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A finite element analysis of a pseudostress formulation for the Stokes eigenvalue problem

Salim Meddahi
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, Spain
salim@uniovi.es

David Mora*
Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, Concepción, Chile
*Corresponding author: dmora@ubiobio.cl

and

Rodolfo Rodríguez
CI²MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
rodolfo@ing-mat.udec.cl

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In this paper we analyse a finite element approximation of the Stokes eigenvalue problem. We introduce a variational formulation relying only on the pseudostress tensor and propose a discretization by means of the lowest-order Brezzi–Douglas–Marini mixed finite element. However, similar results hold true for other $H(\mathrm{div})$-conforming elements, like Raviart–Thomas elements. We show that the resulting scheme provides a correct approximation of the spectrum and prove optimal-order error estimates. Finally, we report some numerical tests supporting our theoretical results.

Keywords: Stokes equations; pseudostress; eigenvalue problem; mixed finite elements; error estimates.

1. Introduction

The finite element approximation of eigenvalue problems is a subject of great interest from both the practical and theoretical points of view. We refer the reader to Babuška & Osborn (1991) and Boffi (2010) and the references therein for the state of the art in this subject area. We are particularly interested in the finite element analysis of the Stokes eigenvalue problem. The practical interest in Stokes eigenvalues and eigenmodes is discussed, for instance, in Leriche & Labrosse (2004). One motivation is, for example, the study of a plate buckling problem. Indeed, it is well known that when a thin (Kirchhoff) plate is subject to clamped boundary conditions, it admits an equivalent formulation in terms of a Stokes problem (see, for instance, Mercier *et al.*, 1981, Section 7(d) and Chen & Lin, 2006).

Two formulations of the Stokes eigenvalue problem were analysed in Mercier *et al.* (1981, Section 7(d,e)). More recently, an alternative study was presented in Boffi *et al.* (1997) (see also Boffi, 2010, Part 3 and the references therein). Most of these approaches rely on the usual velocity–pressure formulation, which was also used in Lovadina *et al.* (2009) to perform an *a posteriori* error analysis of the Stokes eigenvalue problem.

The aim of this paper is to propose an alternative formulation of this problem. We follow the strategy used in Cai *et al.* (2010) and Gatica *et al.* (2010, 2012) for the steady-state Stokes problem and introduce the so-called pseudostress tensor as a variable. This leads to the pseudostress–velocity formulation introduced for the first time (without any additional stabilization term) in Cai *et al.* (2010). This formulation is preferable to the classical velocity–pressure formulation when an accurate calculation of the stress is needed, because it leads to a locally conservative scheme. Moreover, this formulation transforms an essential boundary condition on velocity into a more convenient natural boundary condition.

One possible disadvantage of using the pseudostress–velocity formulation is that it increases the number of degrees of freedom of the resulting algebraic linear system of equations. However, for the eigenvalue variational formulation, the fact that we will be able to eliminate the pressure and velocity fields, keeping the pseudostress as the only unknown, will be a partial remedy. Last but not least, let us mention that the velocity, the stress and the pressure fields can be easily postprocessed without affecting the accuracy of the approximation.

For the sake of simplicity, we will illustrate our spectral approximation theory with a particular finite element. Since the pseudostress will be sought in $H(\mathbf{div};\Omega)$, we have chosen the lowest-order Brezzi–Douglas–Marini (BDM) mixed finite element. However, similar results hold true for other $H(\mathbf{div};\Omega)$-conforming elements. In particular, we have checked that all the forthcoming analysis remains valid for the lowest-order Raviart–Thomas (RT) element. Only one minor difference appears in the spectral characterization of the corresponding discrete problem (which will be further elaborated on below) but it does not affect the subsequent analysis at all.

The well-known abstract spectral approximation theory (see Babuška & Osborn, 1991) cannot be used to deal with the analysis of our problem. Indeed, the kernel of the bilinear form on the left-hand side of the variational formulation has in our case an infinite-dimensional kernel. Although the standard shift strategy allows a solution operator to be defined, this is not compact and its nontrivial essential spectrum may in such cases lead to spectral pollution at the discrete level. However, we follow Meddahi *et al.* (2013) and take advantage of the classical theory developed in Descloux *et al.* (1978a,b) for noncompact operators to prove that our numerical scheme provides a safe approximation of the eigenvalues at an optimal convergence rate.

The outline of this article is as follows: we introduce in Section 2 the variational formulation of the eigenvalue Stokes problem and define a solution operator. Section 3 is devoted to the spectral characterization. In Section 4, we introduce the discrete eigenvalue problem and describe the spectrum of a discrete solution operator. In Section 5, we prove that the numerical scheme provides a correct spectral approximation and establish optimal-order error estimates for the eigenvalues and eigenfunctions. Finally, we report in Section 6 a set of numerical tests with both BDM and RT elements to confirm that the method is not polluted with spurious modes and to show that the experimental rates of convergence are in accordance with the theoretical ones.

We end this section with some notation that will be used below. Given any Hilbert space $\mathcal{V}$, let $\mathcal{V}^n$ and $\mathcal{V}^{n\times n}$ denote, respectively, the space of vectors and tensors of order n ($n=2$ or 3) with entries in $\mathcal{V}$. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\sigma} := (\sigma_{ij}) \in \mathbb{R}^{n\times n}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathrm{T}} := (\tau_{ji})$, the tensor inner product $\boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^{n} \tau_{ij}\sigma_{ij}$, the trace $\operatorname{tr}\boldsymbol{\tau} := \sum_{i=1}^{n} \tau_{ii}$ and the deviatoric tensor $\boldsymbol{\tau}^{\mathrm{D}} := \boldsymbol{\tau} - (1/n)(\operatorname{tr}\boldsymbol{\tau})\mathbf{I}$, where $\mathbf{I}$ stands for the identity matrix of $\mathbb{R}^{n\times n}$.

Let Ω be a generic Lipschitz bounded domain of $\mathbb{R}^n$. For $s \geqslant 0$, $\|\cdot\|_{s,\Omega}$ stands indistinctly for the norm of the Hilbertian Sobolev spaces $H^s(\Omega)$, $H^s(\Omega)^n$ or $H^s(\Omega)^{n\times n}$, with the convention

$H^0(\Omega) := L^2(\Omega)$. We also define for $s \geqslant 0$ the Hilbert space $H^s(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in H^s(\Omega)^{n\times n}: \ \mathbf{div}\,\boldsymbol{\tau} \in H^s(\Omega)^n\}$, whose norm is given by $\|\boldsymbol{\tau}\|^2_{H^s(\mathbf{div};\Omega)} := \|\boldsymbol{\tau}\|^2_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{s,\Omega}$ and define $H(\mathbf{div};\Omega) := H^0(\mathbf{div};\Omega)$.

Finally, we employ $\mathbf{0}$ to denote a generic null vector or tensor and C to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The spectral problem

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a bounded and connected Lipschitz domain. We assume that its boundary $\partial\Omega$ admits a disjoint partition $\partial\Omega = \Gamma \cup \Sigma$ and denote by $\boldsymbol{n}$ the outward unit vector normal to $\partial\Omega$.

The Stokes eigenvalue problem is formulated as follows (see Lovadina *et al.*, 2009): find nontrivial $(\lambda, \boldsymbol{u}, p)$ such that

$$\begin{cases} -\,\mathbf{div}(\nabla \boldsymbol{u}) + \nabla p = \lambda \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on } \Gamma, \\ (\nabla \boldsymbol{u} - p\mathbf{I})\boldsymbol{n} = \mathbf{0} & \text{on } \Sigma. \end{cases}$$

Our aim is to employ a dual–mixed approach to derive a variational formulation of this problem. To this end, we introduce the pseudostress tensor (see Cai *et al.*, 2010)

$$\boldsymbol{\sigma} := \nabla \boldsymbol{u} - p\mathbf{I},$$

and reformulate the problem above in terms of this variable as follows: find nontrivial $(\lambda, \boldsymbol{\sigma}, \boldsymbol{u})$ such that

$$\begin{cases} -\,\mathbf{div}\,\boldsymbol{\sigma} = \lambda \boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{\sigma}^{\mathrm{D}} - \nabla \boldsymbol{u} = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on } \Gamma, \\ \boldsymbol{\sigma}\boldsymbol{n} = \mathbf{0} & \text{on } \Sigma. \end{cases} \tag{2.1}$$

We point out that the pressure p has disappeared from the formulation but can be easily recovered since

$$p = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}).$$

We note that the vector space

$$\mathcal{W} := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega): \ \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \ \text{ on } \Sigma\}$$

endowed with the $H(\mathbf{div};\Omega)$ inner product is a Hilbert space. Testing the second equation of (2.1) with $\boldsymbol{\tau} \in \mathcal{W}$ and integrating by parts yield

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{D}} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{u} \cdot \mathbf{div}\,\boldsymbol{\tau} = 0.$$

Next, we eliminate $\boldsymbol{u}$ from the last identity by using the first equation of (2.1) to obtain

$$\int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} = \lambda \int_\Omega \boldsymbol{\sigma}^{\mathrm{D}} : \boldsymbol{\tau}^{\mathrm{D}} \quad \forall \boldsymbol{\tau} \in \mathcal{W}.$$

Consequently, the pseudostress Stokes eigenvalue variational formulation reads as follows.

