h(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}\boldsymbol{x}_h)}{|\!|\!|\boldsymbol{x}_h|\!|\!|}\right) = 0. \tag{41}$$

Then, if m is the multiplicity of an eigenvalue $\mu \notin \{0,1\}$ of $\widetilde{\boldsymbol{T}}$, there exists exactly m eigenvalues $\{\mu_{i,h},\quad i = 1\cdots,m\}$ of $\boldsymbol{T}_h$ such that

$$\lim_{h\to 0}\max_{1\leq i\leq m}|\mu - \mu_{i,h}| = 0.$$

Moreover, if $\mathbb{E}(\mu)$ is the eigenspace corresponding to μ and $\mathbb{E}_h(\mu)$ is the $\boldsymbol{T}_h$-invariant subspace of $\mathbb{X}_h$ spanned by the eigenspaces corresponding to $\{\mu_{i,h},\quad i = 1\cdots,m\}$ then

$$\lim_{h\to 0}\widehat{\delta}(\mathbb{E}(\mu),\mathbb{E}_h(\mu)) = 0.$$

Proof. We deduce from Theorem 4.1 that $\lim_{h\to 0}\left\|\widetilde{\boldsymbol{T}} - \boldsymbol{T}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} = 0$ and the result is a consequence of Section 2 of [11]. □

Theorem 4.3. *Under the condition (41), if m is the multiplicity of an eigenvalue $\mu \notin \{0,1\}$ of $\widetilde{\boldsymbol{T}}$ then, for h sufficiently small, there exists a constant $C>0$ independent of h such that*

$$\widehat{\delta}(\mathbb{E}(\mu),\mathbb{E}_h(\mu)) \le C\Big(\left\|\widetilde{\boldsymbol{P}}-\boldsymbol{P}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} + \delta(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}}),\mathbb{X}_h) + \sup_{\boldsymbol{x}_h\in\mathbb{X}_h}\frac{\Upsilon_h(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}\boldsymbol{x}_h)}{|||\boldsymbol{x}_h|||}\Big).$$

Proof. It is well-known that $\boldsymbol{G}$ is a projector in $\widetilde{\mathbb{X}}$ with range $\mathbb{E}(\mu)$ and we deduce from Theorem 4.2 that, for h sufficiently enough, $\boldsymbol{G}_h$ is a projector in $\mathbb{X}_h$ with range $\mathbb{E}_h(\mu)$, cf. [11]. Hence, for all $\boldsymbol{x}_h\in\mathbb{E}_h(\mu)$, we have $\boldsymbol{G}_h\boldsymbol{x}_h=\boldsymbol{x}_h$, whereas $\boldsymbol{G}\boldsymbol{x}_h\in\mathbb{E}(\mu)$. It follows that,

$$\delta(\boldsymbol{x}_h,\mathbb{E}(\mu)) \le \|\boldsymbol{G}_h\boldsymbol{x}_h-\boldsymbol{G}\boldsymbol{x}_h\| \le \|\boldsymbol{G}_h-\boldsymbol{G}\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}\,|||\boldsymbol{x}_h|||$$

for all $\boldsymbol{x}_h\in\mathbb{E}_h(\mu)$. We deduce from (40) that there exist constants $C>0$ and $h_0>0$ such that, for all $h<h_0$,

$$\delta(\mathbb{E}_h(\mu),\mathbb{E}(\mu)) \le C\left\|\widetilde{\boldsymbol{T}}-\boldsymbol{T}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}. \tag{42}$$

On the other hand, we notice that $\boldsymbol{G}\boldsymbol{x}=\boldsymbol{x}$ for all $\boldsymbol{x}\in\mathbb{E}(\mu)$. Then, for all $\boldsymbol{y}_h\in\mathbb{X}_h$ and for h small enough,

$$\begin{aligned}|||\boldsymbol{x}-\boldsymbol{G}_h\boldsymbol{y}_h||| &\le |||\boldsymbol{G}(\boldsymbol{x}-\boldsymbol{y}_h)||| + |||(\boldsymbol{G}-\boldsymbol{G}_h)\boldsymbol{y}_h||| \le \\ &\|\boldsymbol{G}\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})}\,|||(\boldsymbol{x}-\boldsymbol{y}_h)||| + \|(\boldsymbol{G}-\boldsymbol{G}_h)\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}\,|||\boldsymbol{y}_h||| \\ &\le (1+\|\boldsymbol{G}\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})})\,|||\boldsymbol{x}-\boldsymbol{y}_h||| + |||\boldsymbol{G}-\boldsymbol{G}_h|||_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}\,|||\boldsymbol{x}|||,\end{aligned}$$

where the last inequality follows from the triangle inequality, (40), Lemma 4.1 and (41). It follows that

$$\begin{aligned}\frac{\inf_{\boldsymbol{x}_h\in\mathbb{E}_h(\mu)}|||\boldsymbol{x}-\boldsymbol{x}_h|||}{|||\boldsymbol{x}|||} &= \frac{\inf_{\boldsymbol{y}_h\in\mathbb{X}_h}|||\boldsymbol{x}-\boldsymbol{G}_h\boldsymbol{y}_h|||}{|||\boldsymbol{x}|||} \le \\ &(1+\|\boldsymbol{G}\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})})\frac{\inf_{\boldsymbol{y}_h\in\mathbb{X}_h}|||\boldsymbol{x}-\boldsymbol{y}_h|||}{|||\boldsymbol{x}|||} + \|\boldsymbol{G}-\boldsymbol{G}_h\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}, \quad \forall\boldsymbol{x}\in\mathbb{E}(\mu).\end{aligned}$$

