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Locking-free finite element method for a bending moment formulation of Timoshenko beams

Felipe Lepe [a], David Mora [a,b,*], Rodolfo Rodríguez [b,c]

[a] Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile
[b] Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, Concepción, Chile
[c] Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

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ABSTRACT

In this paper we study a finite element formulation for Timoshenko beams. It is known that standard finite elements applied to this model lead to wrong results when the thickness of the beam is small. Here, we consider a mixed formulation in terms of the transverse displacement, rotation, shear stress and bending moment. By using the classical Babuška–Brezzi theory, it is proved that the resulting variational formulation is well posed. We discretize it by continuous piecewise linear finite elements for the shear stress and bending moment, and discontinuous piecewise constant finite elements for the displacement and rotation. We prove an optimal (linear) order of convergence in terms of the mesh size for the natural norms and a double order (quadratic) in L^2-norms for the shear stress and the bending moment. These estimates involve constants and norms of the solution that are proved to be bounded independently of the beam thickness, which ensures the locking-free character of the method. Numerical tests are reported in order to support our theoretical results.

1. Introduction

Beams used in practice, like in buildings and bridges as well as in aircrafts, cars, ships, etc., commonly present continuous and discontinuous variations of the geometry and the physical parameters. They may also have appreciable thickness where the shear stress is not negligible. As a result, the thick beam model based on the Timoshenko theory have gained more popularity (see for instance [1–6]).

In this paper we study the numerical approximation of the bending of a non-homogeneous beam modeled by Timoshenko equations. Despite its simplicity, the numerical approximation of this problem often presents some difficulties. Indeed, it is very well known that standard finite element methods applied to models of thin structures, like beams, rods and plates, are subject to the so-called locking phenomenon. This means that they produce very unsatisfactory results when the thickness is small with respect to the other dimensions of the structure (see [7]).

Several strategies have been undertaken to overdraw the locking phenomenon. We refer to [8] for a detailed discussion on different approaches. On the other hand, several numerical methods have been mathematically proved to be locking-free and optimally convergent for Timoshenko beams by resorting to mixed formulations, discontinuous Galerkin methods, p and h–p versions of the finite element method or reduced integration; let us mention, for instance [9–16].

* Corresponding author at: Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile. Tel.: +56 41 3111146.
E-mail addresses: flepe@ubiobio.cl (F. Lepe), dmora@ubiobio.cl (D. Mora), rodolfo@ing-mat.udec.cl (R. Rodríguez).

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The purpose of this paper is to propose a mixed finite element method for a bending moment formulation of non-homogeneous Timoshenko beams and to provide an a priori error analysis. We introduce the bending moment together with the shear stress as new unknowns in the model (we note that the former usually represents a quantity of major interest in engineering applications, see for instance [17,18]), which together with the rotation and the transverse displacement lead us to a mixed variational formulation. One advantage of such a formulation is that the bending moment and the shear stress are computed directly instead of by means of a post-process, which may produce loss of accuracy. This approach follows the strategy used in [19] for Reissner–Mindlin plates, where a finite element method was introduced for the approximation of the bending of a clamped plate. However, the one-dimensional character of the problem allows us to give simpler proofs valid in a more general context. In particular, the results of this paper are valid for non-homogeneous beams, whose physical and geometrical properties may be discontinuous, and the error estimates are fully independent of the beam thickness. To cover such cases, a key point in our analysis is an improved regularity result, which we are able to prove by exploiting, once more, the one-dimensional character of the problem.

Using the Babuška–Brezzi theory, we show that the proposed mixed variational formulation is well posed and stable in the natural norms of the considered Sobolev spaces. For the numerical approximation, piecewise linear and continuous finite elements are used for the bending moment and the shear stress and piecewise constants for the transverse displacement and rotation. We prove a uniform inf–sup condition with respect to the discretization parameter h and the thickness t. The method is proved to have an optimal (linear) order of convergence in terms of the mesh size h for the natural norms and a double order (quadratic) in L^2-norms for the shear stress and the bending moment. Moreover, the obtained estimates only depend on norms of quantities which are known to be bounded independently of t. Therefore, the method turns out to be thoroughly locking-free.