Problem 2.1. Find $\lambda \in \mathbb{R}$ and $\mathbf{0} \neq \boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$ such that

$$\int_{\Omega} \operatorname{\mathbf{div}} \boldsymbol{\sigma} \cdot \operatorname{\mathbf{div}} \boldsymbol{\tau} = \lambda \int_{\Omega} \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

It is convenient to use a shift argument to rewrite this eigenvalue problem. Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be the bounded bilinear forms defined for any $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$ by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_{\Omega} \operatorname{\mathbf{div}} \boldsymbol{\sigma} \cdot \operatorname{\mathbf{div}} \boldsymbol{\tau} + \int_{\Omega} \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}},$$
$$b(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_{\Omega} \boldsymbol{\sigma}^{\mathtt{D}} : \boldsymbol{\tau}^{\mathtt{D}}.$$

Then, Problem 2.1 can be written in the following equivalent form.

Problem 2.2. Find $\lambda \in \mathbb{R}$ and $\mathbf{0} \neq \boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$ such that

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\lambda + 1) b(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

Now we introduce the solution operator

$$\boldsymbol{T} : \boldsymbol{\mathcal{W}} \longrightarrow \boldsymbol{\mathcal{W}},$$
$$\boldsymbol{f} \longmapsto \boldsymbol{T}\boldsymbol{f} := \boldsymbol{\sigma}^*,$$

where $\boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{W}}$ is the solution of the source problem

$$a(\boldsymbol{\sigma}^*, \boldsymbol{\tau}) = b(\boldsymbol{f}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}. \tag{2.2}$$

The following lemma, whose proof is essentially identical to that of Meddahi *et al.* (2013, Lemma 2.1), allows us to establish the well-posedness of problem (2.2).

LEMMA 2.3 There exists a constant $\alpha > 0$, depending only on Ω, such that

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \geqslant \alpha \|\boldsymbol{\tau}\|^2_{H(\operatorname{\mathbf{div}};\Omega)} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}.$$

We deduce from this lemma that the linear operator $\boldsymbol{T}$ is well defined and bounded. As will be shown below (cf. Lemma 3.5(ii)), $\mu = 0$ is an eigenvalue of $\boldsymbol{T}$. All the remaining eigenvalues of this operator are related with those of Problem 2.1. In fact, note that $(\lambda, \boldsymbol{\sigma}) \in \mathbb{R} \times \boldsymbol{\mathcal{W}}$ solves Problem 2.1 if and only if $(1/(1+\lambda), \boldsymbol{\sigma})$ is an eigenpair of $\boldsymbol{T}$ with a nonvanishing eigenvalue, i.e., if and only if

$$\boldsymbol{T}\boldsymbol{\sigma} = \mu \boldsymbol{\sigma} \quad \text{with } \mu := \frac{1}{1+\lambda} \neq 0 \text{ and } \boldsymbol{\sigma} \neq \mathbf{0}.$$

Moreover, it is easy to check that $\boldsymbol{T}$ is self-adjoint with respect to the inner product $a(\cdot,\cdot)$ in $\boldsymbol{\mathcal{W}}$. Indeed, given $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{\mathcal{W}}$, because of the definition of $\boldsymbol{T}$ and the symmetry of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, there holds

$$a(\boldsymbol{T}\boldsymbol{f}, \boldsymbol{g}) = b(\boldsymbol{f}, \boldsymbol{g}) = b(\boldsymbol{g}, \boldsymbol{f}) = a(\boldsymbol{T}\boldsymbol{g}, \boldsymbol{f}) = a(\boldsymbol{f}, \boldsymbol{T}\boldsymbol{g}).$$

3. Spectral characterization

Our next goal is to describe the spectrum $\mathrm{sp}(\boldsymbol{T})$ of the solution operator. To this end, we define

$$\mathcal{K} := \{\boldsymbol{\tau} \in \mathcal{W} : \ \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \ \text{ in } \Omega\}.$$

It is straightforward to check that $\boldsymbol{T}|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ reduces to the identity. Thus, $\mu = 1$ is an eigenvalue of $\boldsymbol{T}$ and, from its definition, $\boldsymbol{\sigma}$ is an associated eigenfunction if and only if

$$\int_{\Omega} \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} = 0 \quad \forall \boldsymbol{\tau} \in \mathcal{W}.$$

Consequently we have proved the following result.

LEMMA 3.1 The operator $\boldsymbol{T}$ admits the eigenvalue $\mu = 1$ and its associated eigenspace is $\mathcal{K}$.

Let us introduce now the auxiliary operator

$$\begin{aligned} \boldsymbol{P} : \mathcal{W} &\longrightarrow \mathcal{W}, \\ \boldsymbol{\sigma} &\longmapsto \boldsymbol{P}\boldsymbol{\sigma} := \tilde{\boldsymbol{\sigma}}, \end{aligned}$$

where $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{u}}) \in \mathcal{W} \times L^2(\Omega)^n$ is the solution of the following mixed problem:

$$\int_{\Omega} \tilde{\boldsymbol{\sigma}}^{\mathrm{D}} : \boldsymbol{\tau}^{\mathrm{D}} + \int_{\Omega} \tilde{\boldsymbol{u}} \cdot \mathbf{div}\,\boldsymbol{\tau} = 0 \quad \forall \boldsymbol{\tau} \in \mathcal{W}, \tag{3.1}$$

$$\int_{\Omega} \boldsymbol{v} \cdot \mathbf{div}\,\tilde{\boldsymbol{\sigma}} = \int_{\Omega} \boldsymbol{v} \cdot \mathbf{div}\,\boldsymbol{\sigma} \quad \forall \boldsymbol{v} \in L^2(\Omega)^n. \tag{3.2}$$

The Babuška–Brezzi theory shows that this problem is well posed. In fact, it is well known that the inf–sup condition

$$\sup_{\boldsymbol{\tau} \in \mathcal{W}} \frac{\int_{\Omega} \boldsymbol{v} \cdot \mathbf{div}\,\boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geqslant \beta \|\boldsymbol{v}\|_{0,\Omega} \quad \forall \boldsymbol{v} \in L^2(\Omega)^n$$

holds true and Lemma 2.3 guarantees that the bilinear form $\int_{\Omega} \boldsymbol{\sigma}^{\mathrm{D}} : \boldsymbol{\tau}^{\mathrm{D}}$ is elliptic on the kernel $\{\boldsymbol{\tau} \in \mathcal{W} : \ \int_{\Omega} \boldsymbol{v} \cdot \mathbf{div}\,\boldsymbol{\tau} = 0 \ \forall \boldsymbol{v} \in L^2(\Omega)^n\} = \mathcal{K}$. Therefore, the linear operator $\boldsymbol{P}$ is well defined and bounded. Problem (3.1–3.2) is none other than the dual–mixed formulation of the following Stokes problem with external body force $-\,\mathbf{div}\,\boldsymbol{\sigma}$:

$$-\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}} = -\,\mathbf{div}\,\boldsymbol{\sigma} \qquad \text{in } \Omega, \tag{3.3}$$

$$\tilde{\boldsymbol{\sigma}}^{\mathrm{D}} - \nabla \tilde{\boldsymbol{u}} = \mathbf{0} \qquad \text{in } \Omega, \tag{3.4}$$

$$\tilde{\boldsymbol{u}} = \mathbf{0} \qquad \text{on } \Gamma, \tag{3.5}$$

$$\tilde{\boldsymbol{\sigma}}\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Sigma. \tag{3.6}$$

In fact, it is straightforward to check that $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{u}}) \in H(\mathbf{div};\Omega) \times H^1(\Omega)^n$ satisfies these equations if and only if $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{u}}) \in \mathcal{W} \times L^2(\Omega)^n$ is the solution to (3.1–3.2).