Using (40) and the fact that $\boldsymbol{T}\circ\boldsymbol{P}\boldsymbol{x}=\boldsymbol{T}\boldsymbol{x}=\mu\boldsymbol{x}$ for any $\boldsymbol{x}\in\mathbb{E}(\mu)$ yield

$$\begin{aligned}\delta(\mathbb{E}(\mu),\mathbb{E}_h(\mu)) &\le \frac{1+\|\boldsymbol{G}\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})}}{\mu}\delta(\boldsymbol{T}\circ\boldsymbol{P}(\mathbb{X}),\mathbb{X}_h) + \|\boldsymbol{G}-\boldsymbol{G}_h\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} \\ &\le \frac{1+\|\boldsymbol{G}\|_{\mathcal{L}(\widetilde{\mathbb{X}},\widetilde{\mathbb{X}})}}{\mu}\,\delta(\boldsymbol{T}\circ\boldsymbol{P}(\mathbb{X}),\mathbb{X}_h) + \left\|\widetilde{\boldsymbol{T}}-\boldsymbol{T}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}.\end{aligned}$$

The result follows now from Lemma 4.1 by noticing that both $\delta(\boldsymbol{T}\circ\boldsymbol{P}(\mathbb{X}),\mathbb{X}_h)$ and $\delta(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\mathbb{X}_h),\mathbb{X}_h)$ are smaller than $\delta(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}}),\mathbb{X}_h)$. ∎

Theorem 4.4. *Under the condition (41), if m is the multiplicity of an eigenvalue $\mu\notin\{0,1\}$ of $\widetilde{\boldsymbol{T}}$ then there exists a constant $C>0$ independent of h such that*

$$\sup_{1\le i\le m}|\mu-\mu_{i,h}| \le C\bigg(\Big(\left\|\widetilde{\boldsymbol{P}}-\boldsymbol{P}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})} + \delta(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}}),\mathbb{X}_h)\Big)^2 + \sup_{\boldsymbol{x}\in\mathbb{E}(\mu)}\frac{\Upsilon_h(\boldsymbol{x})}{|||\boldsymbol{x}|||}\bigg).$$

Proof. Let $\boldsymbol{u}_{i,h}$ be an eigenfunction corresponding to $\mu_{i,h}$ such that $\|\boldsymbol{x}_{i,h}\|=1$. We know from (42) that, if h is sufficiently small,

$$\delta(\boldsymbol{x}_{i,h},\mathbb{E}(\mu)) \le C\left\|\widetilde{\boldsymbol{T}}-\boldsymbol{T}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}.$$

Then, there exists an eigenfunction $\boldsymbol{x} := ((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \mathbb{E}(\mu)$ satisfying

$$|||\boldsymbol{x}_{i,h} - \boldsymbol{x}||| \leq C \left\| \widetilde{\boldsymbol{T}} - \boldsymbol{T}_h \right\|_{\mathcal{L}(\mathbb{X}_h, \widetilde{\mathbb{X}})},$$

which proves that $|||\boldsymbol{x}|||$ is bounded from below and above by constant independent of h. Proceeding as in the proof of Lemma 3.1 we obtain that

$$\mathbb{A}(\boldsymbol{x}, \boldsymbol{y}_h) = \lambda \mathbb{B}(\boldsymbol{x}, \boldsymbol{y}_h) - \int_\Sigma \frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\sigma} \cdot (q_h \boldsymbol{\nu} + \boldsymbol{\tau}_h \boldsymbol{\nu}) \tag{43}$$

for all $\boldsymbol{y}_h := ((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \mathbb{X}_h$. With the aid of (43), it is easy to show that the identity

$$\mathbb{A}(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h}) - \lambda \mathbb{B}(\boldsymbol{x} - \boldsymbol{x}_{i,h}, \boldsymbol{x} - \boldsymbol{x}_{i,h}) + 2 \int_\Sigma \frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}} \boldsymbol{\sigma} \cdot (p_{i,h} \boldsymbol{\nu} + \boldsymbol{\sigma}_{i,h} \boldsymbol{\nu}) = (\lambda_{i,h} - \lambda) \mathbb{B}(\boldsymbol{x}_{i,h}, \boldsymbol{x}_{i,h})$$

holds true. Now, as $\mathbb{E}(\mu)$ is finite-dimensional, there exists $c > 0$, independent of h, such that $\mathbb{B}(\boldsymbol{x}, \boldsymbol{x}) \geq c$. This proves that $\mathbb{B}(\boldsymbol{x}_{ih}, \boldsymbol{x}_{ih}) \geq \frac{c}{2}$ for h sufficiently small. The result follows now from the fact that $\mathbb{A}$ and $\mathbb{B}$ are continuous bilinear forms on $\widetilde{\mathbb{X}}$. □