The outline of this paper is as follows. In Section 2, we recall the differential equations governing the problem, and state a mixed variational formulation. Then, we prove the unique solvability and stability properties of the proposed formulation and some regularity results. In Section 3, we present the finite element discretization of our variational formulation, for which we prove a discrete inf–sup condition uniformly with respect to the beam thickness t and the mesh parameter h. Then, we establish the linear convergence of the method for the natural norms and a quadratic order in L^2-norm for the shear stress and the bending moment. In Section 4, we report some numerical tests which confirm the theoretical error estimates and allow us to assess the performance of the proposed method. Finally, we summarize some conclusions in Section 5.

We will use standard notations for Sobolev spaces, norms and seminorms. For $l \geq 0$, $\|\cdot\|_{l,\mathrm{I}}$ stands for the norm of the Hilbertian Sobolev space $H^l(\mathrm{I})$, with the convention $H^0(\mathrm{I}) := L^2(\mathrm{I})$. Moreover, we will denote with c and C, with or without subscripts, tildes or hats, a generic positive constant, possibly different at different occurrences, but always independent of the beam thickness t and the mesh parameter h.

2. Timoshenko beam model

Let us consider an elastic beam which satisfies the Timoshenko hypotheses for the admissible displacements. We assume that the geometry and the physical parameters of the beam may change along the axial direction. The deformation of the beam is described in terms of the transverse displacement w and the rotation of the transverse fibers β. Let x be the coordinate in the axial direction.

The equations for the bending of a clamped Timoshenko beam subjected to distributed load $p(x)$ and moment loading $m(x)$ reads as follows (see [20–22]):

Find $(\beta, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} E(x)\mathbb{I}(x)\beta'(x)\eta'(x)\,dx + \int_{\mathrm{I}} G(x)A(x)k_c(x)(\beta(x) - w'(x))(\eta(x) - v'(x))\,dx$$
$$= \int_{\mathrm{I}} m(x)\eta(x)\,dx + \int_{\mathrm{I}} p(x)v(x)\,dx \tag{2.1}$$

for all $(\eta, v) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$, where $\mathrm{I} := (0, L)$, L being the length of the beam, $E(x)$ is the Young modulus, $\mathbb{I}(x)$ the moment of inertia of the cross-section, $A(x)$ the area of the cross-section, $G(x) := E(x)/(2(1+\nu(x)))$ the shear modulus, with $\nu(x)$ the Poisson ratio, and $k_c(x)$ a correction factor. We consider that $E(x)$, $\mathbb{I}(x)$, $A(x)$ and $\nu(x)$ are piecewise smooth in the interval I, the most usual case being when all these coefficients are piecewise constant. Moreover, primes denote derivatives with respect to the x-coordinate.

It is well known that standard finite element procedures, when used in formulations such as (2.1) for very thin structures, are subject to numerical locking, a phenomenon induced by the difference of magnitude between the coefficients in front of the different terms (see [9]). The appropriate framework for analyzing this difficulty is obtained by rescaling formulation (2.1) so as to identify a family of problems with a well-posed limit as the thickness becomes infinitely small. With this aim, we introduce the following non-dimensional parameter, characteristic of the thickness of the beam,

$$t^2 := \frac{1}{L}\int_{\mathrm{I}} \frac{\mathbb{I}(x)}{A(x)L^2}\,dx,$$

which we assume may take values in the range $(0, t_{\max}]$.

We define

$$g(x) := \frac{m(x)}{t^3}, \qquad f(x) := \frac{p(x)}{t^3}, \qquad \hat{\mathbb{I}}(x) := \frac{\mathbb{I}(x)}{t^3}, \qquad \hat{A}(x) := \frac{A(x)}{t},$$
$$\mathsf{E}(x) := E(x)\hat{\mathbb{I}}(x) \quad \text{and} \quad \kappa(x) := G(x)\hat{A}(x)k_c(x).$$

Then, problem (2.1) can be equivalently written as follows:
Find $(\beta, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} \mathsf{E}(x)\beta'(x)\eta'(x)\,dx + \frac{1}{t^2}\int_{\mathrm{I}} \kappa(x)(\beta(x) - w'(x))(\eta(x) - v'(x))\,dx = \int_{\mathrm{I}} g(x)\eta(x)\,dx + \int_{\mathrm{I}} f(x)v(x)\,dx \tag{2.2}$$

for all $(\eta, v) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$.