Owing to the regularity result for the classical Stokes problem (see, for instance, Girault & Raviart, 1986; Fabes *et al.*, 1988; Savaré, 1998), we know that the solution $\tilde{\boldsymbol{u}}$ to (3.3–3.6) belongs to $H^{1+s}(\Omega)^n$

for some $s \in (0,1]$ depending only on the geometry of Ω and

$$\|\tilde{\boldsymbol{u}}\|_{1+s,\Omega} \leqslant C\|\,\mathbf{div}\,\boldsymbol{\sigma}\,\|_{0,\Omega},$$

with $C > 0$ independent of $\boldsymbol{\sigma}$. From now on, $s \in (0,1]$ denotes a constant such that this inequality holds true. As a consequence of this regularity result, we can state the following lemma.

LEMMA 3.2 There exists $C > 0$ such that, for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$, if $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{u}}) \in \boldsymbol{\mathcal{W}} \times L^2(\Omega)^n$ is the solution to equations (3.1–3.2), then

$$\|\tilde{\boldsymbol{\sigma}}\|_{s,\Omega} + \|\tilde{\boldsymbol{u}}\|_{1+s,\Omega} \leqslant C\|\,\mathbf{div}\,\boldsymbol{\sigma}\,\|_{0,\Omega}.$$

Consequently, $\boldsymbol{P}(\boldsymbol{\mathcal{W}}) \subset H^s(\Omega)^{n\times n}$.

It is easy to check that the operator $\boldsymbol{P}$ is idempotent and that its kernel is given by $\boldsymbol{\mathcal{K}}$. Therefore, since $\boldsymbol{P}$ is a projector, the direct sum $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{K}} \oplus \boldsymbol{P}(\boldsymbol{\mathcal{W}})$ holds true. Moreover, it is also easy to check that $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{P}(\boldsymbol{\mathcal{W}})$ are orthogonal with respect to the inner product $a(\cdot,\cdot)$ of $\boldsymbol{\mathcal{W}}$. Therefore, the following result is a well-known consequence of the fact that $\boldsymbol{T}$ is self-adjoint with respect to the same inner product.

LEMMA 3.3 The subspace $\boldsymbol{P}(\boldsymbol{\mathcal{W}})$ is invariant for $\boldsymbol{T}$.

The properties of $\boldsymbol{T}$, as an operator from $\boldsymbol{P}(\boldsymbol{\mathcal{W}})$ into itself, are established in the following result.

PROPOSITION 3.4 The self-adjoint operator $\boldsymbol{T}$ satisfies

$$\boldsymbol{T}(\boldsymbol{P}(\boldsymbol{\mathcal{W}})) \subset \{\boldsymbol{\tau} \in H^s(\Omega)^{n\times n} :\ \mathbf{div}\,\boldsymbol{\tau} \in H^1(\Omega)^n\}, \tag{3.7}$$

and there exists $C > 0$ such that, for all $\boldsymbol{f} \in \boldsymbol{P}(\boldsymbol{\mathcal{W}})$, if $\boldsymbol{\sigma}^* = \boldsymbol{T}\boldsymbol{f}$, then

$$\|\boldsymbol{\sigma}^*\|_{s,\Omega} + \|\,\mathbf{div}\,\boldsymbol{\sigma}^*\|_{1,\Omega} \leqslant C\|\boldsymbol{f}\|_{H(\mathbf{div};\Omega)}. \tag{3.8}$$

Consequently, the operator $\boldsymbol{T}|_{\boldsymbol{P}(\boldsymbol{\mathcal{W}})} : \boldsymbol{P}(\boldsymbol{\mathcal{W}}) \to \boldsymbol{P}(\boldsymbol{\mathcal{W}})$ is compact.

Proof. According to Lemma 3.3, $\boldsymbol{T}|_{\boldsymbol{P}(\boldsymbol{\mathcal{W}})} : \boldsymbol{P}(\boldsymbol{\mathcal{W}}) \to \boldsymbol{P}(\boldsymbol{\mathcal{W}})$ is correctly defined. Let $\boldsymbol{f} \in \boldsymbol{P}(\boldsymbol{\mathcal{W}})$ and $\boldsymbol{\sigma}^* = \boldsymbol{T}\boldsymbol{f}$. Testing (2.2) with $\boldsymbol{\tau} \in \mathcal{D}(\Omega)^{n\times n} \subset \boldsymbol{\mathcal{W}}$ yields

$$\boldsymbol{\sigma}^{*\mathrm{D}} - \nabla(\mathbf{div}\,\boldsymbol{\sigma}^*) = \boldsymbol{f}^{\mathrm{D}},$$

which proves that $\mathbf{div}\,\boldsymbol{\sigma}^* \in H^1(\Omega)^n$.

On the other hand, from Lemmas 3.3 and 3.2, $\boldsymbol{\sigma}^* \in \boldsymbol{T}(\boldsymbol{P}(\boldsymbol{\mathcal{W}})) \subset \boldsymbol{P}(\boldsymbol{\mathcal{W}}) \subset H^s(\Omega)^{n\times n}$, so that (3.7) holds true and the estimate (3.8) follows from Lemma 3.2. Finally, the compactness of the operator is a consequence of the compact embedding $\{\boldsymbol{\sigma}^* \in H^s(\Omega)^{n\times n} :\ \mathbf{div}\,\boldsymbol{\sigma}^* \in H^1(\Omega)^n\} \cap \boldsymbol{\mathcal{W}} \hookrightarrow \boldsymbol{\mathcal{W}}$. □

We are now in a position to provide a spectral characterization of $\boldsymbol{T}$.

THEOREM 3.5 The spectrum of $\boldsymbol{T}$ decomposes as follows: $\mathrm{sp}(\boldsymbol{T}) = \{0,1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, where

(i) $\mu = 1$ is an infinite-multiplicity eigenvalue of $\boldsymbol{T}$ and its associated eigenspace is $\boldsymbol{\mathcal{K}}$;

(ii) $\mu = 0$ is an eigenvalue of $\boldsymbol{T}$ and its associated eigenspace is

$$\boldsymbol{\mathcal{Z}} := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}} :\ \boldsymbol{\tau}^{\mathrm{D}} = \boldsymbol{0}\} = \{q\mathbf{I} :\ q \in H^1(\Omega) \text{ and } q = 0 \text{ on } \Sigma\};$$

(iii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0,1)$ is a sequence of nondefective finite-multiplicity eigenvalues of $\boldsymbol{T}$ which converge to 0 and the corresponding eigenspaces lie in $\boldsymbol{P}(\mathcal{W})$.

Proof. The decomposition of $\mathrm{sp}(\boldsymbol{T})$ follows immediately from the classical spectral characterization of compact operators and the facts that $\mathcal{W} = \mathcal{K} \oplus \boldsymbol{P}(\mathcal{W})$, $\boldsymbol{T}|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ reduces to the identity and $\boldsymbol{T}|_{\boldsymbol{P}(\mathcal{W})} : \boldsymbol{P}(\mathcal{W}) \to \boldsymbol{P}(\mathcal{W})$ is compact (cf. Proposition 3.4).

Property (i) was established in Lemma 3.1.

On the other hand, it is easy to check that $\mathcal{Z}$ is the eigenspace of $\boldsymbol{T}$ associated with $\mu = 0$. Thus, property (ii) follows by noting that $\boldsymbol{\tau}^{\mathrm{D}} = \boldsymbol{0}$ if and only if $\boldsymbol{\tau} = q\mathbf{I}$, with $q = (1/n)\mathrm{tr}\,\boldsymbol{\tau} \in L^2(\Omega)$, $\nabla q = \mathbf{div}\,\boldsymbol{\tau} \in L^2(\Omega)^n$ and $q\boldsymbol{n} = \boldsymbol{\tau}\boldsymbol{n}$ on Σ.

Finally, property (iii) follows from Proposition 3.4, the fact that $\boldsymbol{T}$ is self-adjoint and the spectral characterization of self-adjoint compact operators. □

As an immediate consequence of Proposition 3.4, we have the following additional regularity result for the eigenfunctions of $\boldsymbol{T}$ associated to eigenvalues $\mu \in (0,1)$.

COROLLARY 3.6 Let $\boldsymbol{\sigma} \in \mathcal{W}$ be an eigenfunction of $\boldsymbol{T}$ associated with an eigenvalue $\mu \in (0,1)$. Then, $\boldsymbol{\sigma} \in H^s(\Omega)^{n\times n}$, $\mathbf{div}\,\boldsymbol{\sigma} \in H^1(\Omega)^n$ and

$$\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\,\mathbf{div}\,\boldsymbol{\sigma}\|_{1,\Omega} \leqslant C\|\boldsymbol{\sigma}\|_{H(\mathbf{div};\Omega)},$$

with $C > 0$ depending on the eigenvalue.

4. The discrete problem

Let $\{\mathcal{T}_h(\Omega)\}_{h>0}$ be a shape-regular family of triangulations with mesh size h of the polyhedral (polygonal) domain Ω by tetrahedra (triangles) T. In what follows, given an integer $k \geqslant 0$ and a subset S of $\mathbb{R}^n$, $\mathcal{P}_k(S)$ denotes the space of polynomials defined in S of total degree less than or equal to k.