5. **Asymptotic error estimates.** We begin this section by recalling some well-known approximation properties of the finite element spaces introduced above. Given $s \in (0,1]$, let $\boldsymbol{\Pi}_h : \mathrm{H}^s(\Omega_{\mathrm{S}})^{n \times n} \cap \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}_h$ be the usual lowest-order Raviart-Thomas interpolation operator (see [9]), which is characterized by the identities

$$\int_F (\boldsymbol{\Pi}_h \boldsymbol{\tau}) \boldsymbol{\nu}_F \cdot \boldsymbol{\zeta} = \int_F \boldsymbol{\tau} \boldsymbol{\nu}_F \cdot \boldsymbol{\zeta} \qquad \forall \boldsymbol{\zeta} \in \mathcal{P}_0(F)^n$$

for all faces (edges) F of elements $T \in \mathcal{T}_h(\Omega_{\mathrm{S}})$, with $\boldsymbol{\nu}_F$ being a unit vector normal to the face (edge) F. It is well known that $\boldsymbol{\Pi}_h$ is a bounded linear operator and that the following commuting diagram property holds true (cf. [9]):

$$\operatorname{\mathbf{div}}(\boldsymbol{\Pi}_h \boldsymbol{\tau}) = \boldsymbol{L}_h(\operatorname{\mathbf{div}} \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\Omega_{\mathrm{S}})^{n \times n} \cap \mathrm{H}(\operatorname{\mathbf{div}}; \Omega_{\mathrm{S}}), \tag{44}$$

where $\boldsymbol{L}_h : \mathrm{L}^2(\Omega_{\mathrm{S}})^n \to \boldsymbol{\mathcal{U}}_h$ is the $\mathrm{L}^2(\Omega_{\mathrm{S}})^n$-orthogonal projector. In addition, it is well-known that the arguments leading to [14, Theorem 3.16] allow showing that there exists $C > 0$, independent of h, such that

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{0,\Omega_{\mathrm{S}}} \leq C h^s \left(\|\boldsymbol{\tau}\|_{s,\Omega_{\mathrm{S}}} + \|\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{0,\Omega_{\mathrm{S}}} \right) \quad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\Omega_{\mathrm{S}})^{n \times n} \cap \mathrm{H}(\operatorname{\mathbf{div}}; \Omega_{\mathrm{S}}). \tag{45}$$

Finally, we denote by $\boldsymbol{R}_h : \boldsymbol{\mathcal{Q}} \to \boldsymbol{\mathcal{Q}}_h$ the orthogonal projector with respect to the $\mathrm{L}^2(\Omega_{\mathrm{S}})^{n \times n}$-norm and by $\pi_h : \mathrm{H}^1(\Omega_{\mathrm{F}}) \to \mathcal{V}_h$ the orthogonal projector with respect to the $\mathrm{H}^1(\Omega_{\mathrm{F}})$-norm. Then, for any $s \in (0,1]$, we have

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{\mathrm{H}(\operatorname{\mathbf{div}};\Omega_{\mathrm{S}})} \leq C h^s \|\boldsymbol{\tau}\|_{\mathrm{H}^s(\operatorname{\mathbf{div}};\Omega_{\mathrm{S}})} \qquad \forall \boldsymbol{\tau} \in \mathrm{H}^s(\operatorname{\mathbf{div}}; \Omega_{\mathrm{S}}) \cap \boldsymbol{\mathcal{W}}, \tag{46}$$

$$\|\boldsymbol{r} - \boldsymbol{R}_h \boldsymbol{r}\|_{0,\Omega_{\mathrm{S}}} \leq C h^s \|\boldsymbol{r}\|_{s,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{r} \in \mathrm{H}^s(\Omega_{\mathrm{S}})^{n \times n} \cap \boldsymbol{\mathcal{Q}}, \tag{47}$$

$$\|\boldsymbol{v} - \boldsymbol{L}_h \boldsymbol{v}\|_{0,\Omega_{\mathrm{S}}} \leq C h^s \|\boldsymbol{v}\|_{s,\Omega_{\mathrm{S}}} \qquad \forall \boldsymbol{v} \in \mathrm{H}^s(\Omega_{\mathrm{S}})^n, \tag{48}$$

$$\|q - \pi_h q\|_{1,\Omega_{\mathrm{F}}} \leq C h^s \|q\|_{1+s,\Omega_{\mathrm{F}}} \qquad \forall q \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}}). \tag{49}$$

Notice that (46) is actually a straightforward consequence of (45), (44), and (48).

The following estimate holds true.