Now, we assume that there exist constants $\underline{\mathsf{E}}, \overline{\mathsf{E}}, \underline{\kappa}, \overline{\kappa} \in \mathbb{R}^+$ such that

$$\begin{aligned} &\overline{\mathsf{E}} \geq \mathsf{E}(x) \geq \underline{\mathsf{E}} > 0 \quad \forall x \in \mathrm{I}, \\ &\overline{\kappa} \geq \kappa(x) \geq \underline{\kappa} > 0 \quad \forall x \in \mathrm{I}. \end{aligned} \tag{2.3}$$

In such a case, for each $t > 0$, the bilinear form on the left hand side of (2.2) is elliptic and hence this problem has a unique solution.

The aim of this paper is to consider a bending moment formulation of this problem. With this end, by introducing the bending moment $M(x) := \mathsf{E}(x)\beta'(x)$ and the shear stress $V(x) := t^{-2}\kappa(x)(\beta(x) - w'(x))$ as new unknowns in the model, problem (2.2) can be equivalently rewritten as follows (see [12]):

$$\begin{cases} M(x) = \mathsf{E}(x)\beta'(x) & \text{in I}, \\ -M'(x) + V(x) = g(x) & \text{in I}, \\ V'(x) = f(x) & \text{in I}, \\ V(x) = t^{-2}\kappa(x)(\beta(x) - w'(x)) & \text{in I}, \\ w(0) = \beta(0) = w(L) = \beta(L) = 0. \end{cases} \tag{2.4}$$

Testing the different equations in (2.4) with adequate functions and integrating by parts, we obtain the following variational formulation, where from now on we omit the dependence on the axial variable x:
Find $((M, V), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ *such that*

$$\begin{aligned} &\int_{\mathrm{I}} \frac{M\tau}{\mathsf{E}} + t^2 \int_{\mathrm{I}} \frac{V\xi}{\kappa} + \int_{\mathrm{I}} \beta(\tau' - \xi) - \int_{\mathrm{I}} w\xi' = 0 \quad \forall(\tau, \xi) \in \mathbb{H}, \\ &\int_{\mathrm{I}} \eta(M' - V) - \int_{\mathrm{I}} vV' = -\int_{\mathrm{I}} g\eta - \int_{\mathrm{I}} fv \quad \forall(\eta, v) \in \mathbb{Q}, \end{aligned} \tag{2.5}$$

where

$$\mathbb{H} := H^1(\mathrm{I}) \times H^1(\mathrm{I}) \quad \text{and} \quad \mathbb{Q} := L^2(\mathrm{I}) \times L^2(\mathrm{I}),$$

each one endowed with the corresponding product norm. Finally, we endow $\mathbb{H} \times \mathbb{Q}$ with the corresponding product norm, too.

We rewrite the variational problem (2.5) as follows:
Find $((M, V), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ *such that*

$$a((M, V), (\tau, \xi)) + b((\tau, \xi), (\beta, w)) = 0 \quad \forall(\tau, \xi) \in \mathbb{H}, \tag{2.6}$$
$$b((M, V), (\eta, v)) = F(\eta, v) \quad \forall(\eta, v) \in \mathbb{Q}, \tag{2.7}$$

where the bilinear forms $a : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ and $b : \mathbb{H} \times \mathbb{Q} \to \mathbb{R}$ and the linear functional $F : \mathbb{Q} \to \mathbb{R}$ are defined by

$$a((M, V), (\tau, \xi)) := \int_{\mathrm{I}} \frac{M\tau}{\mathsf{E}} + t^2 \int_{\mathrm{I}} \frac{V\xi}{\kappa}, \tag{2.8}$$

$$b((\tau, \xi), (\eta, v)) := \int_{\mathrm{I}} \eta(\tau' - \xi) - \int_{\mathrm{I}} v\xi', \tag{2.9}$$

and

$$F(\eta, v) := -\int_{\mathrm{I}} g\eta - \int_{\mathrm{I}} fv,$$

for all $(M, V), (\tau, \xi) \in \mathbb{H}$ and $(\eta, v) \in \mathbb{Q}$.

Next, we will prove that problem (2.6)–(2.7) satisfies the hypotheses of the Babuška–Brezzi theory, which yields unique solvability and continuous dependence on the data of this variational formulation.

We first observe that the bilinear forms a and b and the linear functional F are bounded with constants independent of the beam thickness t.