For the discrete version of $\mathcal{W}$ we use standard $H(\mathbf{div};\Omega)$-conforming finite elements; in particular, we choose the classical BDM elements (see Brezzi & Fortin, 1991), namely

$$\mathcal{W}_h := \{\boldsymbol{\tau}_h \in \mathcal{W} :\ \boldsymbol{\tau}_h|_T \in \mathcal{P}_1(T)^{n\times n}\ \forall T \in \mathcal{T}_h(\Omega)\}.$$

In addition, for the analysis below, we will also need the space

$$\mathcal{U}_h := \{\boldsymbol{v}_h \in L^2(\Omega)^n :\ \boldsymbol{v}_h|_T \in \mathcal{P}_0(T)^n\ \forall T \in \mathcal{T}_h(\Omega)\}.$$

Let us recall some well-known approximation properties of the finite element spaces introduced above. Given $s \in (0,1]$, we denote by $\boldsymbol{\Pi}_h : H^s(\Omega)^{n\times n} \cap \mathcal{W} \to \mathcal{W}_h$ the usual BDM interpolation operator (see Brezzi & Fortin, 1991) which, for sufficiently smooth $\boldsymbol{\tau} \in \mathcal{W}$, is characterized by the conditions

$$\int_F (\boldsymbol{\Pi}_h\boldsymbol{\tau})\boldsymbol{n}_F \cdot \boldsymbol{q} = \int_F \boldsymbol{\tau}\boldsymbol{n}_F \cdot \boldsymbol{q} \quad \forall \boldsymbol{q} \in \mathcal{P}_1(F)^n$$

for all faces (edges) F of $T \in \mathcal{T}_h(\Omega)$, where $\boldsymbol{n}_F$ is a unit vector normal to the face (edge) F. The following commuting diagram property holds true (see Brezzi & Fortin, 1991):

$$\mathbf{div}(\boldsymbol{\Pi}_h\boldsymbol{\tau}) = \boldsymbol{L}_h(\mathbf{div}\,\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in H^s(\Omega)^{n\times n} \cap H(\mathbf{div};\Omega), \tag{4.1}$$

where $\boldsymbol{L}_h : L^2(\Omega)^n \to \boldsymbol{\mathcal{U}}_h$ is the $L^2(\Omega)^n$-orthogonal projector. In addition, by repeating the arguments from the proof of Hiptmair (2002, Theorem 3.16), it is easy to show that there exists $C > 0$, independent of h, such that

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{0,\Omega} \leqslant Ch^s(\|\boldsymbol{\tau}\|_{s,\Omega} + \|\,\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}) \quad \forall \boldsymbol{\tau} \in H^s(\Omega)^{n\times n} \cap H(\mathbf{div};\Omega). \tag{4.2}$$

Moreover, for any $s \in (0, 1]$, there holds

$$\|\boldsymbol{v} - \boldsymbol{L}_h\boldsymbol{v}\|_{0,\Omega} \leqslant Ch^s\|\boldsymbol{v}\|_{s,\Omega} \quad \forall \boldsymbol{v} \in H^s(\Omega)^n, \tag{4.3}$$

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s\|\boldsymbol{\tau}\|_{H^s(\mathbf{div};\Omega)} \quad \forall \boldsymbol{\tau} \in H^s(\mathbf{div};\Omega) \cap \boldsymbol{\mathcal{W}}. \tag{4.4}$$

Actually, (4.4) is a straightforward consequence of (4.1–4.3).

Now we introduce the discrete counterpart of Problem 2.1.

Problem 4.1. Find $\lambda_h \in \mathbb{R}$ and $\mathbf{0} \neq \boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$ such that

$$\int_\Omega \mathbf{div}\,\boldsymbol{\sigma}_h \cdot \mathbf{div}\,\boldsymbol{\tau}_h = \lambda_h \int_\Omega \boldsymbol{\sigma}_h^{\mathrm{D}} : \boldsymbol{\tau}_h^{\mathrm{D}} \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h.$$

As in the continuous case, we resort to a shift argument to write this problem in the following equivalent form, with the bilinear forms a and b as defined in Section 2.

Problem 4.2. Find $\lambda_h \in \mathbb{R}$ and $\mathbf{0} \neq \boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$ such that

$$a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = (\lambda_h + 1)b(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h.$$

Now we are in a position to define the discrete version of the operator $\boldsymbol{T}$:

$$\tilde{\boldsymbol{T}}_h : \boldsymbol{\mathcal{W}} \longrightarrow \boldsymbol{\mathcal{W}},$$
$$\boldsymbol{f} \longmapsto \tilde{\boldsymbol{T}}_h\boldsymbol{f} := \boldsymbol{\sigma}_h^*,$$

where $\boldsymbol{\sigma}_h^* \in \boldsymbol{\mathcal{W}}_h$ is the solution of the discrete source problem

$$a(\boldsymbol{\sigma}_h^*, \boldsymbol{\tau}_h) = b(\boldsymbol{f}, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h.$$

Because of Lemma 2.3 and the Lax–Milgram theorem, $\tilde{\boldsymbol{T}}_h$ is well defined and uniformly bounded with respect to h. Moreover, the Cea lemma ensures the existence of a constant $C > 0$, independent of h, such that, for all $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$,

$$\|\boldsymbol{T}\boldsymbol{\sigma} - \tilde{\boldsymbol{T}}_h\boldsymbol{\sigma}\|_{H(\mathbf{div};\Omega)} \leqslant C \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{T}\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}. \tag{4.5}$$

Note that, since $\tilde{\boldsymbol{T}}_h(\boldsymbol{\mathcal{W}}) \subset \boldsymbol{\mathcal{W}}_h$, we are allowed to introduce the operator

$$\boldsymbol{T}_h := \tilde{\boldsymbol{T}}_h|_{\boldsymbol{\mathcal{W}}_h} : \boldsymbol{\mathcal{W}}_h \longrightarrow \boldsymbol{\mathcal{W}}_h.$$

It is well known that $\mathrm{sp}(\tilde{\boldsymbol{T}}_h) = \mathrm{sp}(\boldsymbol{T}_h) \cup \{0\}$ (see, for instance, Bermúdez *et al.*, 1995, Lemma 4.1).

Once more, as in the continuous case, $(\lambda_h, \boldsymbol{\sigma}_h) \in \mathbb{R} \times \boldsymbol{\mathcal{W}}_h$ solves Problem 4.1 if and only if $(1/(1+\lambda_h), \boldsymbol{\sigma}_h)$ is an eigenpair of $\boldsymbol{T}_h$ with a nonvanishing eigenvalue, i.e., if and only if

$$\boldsymbol{T}_h\boldsymbol{\sigma}_h = \mu_h\boldsymbol{\sigma}_h \quad \text{with } \mu_h := \frac{1}{1+\lambda_h} \neq 0 \text{ and } \boldsymbol{\sigma}_h \neq \boldsymbol{0}.$$

Moreover, it can immediately be checked that $\boldsymbol{T}_h$ is also self-adjoint with respect to $a(\cdot,\cdot)$.

To describe the spectrum of this operator, we will proceed as in the continuous case and decompose $\boldsymbol{\mathcal{W}}_h$ into a convenient direct sum. To this end, we define

$$\boldsymbol{\mathcal{K}}_h := \boldsymbol{\mathcal{K}} \cap \boldsymbol{\mathcal{W}}_h = \{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h : \ \mathbf{div}\,\boldsymbol{\tau}_h = \boldsymbol{0} \ \text{ in } \Omega\},$$

and note that, once more, $\boldsymbol{T}_h|_{\boldsymbol{\mathcal{K}}_h} : \boldsymbol{\mathcal{K}}_h \to \boldsymbol{\mathcal{K}}_h$ reduces to the identity. Moreover, we have the following discrete analogue to Lemma 3.1.

LEMMA 4.3 The operator $\boldsymbol{T}_h$ admits the eigenvalue $\mu_h = 1$ and $\boldsymbol{\mathcal{K}}_h$ is the associated eigenspace.