Lemma 5.1. *There exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{\Pi}_h \boldsymbol{E} q\|_{0,\Omega_{\rm S}} \leq C\,\|q\|_{1,\Omega_{\rm F}} \qquad \forall q \in {\rm H}^1(\Omega_{\rm F}),$$

$$\|\boldsymbol{E} q - \boldsymbol{\Pi}_h \boldsymbol{E} q\|_{0,\Omega_{\rm S}} \leq C h^{t_{\rm S}}\,\|q\|_{1,\Omega_{\rm F}} \qquad \forall q \in {\rm H}^1(\Omega_{\rm F}).$$

Proof. See [17, Lemma 5.1]. □

Next, we introduce the discrete counterparts of $\boldsymbol{E}$ and $\widehat{\boldsymbol{E}}$, defined for any $q \in {\rm H}^1(\Omega_{\rm F})$ by

$$\boldsymbol{E}_h q := \boldsymbol{\Pi}_h \boldsymbol{E}(\pi_h q) \in \boldsymbol{\mathcal{W}}_h \qquad \text{and} \qquad \widehat{\boldsymbol{E}}_h q := (\boldsymbol{E}_h q, \pi_h q)\,. \tag{50}$$

It is clear that $\widehat{\boldsymbol{E}}_h q \in \boldsymbol{\mathcal{Y}}_h$ for all $q \in {\rm H}^1(\Omega_{\rm F})$. Indeed, as $\boldsymbol{\nu}$ is piecewise constant on Σ,

$$(\boldsymbol{E}_h q)\boldsymbol{\nu} = \boldsymbol{\varrho}_h(\boldsymbol{E}(\pi_h q)\boldsymbol{\nu}) = -\boldsymbol{\varrho}_h\big((\pi_h q)\boldsymbol{\nu}\big) = -\big(\varrho_h(\pi_h q)\big)\boldsymbol{\nu}$$

where $\boldsymbol{\varrho}_h : {\rm L}^2(\Sigma)^n \to \mathcal{P}_0(\Sigma_{\rm S}^h)^n$ stands for the vectorial counterpart of ϱ_h. Moreover, we have the following result.

Lemma 5.2. *There exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{E} q - \boldsymbol{E}_h q\|_{{\rm H}(\mathbf{div};\Omega_{\rm S})} \leq C\left(h^{t_{\rm S}}\,\|q\|_{1,\Omega_{\rm F}} + \|q - \pi_h q\|_{1,\Omega_{\rm F}}\right) \qquad \forall q \in {\rm H}^1(\Omega_{\rm F}).$$

Proof. Since $\mathbf{div}\,\boldsymbol{E} q = \mathbf{div}\,\boldsymbol{E}_h q = \mathbf{0}$, we only have to estimate the ${\rm L}^2(\Omega_{\rm S})$-norm. To this end, we add and subtract $\boldsymbol{\Pi}_h \boldsymbol{E} q$ and use the triangle inequality to obtain

$$\|\boldsymbol{E} q - \boldsymbol{E}_h q\|_{0,\Omega_{\rm S}} \leq \|\boldsymbol{E} q - \boldsymbol{\Pi}_h \boldsymbol{E} q\|_{0,\Omega_{\rm S}} + \|\boldsymbol{\Pi}_h \boldsymbol{E}(q - \pi_h q)\|_{0,\Omega_{\rm S}}\,.$$

Hence, the proof follows from the two estimates in Lemma 5.1. □

Our aim now is to show that, if $\boldsymbol{\tau}$ is sufficiently smooth, then $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ can be approximated well from $\boldsymbol{\mathcal{Y}}_h$.

Lemma 5.3. *Let $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\tau} \in {\rm H}^{t_{\rm S}}(\Omega_{\rm S})^{n\times n}$ and let*

$$(\boldsymbol{\tau}_h, q_h) := (\boldsymbol{\Pi}_h \boldsymbol{\tau} + (\boldsymbol{E}_h q - \boldsymbol{\Pi}_h \boldsymbol{E} q)\,, \pi_h q)\,.$$

Then, $(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\mathcal{Y}}_h$ and

$$\|(\boldsymbol{\tau}, q) - (\boldsymbol{\tau}_h, q_h)\| \leq C\left[\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{{\rm H}(\mathbf{div};\Omega_{\rm S})} + \|q - \pi_h q\|_{1,\Omega_{\rm F}}\right].$$

Proof. First notice that

$$\boldsymbol{\tau}_h \boldsymbol{\nu} + \varrho_h(q_h)\boldsymbol{\nu} = \boldsymbol{\Pi}_h\left(\boldsymbol{\tau} - \boldsymbol{E} q\right)\boldsymbol{\nu} + \left(\boldsymbol{E}_h q \boldsymbol{\nu} + \varrho_h(\pi_h q)\boldsymbol{\nu}\right) = \mathbf{0} \qquad \text{on } \Sigma.$$

Indeed, from the definition of $\boldsymbol{E}_h$ (cf. (50)), it is clear that $((\boldsymbol{E}_h q)\boldsymbol{\nu} + \varrho_h(\pi_h q)\boldsymbol{\nu})$ vanishes on Σ and so does $\boldsymbol{\Pi}_h\left(\boldsymbol{\tau} - \boldsymbol{E} q\right)\boldsymbol{\nu}$, because $\left(\boldsymbol{\tau} - \boldsymbol{E} q\right)\boldsymbol{\nu} = \boldsymbol{\tau}\boldsymbol{\nu} + q\boldsymbol{\nu} = \mathbf{0}$ on Σ for any $(\boldsymbol{\tau}, q) \in \boldsymbol{\mathcal{Y}}$.