Let

$$K := \{(\tau, \xi) \in \mathbb{H} : \; b((\tau, \xi), (\eta, v)) = 0 \; \forall (\eta, v) \in \mathbb{Q}\}$$

be the so-called continuous kernel; hence (cf. (2.9))

$$K = \left\{(\tau, \xi) \in \mathbb{H} : \; (\tau' - \xi) = 0, \text{ and } \xi' = 0 \text{ in I}\right\} = \left\{(\tau, \tau') : \tau \in \mathbb{P}_1(\mathrm{I})\right\}.$$

The following lemma shows that the bilinear form a is $\mathbb{H}$-elliptic in K uniformly in t.

Lemma 2.1. *There exists $\alpha > 0$, independent of t, such that*

$$a((\tau, \xi), (\tau, \xi)) \geq \alpha \|(\tau, \xi)\|_{\mathbb{H}}^2 \quad \forall (\tau, \xi) \in K.$$

Proof. Given $(\tau, \xi) \in K$, from (2.8) and (2.3) we obtain

$$a((\tau, \xi), (\tau, \xi)) \geq \frac{1}{\overline{\mathrm{E}}} \|\tau\|_{0,\mathrm{I}}^2 + \frac{t^2}{\overline{\kappa}} \|\xi\|_{0,\mathrm{I}}^2 \geq \frac{1}{\overline{\mathrm{E}}} \|\tau\|_{0,\mathrm{I}}^2 \geq C \|\tau\|_{1,\mathrm{I}}^2,$$

the last inequality because of the equivalence between $\|\cdot\|_{0,\mathrm{I}}$ and $\|\cdot\|_{1,\mathrm{I}}$ in $K \cong \mathbb{P}_1(\mathrm{I})$, with a constant independent of t. Thus, the result follows from the fact that

$$\|(\tau, \xi)\|_{\mathbb{H}}^2 = \|\tau\|_{1,\mathrm{I}}^2 + \|\tau'\|_{0,\mathrm{I}}^2 \quad \forall (\tau, \xi) \in K.$$

Therefore, we end the proof. □

The next step is to prove an inf–sup condition for the bilinear form b uniformly in t.

Lemma 2.2. *There exists $C > 0$, independent of t, such that*

$$\sup_{\substack{(\tau,\xi) \in \mathbb{H} \\ (\tau,\xi) \neq 0}} \frac{b((\tau, \xi), (\eta, v))}{\|(\tau, \xi)\|_{\mathbb{H}}} \geq C \|(\eta, v)\|_{\mathbb{Q}} \quad \forall (\eta, v) \in \mathbb{Q}.$$

Proof. Let $(\eta, v) \in \mathbb{Q}$. Let $\widetilde{\tau}(r) := \int_0^r \eta(s)\, ds$, $0 \leq r \leq L$. We have that $\widetilde{\tau}' = \eta \in L^2(\mathrm{I})$. Hence $\widetilde{\tau} \in H^1(\mathrm{I})$, and

$$\|\widetilde{\tau}\|_{1,\mathrm{I}} \leq \left(\frac{L^2 + 2}{2}\right)^{1/2} \|\eta\|_{0,\mathrm{I}}.$$

Therefore,

$$\sup_{\substack{(\tau,\xi) \in \mathbb{H} \\ (\tau,\xi) \neq 0}} \frac{b((\tau, \xi), (\eta, v))}{\|(\tau, \xi)\|_{\mathbb{H}}} \geq \frac{b((\widetilde{\tau}, 0), (\eta, v))}{\|\widetilde{\tau}\|_{1,\mathrm{I}}} = \frac{\|\eta\|_{0,\mathrm{I}}^2}{\|\widetilde{\tau}\|_{1,\mathrm{I}}} \geq \left(\frac{2}{L^2 + 2}\right)^{1/2} \|\eta\|_{0,\mathrm{I}}. \tag{2.10}$$

Finally, let $\widetilde{\xi}(r) := -\int_0^r v(s)\, ds$, $0 \leq r \leq L$. The same arguments as above allow us to prove that

$$\|\widetilde{\xi}\|_{1,\mathrm{I}} \leq \left(\frac{L^2 + 2}{2}\right)^{1/2} \|v\|_{0,\mathrm{I}}.$$