We define the discrete version of the auxiliary operator $\boldsymbol{P}$ by

$$\begin{aligned} \boldsymbol{P}_h : \boldsymbol{\mathcal{W}} &\longrightarrow \boldsymbol{\mathcal{W}}_h, \\ \boldsymbol{\sigma} &\longmapsto \boldsymbol{P}_h(\boldsymbol{\sigma}) := \tilde{\boldsymbol{\sigma}}_h, \end{aligned}$$

where $(\tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{u}}_h) \in \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{U}}_h$ is the solution of the following discrete mixed problem:

$$\int_\Omega \tilde{\boldsymbol{\sigma}}_h^{\mathrm{D}} : \boldsymbol{\tau}_h^{\mathrm{D}} + \int_\Omega \tilde{\boldsymbol{u}}_h \cdot \mathbf{div}\,\boldsymbol{\tau}_h = 0 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h, \tag{4.6}$$

$$\int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\tilde{\boldsymbol{\sigma}}_h = \int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\boldsymbol{\sigma} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h. \tag{4.7}$$

We point out that the following inf–sup condition is satisfied with a constant $\hat{\beta}$ independent of h (see Brezzi & Fortin, 1991):

$$\sup_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \frac{\int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}} \geqslant \hat{\beta}\|\boldsymbol{v}_h\|_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h.$$

Moreover, since $\mathbf{div}(\boldsymbol{\mathcal{W}}_h) \subset \boldsymbol{\mathcal{U}}_h$, the ellipticity of the bilinear form $\int_\Omega \boldsymbol{\sigma}^{\mathrm{D}} : \boldsymbol{\tau}^{\mathrm{D}}$ on the discrete kernel $\{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h : \ \int_\Omega \boldsymbol{v}_h \cdot \mathbf{div}\,\boldsymbol{\tau}_h = 0 \ \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h\} = \boldsymbol{\mathcal{K}}_h$ follows from its ellipticity on $\boldsymbol{\mathcal{K}} \supset \boldsymbol{\mathcal{K}}_h$. Hence, as a consequence of the Babuška–Brezzi theory, problem (4.6–4.7) is well posed. The linear operator $\boldsymbol{P}_h$ is then well defined and uniformly bounded with respect to h. Moreover, the following Cea estimate holds true:

$$\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{H(\mathbf{div};\Omega)} + \|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_h\|_{0,\Omega} \leqslant C\left[\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} + \inf_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{U}}_h} \|\tilde{\boldsymbol{u}} - \boldsymbol{v}_h\|_{0,\Omega}\right], \tag{4.8}$$

where $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{u}})$ and $(\tilde{\boldsymbol{\sigma}}_h, \tilde{\boldsymbol{u}}_h)$ are the solutions to (3.1–3.2) and (4.6–4.7), respectively.

The above estimate, combined with the approximation properties (4.3) and (4.4), leads to

$$\|\boldsymbol{P}\boldsymbol{\sigma} - \boldsymbol{P}_h\boldsymbol{\sigma}\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s[\|\tilde{\boldsymbol{\sigma}}\|_{H^s(\mathbf{div};\Omega)} + \|\tilde{\boldsymbol{u}}\|_{s,\Omega}],$$

whenever $\tilde{\boldsymbol{\sigma}} \in H^s(\Omega)^{n\times n}$, $\tilde{\boldsymbol{u}} \in H^s(\Omega)^n$ and $\mathbf{div}\,\tilde{\boldsymbol{\sigma}} \in H^s(\Omega)^n$. We already know from Lemma 3.2 that $\tilde{\boldsymbol{\sigma}} \in H^s(\Omega)^{n\times n}$ and $\tilde{\boldsymbol{u}} \in H^s(\Omega)^n$. However, $\mathbf{div}\,\tilde{\boldsymbol{\sigma}}$ cannot be in $H^s(\Omega)^n$ for an arbitrary $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{W}}$. Indeed, from (3.3), $\mathbf{div}\,\tilde{\boldsymbol{\sigma}} = \mathbf{div}\,\boldsymbol{\sigma}$, which is in general only in $L^2(\Omega)^n$. In spite of this fact, an $\mathcal{O}(h^s)$ convergence for $\|\boldsymbol{P}\boldsymbol{\sigma}_h - \boldsymbol{P}_h\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}$ can be proved for $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$. Fortunately, this is enough to develop the spectral approximation theory of our problem.

Lemma 4.4 There exists $C > 0$ such that, for all $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$,

$$\|\boldsymbol{P}\boldsymbol{\sigma}_h - \boldsymbol{P}_h\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s\|\,\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega}.$$

Proof. For $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$, let $\tilde{\boldsymbol{\sigma}} = \boldsymbol{P}\boldsymbol{\sigma}_h$ and $\tilde{\boldsymbol{\sigma}}_h = \boldsymbol{P}_h\boldsymbol{\sigma}_h$. By virtue of (4.8), (4.3) and Lemma 3.2, we have

$$\|\boldsymbol{P}\boldsymbol{\sigma}_h - \boldsymbol{P}_h\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)} \leqslant C\left[\inf_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{W}}_h}\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} + h\|\,\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega}\right].$$

Now, since $\tilde{\boldsymbol{\sigma}} \in H^s(\Omega)^{n\times n} \cap \boldsymbol{\mathcal{W}}$ (cf. Lemma 3.2), $\boldsymbol{\Pi}_h\tilde{\boldsymbol{\sigma}} \in \boldsymbol{\mathcal{W}}_h$ is well defined and, according to (4.2),

$$\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_h\tilde{\boldsymbol{\sigma}}\|_{0,\Omega} \leqslant Ch^s(\|\tilde{\boldsymbol{\sigma}}\|_{s,\Omega} + \|\,\mathbf{div}\,\tilde{\boldsymbol{\sigma}}\|_{0,\Omega}).$$

On the other hand, from (3.3), $\mathbf{div}\,\tilde{\boldsymbol{\sigma}} = \mathbf{div}\,\boldsymbol{\sigma}_h$ in Ω. Therefore, because of (4.1),

$$\mathbf{div}(\boldsymbol{\Pi}_h\tilde{\boldsymbol{\sigma}}) = \boldsymbol{L}_h(\mathbf{div}\,\tilde{\boldsymbol{\sigma}}) = \boldsymbol{L}_h(\mathbf{div}\,\boldsymbol{\sigma}_h) = \mathbf{div}\,\boldsymbol{\sigma}_h = \mathbf{div}\,\tilde{\boldsymbol{\sigma}},$$

which proves that

$$\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_h\tilde{\boldsymbol{\sigma}}\|_{H(\mathbf{div};\Omega)} = \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}_h\tilde{\boldsymbol{\sigma}}\|_{0,\Omega}.$$

Thus, the result follows from all this and the fact that $\|\tilde{\boldsymbol{\sigma}}\|_{s,\Omega} \leqslant C\|\,\mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega}$ (cf. Lemma 3.2). □

It is easy to check that $\boldsymbol{P}_h|_{\boldsymbol{\mathcal{W}}_h}$ is idempotent and that its kernel is $\boldsymbol{\mathcal{K}}_h$, which allows us to write $\boldsymbol{\mathcal{W}}_h$ as a direct sum of $\boldsymbol{\mathcal{K}}_h$ and $\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)$, i.e., $\boldsymbol{\mathcal{W}}_h = \boldsymbol{\mathcal{K}}_h \oplus \boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)$. Actually, it is easy to show that this decomposition is orthogonal with respect to $a(\cdot,\cdot)$. Consequently, since $\boldsymbol{T}_h$ is self-adjoint with respect to this bilinear form, the subspace $\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)$ is invariant for $\boldsymbol{T}_h$ and the following spectral characterization holds true.

Theorem 4.5 The spectrum of $\boldsymbol{T}_h$ consists of $M := \dim(\boldsymbol{\mathcal{W}}_h)$ nondefective eigenvalues, repeated according to their respective multiplicities. The spectrum decomposes as follows: $\mathrm{sp}(\boldsymbol{T}_h) = \{0,1\} \cup \{\mu_{hk}\}_{k=1}^K$. Moreover,

(i) the eigenspace associated with $\mu_h = 1$ is $\boldsymbol{\mathcal{K}}_h$;

(ii) the eigenspace associated with $\mu_h = 0$ is

$$\boldsymbol{\mathcal{Z}}_h := \{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h :\ \boldsymbol{\tau}_h^{\mathrm{D}} = \mathbf{0}\};$$

(iii) the eigenspaces associated with $\mu_{hk} \in (0,1)$, $k = 1,\ldots,K := M - \dim(\boldsymbol{\mathcal{K}}_h) - \dim(\boldsymbol{\mathcal{Z}}_h)$ lie on $\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)$.