To prove the estimate we use again the definition of $\boldsymbol{E}_h$ to write

$$\begin{aligned}&\|(\boldsymbol{\tau}, q) - (\boldsymbol{\tau}_h, q_h)\| \\ &\qquad \leq \|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{{\rm H}(\mathbf{div};\Omega_{\rm S})} + \|\boldsymbol{\Pi}_h \boldsymbol{E}(\pi_h q - q)\|_{{\rm H}(\mathbf{div};\Omega_{\rm S})} + \|q - \pi_h q\|_{1,\Omega_{\rm F}}\,.\end{aligned}$$

Then, the result follows from the first inequality in Lemma 5.1 and the fact that $\boldsymbol{E}(\pi_h q - q)$ is divergence-free and, hence, so is $\boldsymbol{\Pi}_h \boldsymbol{E}(\pi_h q - q)$. □

Lemma 5.4. *There exists a constant $C > 0$ independent of h such that*

$$\delta(\boldsymbol{T} \circ \widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}}), \mathbb{X}_h) \leq C h^t, \quad \textit{with } t := \min\{t_{\rm S}, t_{\rm F}\}.$$

Proof. On the one hand, we have that $(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}})(\widetilde{\mathbb{X}})\subset\boldsymbol{T}(\mathbb{X})$. On the other hand, it is straightforward that, for any $\widetilde{\boldsymbol{x}}\in\widetilde{\mathbb{X}}$,

$$\mathbb{B}((\boldsymbol{I}-\boldsymbol{T})\circ\widetilde{\boldsymbol{P}}(\widetilde{\boldsymbol{x}}),\boldsymbol{y})=0,\qquad\forall\boldsymbol{y}\in\ker(a)\times\boldsymbol{\mathcal{Q}}.$$

Thus, $(\boldsymbol{I}-\boldsymbol{T})\circ\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}})\subset\boldsymbol{P}(\mathbb{X})$. It follows from Lemmas 2.3 and 2.4 that

$$\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}})\subset\boldsymbol{T}(\mathbb{X})\cap\widetilde{\boldsymbol{P}}(\widetilde{\mathbb{X}})\hookrightarrow\left[\mathrm{H}^{t_{\mathrm{S}}}(\mathbf{div};\Omega_{\mathrm{S}})\times\mathrm{H}^{1+t_{\mathrm{F}}}(\Omega_{\mathrm{F}})\right]\times\mathrm{H}^{t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^{n\times n}$$

and the result is a consequence of Lemma 5.3 and the approximation properties (46)-(49). ☐

Let $\widehat{\boldsymbol{E}}_h$ be the operator defined in (50) and let

$$\begin{aligned}\widetilde{\boldsymbol{P}}_h:\ &\widetilde{\boldsymbol{\mathcal{Y}}}\times\boldsymbol{\mathcal{Q}}\longrightarrow\boldsymbol{\mathcal{Y}}_h\times\boldsymbol{\mathcal{Q}}_h,\\ &((\boldsymbol{\sigma},p),\boldsymbol{r})\longmapsto\widetilde{\boldsymbol{P}}_h((\boldsymbol{\sigma},p),\boldsymbol{r}):=((\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h)+\widehat{\boldsymbol{E}}_h\bar{p},\widetilde{\boldsymbol{r}}_h),\end{aligned}$$

where $(\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}$ and $(\widetilde{\boldsymbol{r}}_h,\widetilde{\boldsymbol{u}}_h)\in\boldsymbol{\mathcal{Q}}_h\times\boldsymbol{\mathcal{U}}_h$ solve the equations

$$d((\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h),(\boldsymbol{\tau}_h,\xi))+B((\boldsymbol{\tau}_h,\xi),(\widetilde{\boldsymbol{r}}_h,\widetilde{\boldsymbol{u}}_h))=-d(\widehat{\boldsymbol{E}}_h\bar{p},(\boldsymbol{\tau}_h,\xi)),\tag{51}$$

$$B((\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h),(\boldsymbol{s}_h,\boldsymbol{v}_h))=\int_{\Omega_{\mathrm{S}}}\mathbf{div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}_h-b(\widehat{\boldsymbol{E}}_h\bar{p},\boldsymbol{s}_h)\tag{52}$$

for all $(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}$ and $(\boldsymbol{s}_h,\boldsymbol{v}_h)\in\boldsymbol{\mathcal{Q}}_h\times\boldsymbol{\mathcal{U}}_h$. It is clear that $\tilde{\boldsymbol{P}}_h|_{\boldsymbol{\mathcal{Y}}_h\times\boldsymbol{\mathcal{Q}}_h}=\boldsymbol{P}_h$.

Equations (51)–(52) constitute a conforming finite element discretization of the mixed problem (21)–(22) used to define $\widetilde{\boldsymbol{P}}$. The uniform discrete inf-sup condition of B for the pair $\{\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}},\boldsymbol{\mathcal{Q}}_h\times\boldsymbol{\mathcal{U}}_h\}$ is an easy consequence of (24). Moreover, [17, Lemma 2.1] guarantees the uniform ellipticity of d on $\boldsymbol{\mathcal{W}}\times\mathrm{H}^1(\Omega_{\mathrm{F}})\supset\ker_h(a)$, whereas the fact that $\mathbf{div}(\boldsymbol{\mathcal{W}}_h)\subset\boldsymbol{\mathcal{U}}_h$ implies that $\ker_h(B)\subset\ker_h(a)$. Hence, as a consequence of the Babuška-Brezzi theory, problem (51)–(52), is well posed. Furthermore, thanks to the definition of $\widehat{\boldsymbol{E}}_h\bar{p}$, the first estimate from Lemma 5.1, and the fact that $\|\pi_h\bar{p}\|_{1,\Omega_{\mathrm{F}}}\le\|\bar{p}\|_{1,\Omega_{\mathrm{F}}}$ (since π_h is a projection), we can claim that the operators $\widetilde{\boldsymbol{P}}_h$ are bounded uniformly with respect to h and the following Strang-like estimate holds true:

$$\begin{aligned}&\|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h)\|+\|\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}}_h\|_{0,\Omega_{\mathrm{S}}}+\|\widetilde{\boldsymbol{r}}-\widetilde{\boldsymbol{r}}_h\|_{0,\Omega_{\mathrm{S}}}\\&\le C\Bigg[\inf_{(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}}\|(\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c})-(\boldsymbol{\tau}_h,\xi)\|+\inf_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{U}}_h}\|\widetilde{\boldsymbol{u}}-\boldsymbol{v}_h\|_{0,\Omega_{\mathrm{S}}}+\inf_{\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\|\widetilde{\boldsymbol{r}}-\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}\\&\qquad+\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\xi)\in\boldsymbol{\mathcal{Y}}_{h,\mathbb{R}}}\frac{|d(\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p},(\boldsymbol{\tau}_h,\xi))|}{\|(\boldsymbol{\tau}_h,\xi)\|}+\sup_{\mathbf{0}\neq\boldsymbol{s}_h\in\boldsymbol{\mathcal{Q}}_h}\frac{|b(\widehat{\boldsymbol{E}}\bar{p}-\widehat{\boldsymbol{E}}_h\bar{p},\boldsymbol{s}_h)|}{\|\boldsymbol{s}_h\|_{0,\Omega_{\mathrm{S}}}}\Bigg],\end{aligned}\tag{53}$$

where $((\widetilde{\boldsymbol{\sigma}}_0,\widetilde{c}),(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{r}}))$ and $((\widetilde{\boldsymbol{\sigma}}_{h,0},\widetilde{c}_h),(\widetilde{\boldsymbol{u}}_h,\widetilde{\boldsymbol{r}}_h))$ are the solutions to (21)–(22) and (51)–(52), respectively. As a consequence, we have the following estimate.

Lemma 5.5. *There exists $C>0$, independent of h, such that*

$$\left\|\widetilde{\boldsymbol{P}}-\boldsymbol{P}_h\right\|_{\mathcal{L}(\mathbb{X}_h,\widetilde{\mathbb{X}})}\le C\,h^{t_{\mathrm{S}}}.$$

Proof. See [17, Lemma 6.3] for more details. ☐

Lemma 5.6. *There exists $C_1 > 0$, independent of h, such that*

$$\sup_{\boldsymbol{x}\in\widetilde{\mathbb{X}}} \frac{\Upsilon_h(\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}\boldsymbol{x})}{|||\boldsymbol{x}|||} \leq C_1\, h.$$

Moreover, if $\mu \notin \{0,1\}$ is an eigenvalue of $\widetilde{\boldsymbol{T}}$. Then, there exists a constant $C_2 > 0$, independent of h such that

$$\sup_{\boldsymbol{x}\in\mathbb{E}(\mu)} \frac{\Upsilon_h(\boldsymbol{x})}{|||\boldsymbol{x}|||} \leq C_2\, h^{1+t_{\mathrm{S}}}$$

Proof. We first notice that by definition $q_h\boldsymbol{\nu} + \boldsymbol{\tau}_h\boldsymbol{\nu} = (q_h - \varrho_h q_h)\boldsymbol{\nu}$. Hence, if $\boldsymbol{T}\circ\widetilde{\boldsymbol{P}}\boldsymbol{x} := ((\boldsymbol{\sigma}^*, p^*), \boldsymbol{r}^*)$, we have that

$$\begin{aligned}\int_\Sigma \frac{1}{\rho_{\mathrm{S}}} \operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\cdot\big(q_h\boldsymbol{\nu} + \boldsymbol{\tau}_h\boldsymbol{\nu}\big) &= \int_\Sigma \frac{1}{\rho_{\mathrm{S}}}\big(\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\cdot\boldsymbol{\nu} - \varrho_h(\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\cdot\boldsymbol{\nu})\big)\big(q_h - \varrho_h q_h\big)\\ = \int_\Sigma \frac{1}{\rho_{\mathrm{S}}}\big(\operatorname{\mathbf{div}}\boldsymbol{\sigma}^* - \boldsymbol{\varrho}_h(\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*)\big)\cdot\boldsymbol{\nu}\big(q_h - \varrho_h q_h\big) &\leq C_3 h\|\operatorname{\mathbf{div}}\boldsymbol{\sigma}^*\|_{1/2,\Sigma}\,\|q_h\|_{1/2,\Sigma}\\ &\leq C_4 h\,|||\boldsymbol{x}|||\,|||((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)|||\,, \end{aligned}\qquad (54)$$

which proves the first estimate of the Lemma.