Proof. Since $\boldsymbol{\mathcal{W}}_h = \boldsymbol{\mathcal{K}}_h \oplus \boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)$, $\boldsymbol{T}_h|_{\boldsymbol{\mathcal{K}}_h} : \boldsymbol{\mathcal{K}}_h \to \boldsymbol{\mathcal{K}}_h$ is the identity and $\boldsymbol{T}_h(\boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)) \subset \boldsymbol{P}_h(\boldsymbol{\mathcal{W}}_h)$, the theorem follows from Lemma 4.3 and the definition of $\boldsymbol{T}_h$. □

Remark 4.6 The eigenspace $\boldsymbol{\mathcal{Z}}_h$ of $\boldsymbol{T}_h$ corresponding to $\mu_h = 0$ can be characterized as follows:

$$\boldsymbol{\mathcal{Z}}_h = \{q_h \mathbf{I} : q_h \in \mathcal{V}_h\},$$

where

$$\mathcal{V}_h := \{q_h \in H^1(\Omega) : q_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h(\Omega) \text{ and } q_h = 0 \text{ on } \Sigma\}.$$

This follows from the same arguments used in the proof of Theorem 3.5(ii) and the fact that $q_h \mathbf{I} \in \boldsymbol{\mathcal{W}}_h$ for all $q_h \in \mathcal{V}_h$.

Note that the latter holds true for the particular elements we have chosen (BDM), but not necessarily for any family of $H(\mathbf{div};\Omega)$-conforming elements. For instance, it does not hold for the lowest-order RT elements. However, this is not a drawback for approximating the most relevant eigenvalues, which are those closest to $\mu = 1$ (and not those closest to $\mu = 0$). In fact, in practice, the most relevant eigenvalues λ of Problem 2.1 are the lowest positive ones, which correspond to the eigenvalues $\mu = 1/(1+\lambda)$ of $\boldsymbol{T}$ closest to $\mu = 1$. Instead, the eigenvalues of $\boldsymbol{T}$ closest to $\mu = 0$ correspond to the largest eigenvalues λ of Problem 2.1, which do not play any significant role in practice.

5. Spectral approximation

To prove that $\boldsymbol{T}_h$ provides a correct spectral approximation of $\boldsymbol{T}$, we will resort to the theory developed in Descloux *et al.* (1978a,b) for noncompact operators. To this end, we first introduce some notation. For any linear operator $\boldsymbol{S} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{W}}$ we define

$$\|\boldsymbol{S}\|_h := \sup_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \frac{\|\boldsymbol{S}\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}}{\|\boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}}.$$

Given $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$ and two closed subspaces $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ of $\boldsymbol{\mathcal{W}}$, we set

$$\delta(\boldsymbol{\tau}, \boldsymbol{\mathcal{Y}}) := \inf_{\boldsymbol{\eta} \in \boldsymbol{\mathcal{Y}}} \|\boldsymbol{\tau} - \boldsymbol{\eta}\|_{H(\mathbf{div};\Omega)}, \quad \delta(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}) := \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}} : \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} = 1} \delta(\boldsymbol{\tau}, \boldsymbol{\mathcal{Y}})$$

and

$$\hat{\delta}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}) := \max\{\delta(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}), \delta(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{X}})\},$$

the latter being the so-called gap between subspaces $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$.

The theory from Descloux *et al.* (1978a) guarantees a good approximation of the spectrum of $\boldsymbol{T}$ if the following two properties are satisfied:

(P1) $\|\boldsymbol{T} - \boldsymbol{T}_h\|_h \to 0$ as $h \to 0$;

(P2) for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{W}}$, $\quad \delta(\boldsymbol{\tau}, \boldsymbol{\mathcal{W}}_h) \to 0$ as $h \to 0$.

Property (P2) follows immediately from the approximation property of the finite element space (4.4) and the density of smooth functions in $\boldsymbol{\mathcal{W}}$. Property (P1) is a consequence of the following lemma.

LEMMA 5.1 There exists $C > 0$, independent of h, such that

$$\|\boldsymbol{T} - \boldsymbol{T}_h\|_h \leqslant Ch^s.$$

Proof. Given $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h$, we have

$$(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{\sigma}_h = (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}_h\boldsymbol{\sigma}_h + (\boldsymbol{T} - \boldsymbol{T}_h)(\mathbf{I} - \boldsymbol{P}_h)\boldsymbol{\sigma}_h = (\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}_h\boldsymbol{\sigma}_h,$$

where the last identity comes from the facts that $(\mathbf{I} - \boldsymbol{P}_h)$ is a projector onto $\mathcal{K}_h \subset \mathcal{K}$ and the restrictions to this subspace of both $\boldsymbol{T}$ and $\boldsymbol{T}_h$ reduce to the identity. Let us now consider the splitting

$$(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{P}_h\boldsymbol{\sigma}_h = \underbrace{(\boldsymbol{T} - \tilde{\boldsymbol{T}}_h)(\boldsymbol{P}_h - \boldsymbol{P})\boldsymbol{\sigma}_h}_{E_1} + \underbrace{(\boldsymbol{T} - \tilde{\boldsymbol{T}}_h)\boldsymbol{P}\boldsymbol{\sigma}_h}_{E_2}.$$

For the first term we use Lemma 4.4 to obtain the estimate

$$\|E_1\|_{H(\mathbf{div};\Omega)} \leqslant (\|\boldsymbol{T}\| + \|\tilde{\boldsymbol{T}}_h\|)\|(\boldsymbol{P}_h - \boldsymbol{P})\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s\|\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)},$$

whereas, by virtue of the Cea estimate (4.5), the second term is bounded as

$$\|E_2\|_{H(\mathbf{div};\Omega)} \leqslant C \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{T}\boldsymbol{P}\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)}.$$

Now, according to Proposition 3.4, if $\boldsymbol{\sigma}^* = \boldsymbol{T}\boldsymbol{P}\boldsymbol{\sigma}_h$, then $\boldsymbol{\sigma}^* \in H^s(\Omega)^{n\times n}$, $\mathbf{div}\,\boldsymbol{\sigma}^* \in H^1(\Omega)^n$ and

$$\|\boldsymbol{\sigma}^*\|_{s,\Omega} + \|\,\mathbf{div}\,\boldsymbol{\sigma}^*\|_{1,\Omega} \leqslant C\|\boldsymbol{P}\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)} \leqslant C\|\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}.$$

We deduce from the last two estimates and the approximation property (4.4) that

$$\|E_2\|_{H(\mathbf{div};\Omega)} \leqslant C \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h} \|\boldsymbol{\sigma}^* - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s\|\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}.$$

Summing up, we have shown that

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s\|\boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)} \quad \forall \boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{W}}_h,$$

with C independent of h, which allows us to conclude the proof. □

The following result is a consequence of property (P1); see Descloux *et al.* (1978a, Theorem 1).

THEOREM 5.2 Let $U \subset \mathbb{C}$ be an open set such that $\operatorname{sp}(\boldsymbol{T}) \subset U$. Then, there exists $h_0 > 0$ such that $\operatorname{sp}(\boldsymbol{T}_h) \subset U$ for all $h < h_0$.

This theorem means that our Galerkin scheme does not introduce spurious modes with eigenvalues interspersed among the positive eigenvalues of Problem 2.1. Indeed, assume that $\mu \in (0, 1)$ is an isolated eigenvalue of $\boldsymbol{T}$ with finite multiplicity m and that $\mathcal{C}$ is an open circle in the complex plane centred at μ with boundary γ, such that μ is the only eigenvalue of $\boldsymbol{T}$ lying in $\mathcal{C}$ and $\gamma \cap \operatorname{sp}(\boldsymbol{T}) = \emptyset$. Then, according to Descloux *et al.* (1978a, Section 2), for h small enough there exist exactly m eigenvalues $\mu_{h1}, \ldots, \mu_{hm}$ of $\boldsymbol{T}_h$ (repeated according to their respective multiplicities) which lie in $\mathcal{C}$. Moreover, these and only these eigenvalues of $\boldsymbol{T}_h$ converge to μ as $h \to 0$.

The next step in our analysis is to apply the results from Descloux *et al.* (1978b) to obtain error estimates. To this aim, we consider the eigenspace $\boldsymbol{\mathcal{E}}$ of $\boldsymbol{T}$ corresponding to μ and the invariant subspace $\boldsymbol{\mathcal{E}}_h$ of $\boldsymbol{T}_h$ spanned by the eigenspaces of $\boldsymbol{T}_h$ corresponding to $\mu_{h1}, \ldots, \mu_{hm}$. Since $\boldsymbol{T}$ and $\boldsymbol{T}_h$ are self-adjoint with respect to $a(\cdot,\cdot)$, we have the following results from Descloux *et al.* (1978b).