On the other hand, if $\boldsymbol{x} := ((\boldsymbol{\sigma}, p), \boldsymbol{r}) \in \mathbb{E}(\mu)$ then $\boldsymbol{T}\circ\boldsymbol{P}(\boldsymbol{x}) = \mu\boldsymbol{x}$ and we deduce again from (23) that there exists a constant $C > 0$ such that

$$\|\boldsymbol{\sigma}\|_{\mathrm{H}^{t_{\mathrm{S}}}(\operatorname{\mathbf{div}};\Omega_{\mathrm{S}})} + \|\boldsymbol{u}\|_{1+t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|\boldsymbol{r}\|_{t_{\mathrm{S}},\Omega_{\mathrm{S}}} + \|p\|_{1+t_{\mathrm{F}},\Omega_{\mathrm{F}}} \leq C\,|||\boldsymbol{x}|||\,,$$

where $\boldsymbol{u}$ is the displacement field given by $\boldsymbol{u} = \frac{\mu}{(\mu-1)\rho_{\mathrm{S}}}\operatorname{\mathbf{div}}\boldsymbol{\sigma} \in H^{1+t_{\mathrm{S}}}(\Omega_{\mathrm{S}})^n$. With this regularity result at hand, we can proceed as in (54) to obtain

$$\begin{aligned}\int_\Sigma \boldsymbol{u}\cdot\boldsymbol{\nu}\big(q_h - \varrho_h q_h\big) &\leq C_6 h^{1+t_{\mathrm{S}}}\|\operatorname{\mathbf{div}}\boldsymbol{\sigma}\|_{1/2+t_{\mathrm{S}},\Sigma}\,\|q_h\|_{1/2,\Sigma}\\ &\leq C_7 h^{1+t_{\mathrm{S}}}\,|||\boldsymbol{x}|||\,|||((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h)|||\,,\quad \forall((\boldsymbol{\tau}_h, q_h), \boldsymbol{s}_h) \in \mathbb{X}_h,\end{aligned}$$

and the second estimate of the Lemma follows. □

We conclude that we have have the following asymptotic convergence for the eigenfunctions and eigenvalues of problem (1)-(7).

Theorem 5.7. *If $\mu \notin \{0,1\}$ is an eigenvalue of $\widetilde{\boldsymbol{T}}$, there exist constants $C > 0$ and $h_0 > 0$ such that, for all $h < h_0$,*

$$\widehat{\delta}(\mathbb{E}(\mu), \mathbb{E}_h(\mu)) \leq Ch^t \quad \textit{and} \quad \max_{1\leq i\leq m}|\lambda - \lambda_{i,h}| \leq Ch^{2t}, \qquad \textit{with } t := \min\{t_{\mathrm{S}}, t_{\mathrm{F}}\}.$$

6. **Numerical results.** We use a two-dimensional benchmark test that is identical to the one carried out in [17]. The geometrical data representing an elastic container (steel) filled with a compressible liquid (water) is shown in Figure 2. The physical parameters are given by:

- Solid density: $\rho_{\mathrm{S}} = 7700\,\mathrm{kg/m^3}$,
- Young modulus: $E = 1.44\times 10^{11}\,\mathrm{Pa}$,
- Poisson ratio: $\nu = 0.35$,
- Fluid density: $\rho_{\mathrm{F}} = 1000\,\mathrm{kg/m^3}$,
- Acoustic speed: $c = 1430\,\mathrm{m/s}$,
- Gravity acceleration: $g = 9.8\,\mathrm{m/s^2}$.

TABLE 1. Lowest computed sloshing frequencies $\omega_{h,k}^{\mathrm{S}}$ (in rad/s).

Mode	$N = 4$	$N = 6$	$N = 8$	$N = 10$	$N = 12$	Order	Extrapolated	[17]
$\omega_{h,1}^{\mathrm{S}}$	5.3196	5.3164	5.3153	5.3148	5.3145	2.00	5.3138	5.3138
$\omega_{h,2}^{\mathrm{S}}$	7.8697	7.8490	7.8417	7.8383	7.8365	2.00	7.8324	7.8324
$\omega_{h,3}^{\mathrm{S}}$	9.7135	9.6560	9.6358	9.6264	9.6213	1.99	9.6097	9.6099

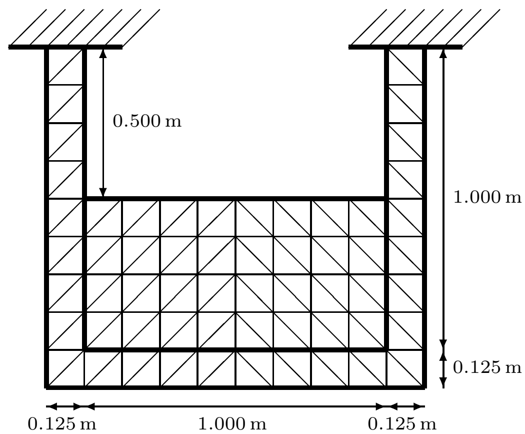

FIGURE 2. Fluid and solid domains. Coarsest mesh ($N = 1$).

We use several meshes which are successive uniform refinements of the coarse initial triangulation shown in Figure 2. The refinement parameter N is the number of element layers across the thickness of the solid ($N = 1$ for the mesh in Figure 2).

We can distinguish between two types of vibrations corresponding to *sloshing* and *elastoacoustic* modes. We refer to [4, 6] for a more detailed discussion on sloshing (or gravity) and elastoacoustic frequencies. We report the lowest computed sloshing vibration frequencies $\omega_{h,k}^{\mathrm{S}}$ in Table 1 and the elastoacoustic vibration frequencies $\omega_{h,k}^{\mathrm{E}}$ in Table 2. The tables also include the estimated orders of convergence, as well as more accurate values of the vibration frequencies extrapolated from the computed ones by means of a least-squares fitting. A double order of convergence can be clearly observed in all cases. We finally notice that our results are in agreement with those obtained in [17] and based on the Arnold-Falk-Winther element [2]. This happens even though the computer cost of the lowest-order PEERS element is lower then that of the Arnold-Falk-Winther (AFW) element. Indeed, the global number of unknowns for PEERS elements is $\approx 12N_v$ while which the number of unknowns for AFW elements is $\approx 20N_v$, where N_v represents the number of vertices in $\mathcal{T}_h(\Omega_{\mathrm{S}})$.

REFERENCES

[1] D. N. Arnold, F. Brezzi and J. Douglas, PEERS: A new mixed finite element method for plane elasticity, *Japan J. Appl. Math.,* **1** (1984), 347–367.

[2] D. N. Arnold, R. S. Falk and R. Winther, Mixed finite element methods for linear elasticity with weakly imposed symmetry, *Math. Comp.,* **76** (2007), 1699–1723.

TABLE 2. Lowest computed elastoacoustic vibration frequencies $\omega^{\mathrm{E}}_{h,k}$ (in rad/s).

Mode	$N = 4$	$N = 6$	$N = 8$	$N = 10$	$N = 12$	Order	Extrapolated	[17]
$\omega^{\mathrm{E}}_{h,1}$	423.60	433.08	436.63	438.41	439.46	1.76	442.12	442.71
$\omega^{\mathrm{E}}_{h,2}$	1415.60	1443.44	1453.59	1458.54	1461.37	1.86	1468.12	1469.45
$\omega^{\mathrm{E}}_{h,3}$	2459.47	2516.73	2538.56	2549.67	2556.26	1.71	2573.58	2578.33
$\omega^{\mathrm{E}}_{h,4}$	2644.78	2703.40	2724.84	2735.34	2741.39	1.84	2756.02	2758.94

[3] A. Bermúdez, R. Durán, M. A. Muschietti, R. Rodríguez and J. Solomin, Finite element vibration analysis of fluid-solid systems without spurious modes, *SIAM J. Numer. Anal.,* **32** (1995), 1280–1295.

[4] A. Bermúdez, R. Durán and R. Rodríguez, Finite element solution of incompressible fluid-structure vibration problems, *Internat. J. Numer. Methods Engrg.,* **40** (1997), 1435–1448.

[5] A. Bermúdez, P. Gamallo, L. Hervella-Nieto, R. Rodríguez and D. Santamarina, *Fluid-structure Acoustic Interaction*, Computational Acoustics of Noise Propagation in Fluids. Finite and Boundary Element Methods (eds. S. Marburg and B. Nolte), Springer, 2008.

[6] A. Bermúdez and R. Rodríguez, Finite element analysis of sloshing and hydroelastic vibrations under gravity, *RAIRO – Math. Model. Numer. Anal. (M^2AN),* **33** (1999), 305–327.

[7] D. Boffi, Finite element approximation of eigenvalue problems, *Acta Numerica,* **19** (2010), 1–120.

[8] D. Boffi, F. Brezzi and M. Fortin, Reduced symmetry elements in linear elasticity, *Comm. Pure Appl. Anal.,* **8** (2009), 95–121.

[9] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Verlag, New York, 1991.

[10] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains: Smoothness and Asymptotics of Solutions*, Lecture Notes in Mathematics, 1341, Springer, Berlin, 1988.

[11] J. Descloux, N. Nassif and J. Rappaz, On spectral approximation. Part 1: The problem of convergence, *RAIRO Anal. Numér.,* **12** (1978), 97–112.

[12] G. N. Gatica, A. Márquez and S. Meddahi, Analysis of the coupling of Lagrange and Arnold–Falk-Winther finite elements for a fluid-solid interaction problem in 3D, *SIAM J. Numer. Anal.,* **50** (2012), 1648–1674.

[13] P. Grisvard, Problèmes aux limites dans les polygones. Mode d'emploi, *EDF, Bulletin de la Direction des Études et Recherches (Serie C),* **1** (1986), 21–59.

[14] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numerica*, **11** (2002), 237–339.

[15] L. Kiefling and G. C. Feng, Fluid-structure finite element vibration analysis, *AIAA J.,* **14** (1976), 199–203.

[16] S. Meddahi, D. Mora and R. Rodríguez, Finite element spectral analysis for the mixed formulation of the elasticity equations, *SIAM J. Numer. Anal.,* **51** (2013), 1041–1063.

[17] S. Meddahi, D. Mora and R. Rodríguez, Finite element analysis for a pressure-stress formulation of a fluid-structure interaction spectral problem, *Comput. Math. Appl.,* **68** (2014), 1733–1750.

[18] H. J.-P. Morand and R. Ohayon, *Fluid Structure Interaction*, J. Wiley & Sons, Chichester, 1995.

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