THEOREM 5.3 There exists a constant $C > 0$, independent of h, such that

$$\hat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \leqslant C\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h).$$

We recall that $\mu \in (0, 1)$ is an eigenvalue of $\boldsymbol{T}$ with multiplicity m if and only if $\lambda := (1/\mu) - 1$ is an eigenvalue of Problem 2.1 with the same multiplicity and that the corresponding eigenfunctions coincide. Analogously, μ_{hi}, $i = 1, \ldots, m$ are the eigenvalues of $\boldsymbol{T}_h$ (repeated according to their respective multiplicities) converging to μ as $h \to 0$ if and only if $\lambda_{hi} := (1/\mu_{hi}) - 1$ are the eigenvalues of Problem 4.1 converging to λ and the corresponding eigenfunctions also coincide. Thus, the theorem above provides an error estimate for the approximation of the eigenfunctions of Problem 2.1 by means of those of Problem 4.1. We have the following result that provides an error estimate for the eigenvalues.

THEOREM 5.4 There exist constants $C > 0$ and $h_0 > 0$ such that, for all $h < h_0$,

$$|\lambda - \lambda_{hi}| \leqslant C\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h)^2, \quad i = 1, \ldots, m.$$

Proof. This result was proved in Descloux *et al.* (1978b, Theorem 3) but, for the sake of completeness, we include here a simple proof adapted to our problem.

Let $\boldsymbol{\sigma}_{hi}$ be an eigenfunction of Problem 4.1 corresponding to λ_{hi}, normalized so that $\|\boldsymbol{\sigma}_{hi}\|_{H(\mathbf{div};\Omega)} = 1$. According to Theorem 5.3, $\delta(\boldsymbol{\sigma}_{hi}, \boldsymbol{\mathcal{E}}) \leqslant C\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h)$. It follows that there exists an eigenfunction $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{E}}$ of Problem 2.1 corresponding to λ such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hi}\|_{H(\mathbf{div};\Omega)} \leqslant C\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h). \tag{5.1}$$

Note that, in spite of the notation, $\boldsymbol{\sigma}$ may actually depend on h.

The key tool to prove the theorem is the following identity:

$$(\lambda_{hi} - \lambda)b(\boldsymbol{\sigma}_{hi}, \boldsymbol{\sigma}_{hi}) = a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hi}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_{hi}) - (\lambda + 1)b(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hi}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_{hi}). \tag{5.2}$$

This identity (which is similar to that from Babuška & Osborn, 1991, Lemma 9.1) follows from straightforward calculations after noting that, since $(\lambda, \boldsymbol{\sigma})$ and $(\lambda_{hi}, \boldsymbol{\sigma}_{hi})$ are solutions to Problems 2.1 and 4.1, respectively,

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\lambda + 1)b(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{W}},$$
$$a(\boldsymbol{\sigma}_{hi}, \boldsymbol{\tau}_h) = (\lambda_{hi} + 1)b(\boldsymbol{\sigma}_{hi}, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{W}}_h.$$

Lemma 2.3 and the fact that $\lambda_{hi} \to \lambda$ as $h \to 0$ yield that there exist $C > 0$ and $h_0 > 0$ such that, for all $h < h_0$,

$$b(\boldsymbol{\sigma}_{hi}, \boldsymbol{\sigma}_{hi}) = \frac{a(\boldsymbol{\sigma}_{hi}, \boldsymbol{\sigma}_{hi})}{\lambda_{hi} + 1} \geqslant \frac{\alpha \|\boldsymbol{\sigma}_{hi}\|^2_{H(\mathbf{div};\Omega)}}{\lambda_{hi} + 1} \geqslant C > 0.$$

Using this estimate to bound the left-hand side of (5.2) from below and the continuity of a and b and (5.1) to bound the right-hand side from above, we derive

$$|\lambda - \lambda_{hi}| \leqslant C\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h)^2, \quad i = 1, \ldots, m,$$

which allows us to conclude the proof. □

The two theorems above yield error estimates depending on $\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h)$. The order of convergence will then depend on the regularity of the eigenfunctions. In particular, using the additional smoothness

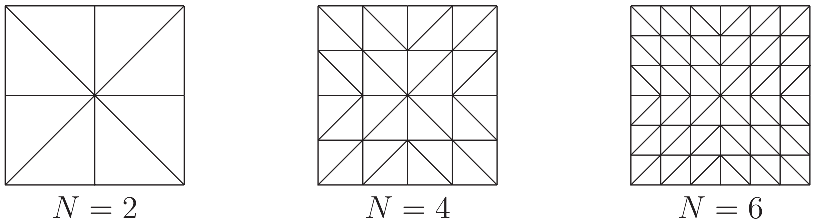

FIG. 1. Uniform meshes.

of the eigenfunctions that we have already proved, we can assert from (4.4) and Corollary 3.6 that if $\mu \in (0, 1)$, then

$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|_{H(\mathbf{div};\Omega)} \leqslant Ch^s \|\boldsymbol{\sigma}\|_{H^s(\mathbf{div};\Omega)} \leqslant Ch^s \|\boldsymbol{\sigma}\|_{H(\mathbf{div};\Omega)} \quad \forall \boldsymbol{\sigma} \in \boldsymbol{\mathcal{E}},$$

which allows us to conclude that, at least, $\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{W}}_h) \leqslant Ch^s$.

6. Numerical results

We report in this section the results of some numerical tests carried out with the method proposed in Section 4, which will allow us to check the theoretical results proved above as well as to assess its efficiency. The numerical method was implemented in a MATLAB code. Using standard basis of BDM elements, Problem 4.2 leads to a generalized matrix eigenvalue problem of the form

$$\mathbb{A}\hat{\boldsymbol{\sigma}} = (\lambda_h + 1)\mathbb{B}\hat{\boldsymbol{\sigma}},$$

where $\hat{\boldsymbol{\sigma}}$ is the vector of components of the eigenfunction in the chosen basis. Matrices $\mathbb{A}$ and $\mathbb{B}$ are both symmetric, with the former positive definite and the latter semidefinite. Therefore, it is possible to apply standard eigensolvers (like the one we have actually used: MATLAB command `eigs`) to the equivalent problem

$$\mathbb{B}\hat{\boldsymbol{\sigma}} = \frac{1}{\lambda_h + 1}\mathbb{A}\hat{\boldsymbol{\sigma}}.$$

Let us remark that the positive definiteness of $\mathbb{A}$ is the reason why we chose Problem 4.2 for the implementation instead of Problem 4.1. In fact, the latter would have led to a semidefinite matrix $\mathbb{A}$ and, hence, to a degenerate generalized matrix eigenvalue problem.

We used uniform meshes such as those shown in Fig. 1. The refinement parameter N used to label each mesh is the number of elements on each edge. In all the tables below the results are obtained by using four meshes with increasing levels of refinement. In each case, we list the lowest computed eigenvalues λ_{hi}. We also report in all tables a column labelled “Order” with the estimated order of convergence and another one labelled “Extrap.” with a more accurate extrapolated approximation of the eigenvalues obtained by means of a least-squares fitting.

6.1 *Test* 1

In this numerical test we took $\Omega := (-1, 1) \times (-1, 1)$ and considered a nonslip boundary condition $\boldsymbol{u} = \mathbf{0}$ on the whole of $\partial\Omega$. We compared our results with those obtained in Lovadina *et al.* (2009) with a velocity–pressure formulation of the Stokes system and a Galerkin method based on the MINI

TABLE 1 *Test* 1. *Lowest eigenvalues* λ_{hi} *of the Stokes problem computed with BDM elements*

	$N=10$	$N=20$	$N=30$	$N=40$	Order	Extrap.	Lovadina *et al.* (2009)
λ_{h1}	13.4657	13.1823	13.1290	13.1103	1.98	13.0860	13.086
$\lambda_{h2}=\lambda_{h3}$	24.2868	23.3472	23.1718	23.1103	1.99	23.0308	23.031
λ_{h4}	34.2444	32.6220	32.3075	32.1963	1.93	32.0443	32.053
λ_{h5}	41.4711	39.2828	38.8666	38.7201	1.96	38.5252	38.532
λ_{h6}	45.9681	42.8124	42.2263	42.0211	2.00	41.7588	41.759

TABLE 2 *Test* 1. *Lowest eigenvalues* λ_{hi} *of the Stokes problem computed with RT elements*

	$N=10$	$N=20$	$N=30$	$N=40$	Order	Extrap.	Lovadina *et al.* (2009)
λ_{h1}	13.0355	13.0691	13.0780	13.0815	1.45	13.0887	13.086
$\lambda_{h2}=\lambda_{h3}$	22.4794	22.8930	22.9692	22.9962	2.00	23.0306	23.031
λ_{h4}	31.6489	31.9330	31.9963	32.0201	1.70	32.0593	32.053
λ_{h5}	36.0607	37.9436	38.2719	38.3857	2.09	38.5207	38.532
λ_{h6}	41.4860	41.7093	41.7365	41.7456	2.63	41.7520	41.759

element. With this aim, we include in the last column of Table 1 the values obtained by extrapolating those reported in Lovadina *et al.* (2009) for the same problem.

It is clear that the eigenvalue approximation order of our method is quadratic and that the results obtained by the two methods agree perfectly well. We also point out that the symmetry of the meshes used allows the double multiplicity of the second eigenvalue to be preserved at the discrete level.

Let us remark that the method discussed in this paper is significantly more expensive than that from Lovadina *et al.* (2009) in terms of computational cost. This happens even though in the latter, which is based on the MINI element, it is not possible to eliminate the cubic bubbles by static condensation in the eigenvalue problem. In fact, since the degrees of freedom corresponding to these bubbles appear in both matrices ($\mathbb{A}$ and $\mathbb{B}$), such elimination would lead to a nonlinear eigenvalue problem which would be much harder to deal with. Because of this, the number of unknowns for the MINI element in this two-dimensional problem is $3N_{\mathrm{V}}+2N_{\mathrm{T}}\approx 7N_{\mathrm{V}}$, where N_{V} denotes the number of vertices and N_{T} the number of elements. In its turn, the number of unknowns for the method analysed in this paper is almost double: $4N_{\mathrm{E}}\approx 12N_{\mathrm{V}}$, where N_{E} denotes the number of edges.

However, as stated in Section 1, the analysis in this paper extends without any significant change to other $H(\mathbf{div};\Omega)$-conforming elements. In particular we have checked that all the results remain valid for the lowest-order RT element. The only difference in the analysis with respect to BDM elements is the lack of a correct approximation of the eigenspace $\mathcal{Z}_h$ pointed out in Remark 4.6. However, as explained in that remark, this does not affect the efficiency of the discretization based on RT elements to approximate the lowest eigenvalues λ of Problem 2.1, which are the most significant in practice.

To check this, we also solved the same problem on the same meshes with the lowest-order RT elements. We report the results in Table 2, which is an analogue of Table 1.

Excellent agreement with the results extrapolated from those obtained with the MINI element discretization from Lovadina *et al.* (2009) can also be clearly observed in Table 2. Let us remark that this happens even though the computational cost of the lowest-order RT discretization is half that of the BDM discretization. In fact, the number of unknowns for RT elements is $2N_{\mathrm{E}}\approx 6N_{\mathrm{V}}$, which is even smaller than that of the MINI element ($7N_{\mathrm{V}}$).

TABLE 3 *Test 2. Lowest buckling coefficients λ_{hi} computed with BDM elements*

	$N=10$	$N=20$	$N=30$	$N=40$	Order	Extrap.	Mora & Rodríguez (2009)
λ_{h1}	52.729	52.441	52.388	52.369	1.99	52.344	52.345
$\lambda_{h2}=\lambda_{h3}$	93.389	92.441	92.265	92.204	2.00	92.125	92.126
λ_{h4}	130.488	128.785	128.466	128.354	1.98	128.207	128.213
λ_{h5}	157.131	154.881	154.461	154.315	1.99	154.124	154.123
λ_{h6}	171.250	168.084	167.498	167.293	2.00	167.029	167.024

6.2 *Test* 2

In this numerical test we compared our method with a finite element scheme proposed and analysed in Mora & Rodríguez (2009) for solving the plate buckling problem. As already mentioned in Section 1, the Stokes eigenvalue problem with a nonslip boundary condition is equivalent to the following, which corresponds to the buckling problem for a uniformly compressed, clamped Kirchhoff plate (see Mercier *et al.*, 1981; Chen & Lin, 2006): find $\lambda \in \mathbb{R}$ and $0 \neq w \in H_0^2(\Omega)$ such that

$$\begin{cases} \Delta^2 w = -\lambda \Delta w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ \nabla w \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where λ is the buckling coefficient and w is the plate transverse displacement, which corresponds in the Stokes problem to a stream function of the velocity field: $\boldsymbol{u} = \mathbf{curl}\, w := (\partial_2 w, -\partial_1 w)$.

The computational domain is now the unit square $\Omega := (0,1) \times (0,1)$. In Table 3, we list the lowest buckling coefficients of problem (6.1) computed by solving Problem 4.2 with BDM elements. We also solved the buckling problem (6.1) by means of the method proposed in Mora & Rodríguez (2009). To allow for comparison, the corresponding extrapolated buckling coefficients are included in the last column of Table 3.

It can be clearly observed from Table 3 that our method computes the buckling coefficients with an optimal quadratic order of convergence and that the agreement with the method from Mora & Rodríguez (2009) is excellent.

Once more, the cost of the method based on BDM elements is much larger than that from Mora & Rodríguez (2009). The number of unknowns for BDM is, as above, $4N_{\mathrm{E}} \approx 12N_{\mathrm{V}}$, whereas that of Mora & Rodríguez (2009) is $4N_{\mathrm{V}}$ (therefore, even smaller than that of RT: $2N_{\mathrm{E}} \approx 6N_{\mathrm{V}}$). However, in this case, the eigenvalue problem to be solved is much simpler than the one arising from the formulation studied in Mora & Rodríguez (2009). In fact, the latter leads to a degenerate generalized matrix eigenvalue problem, which is shown to be well posed in Mora & Rodríguez (2009, Appendix) but that cannot be solved with standard eigensolvers.

6.3 *Test* 3

In this numerical test we applied our method to the Stokes eigenvalue problem with mixed boundary conditions. The computational domain is again the unit square $\Omega := (0,1) \times (0,1)$ and $\Gamma := (0,1) \times \{0\}$. We observe from the results reported in Table 4 that the order of convergence is again quadratic in this case.

Finally, we show in Figs 2 and 3 the eigenfunctions corresponding to the four lowest eigenvalues.

TABLE 4 *Test 3. Lowest eigenvalues λ_{hi} of the Stokes problem with mixed boundary conditions computed with BDM*

	$N=10$	$N=20$	$N=30$	$N=40$	Order	Extrap.
λ_{h1}	2.4708	2.4682	2.4678	2.4676	2.00	2.4674
λ_{h2}	6.2946	6.2835	6.2813	6.2805	1.85	6.2793
λ_{h3}	15.3288	15.2402	15.2232	15.2171	1.94	15.2090
λ_{h4}	22.4812	22.2751	22.2371	22.2237	2.00	22.2065
λ_{h5}	27.3583	27.0518	26.9945	26.9744	1.98	26.9479
λ_{h6}	44.2217	43.4121	43.2619	43.2093	2.00	43.1419

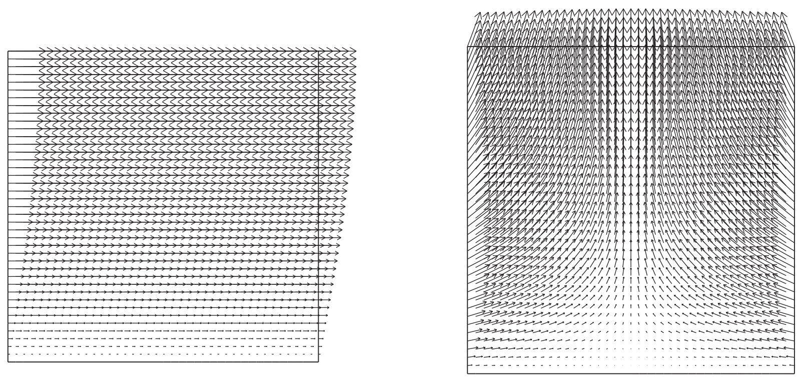

FIG. 2. Test 3. Eigenfunctions of the Stokes problem with mixed boundary condition associated with eigenvalues λ_{h1} (left) and λ_{h2} (right).

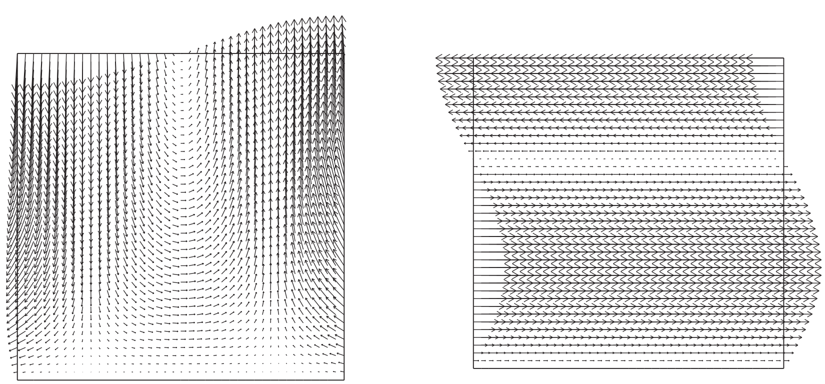

FIG. 3. Test 3. Eigenfunctions of the Stokes problem with mixed boundary condition associated with eigenvalues λ_{h3} (left) and λ_{h4} (right).

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