Hence, it follows that

$$\sup_{\substack{(\tau,\xi) \in \mathbb{H} \\ (\tau,\xi) \neq 0}} \frac{b((\tau, \xi), (\eta, v))}{\|(\tau, \xi)\|_{\mathbb{H}}} \geq \frac{b((0, \widetilde{\xi}), (\eta, v))}{\|\widetilde{\xi}\|_{1,\mathrm{I}}} = \frac{\|v\|_{0,\mathrm{I}}^2 - \int_{\mathrm{I}} \eta \widetilde{\xi}}{\|\widetilde{\xi}\|_{1,\mathrm{I}}} \geq \left(\frac{2}{L^2 + 2}\right)^{1/2} \|v\|_{0,\mathrm{I}} - \|\eta\|_{0,\mathrm{I}}.$$

From this inequality and (2.10), it is immediate to conclude that

$$\sup_{\substack{(\tau,\xi) \in \mathbb{H} \\ (\tau,\xi) \neq 0}} \frac{b((\tau, \xi), (\eta, v))}{\|(\tau, \xi)\|_{\mathbb{H}}} \geq \frac{2}{\sqrt{L^2 + 2}(\sqrt{L^2 + 2} + \sqrt{2})} \|v\|_{0,\mathrm{I}}.$$

Thus, the proof follows from this estimate and (2.10). □

We are now in a position to state the main result of this section which yields the solvability of the continuous problem (2.6)–(2.7).

Theorem 2.3. *There exists a unique solution $((M, V), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ to problem (2.6)–(2.7) and the following continuous dependence result holds:*

$$\|((M, V), (\beta, w))\|_{\mathbb{H}\times\mathbb{Q}} \le C(\|g\|_{0,\mathrm{I}} + \|f\|_{0,\mathrm{I}}),$$

where $C > 0$ is independent of t.

Proof. By virtue of Lemmas 2.1 and 2.2, the proof follows from a straightforward application of [23, Theorem II.1.1]. □

The following result establishes some additional regularity of the solution to problem (2.6)–(2.7). This result will be the key point to prove the convergence of the proposed method.

Proposition 2.1. *Suppose that $f, g \in H^l(\mathrm{I})$, $l = 0, 1$. Let $((M, V), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ be the solution to problem (2.5). Then, there exists a constant $C > 0$, independent of t, g and f, such that*

$$\|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|V\|_{l+1,\mathrm{I}} + \|M\|_{l+1,\mathrm{I}} \le C(\|g\|_{l,\mathrm{I}} + \|f\|_{l,\mathrm{I}}).$$

Proof. Because of the equivalence between problems (2.5) and (2.2), we will use the latter to prove the result.
Consider the following decomposition for the scaled shear stress:

$$V = \psi' + k, \tag{2.11}$$

with $k := (\frac{1}{L}\int_{\mathrm{I}} V) \in \mathbb{R}$, so that $\psi \in H_0^1(\mathrm{I})$. We have that problem (2.2) and the following are also equivalent: find $(\psi, \beta, k, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$ such that

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \psi' v' = -\int_{\mathrm{I}} f v \quad \forall v \in H_0^1(\mathrm{I}), \\ \displaystyle\int_{\mathrm{I}} \mathsf{E}\beta'\eta' + \int_{\mathrm{I}} k\eta = \int_{\mathrm{I}} g\eta - \int_{\mathrm{I}} \psi'\eta \quad \forall \eta \in H_0^1(\mathrm{I}), \\ \displaystyle\int_{\mathrm{I}} \beta q - t^2 \int_{\mathrm{I}} \frac{kq}{\kappa} = t^2 \int_{\mathrm{I}} \frac{\psi' q}{\kappa} \quad \forall q \in \mathbb{R}, \\ \displaystyle\int_{\mathrm{I}} w'\delta' = \int_{\mathrm{I}} \beta\delta' - t^2 \int_{\mathrm{I}} \frac{\delta' k}{\kappa} - t^2 \int_{\mathrm{I}} \frac{\psi'\delta'}{\kappa} \quad \forall \delta \in H_0^1(\mathrm{I}). \end{cases} \tag{2.12}$$

For this problem, we have that for any $t \in (0, t_{\max}]$ and $f, g \in H^l(\mathrm{I})$, $l = 0, 1$, there exists a unique solution $(\psi, \beta, k, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$. Moreover, there exists a constant $C > 0$ independent of t and f, such that

$$\|\psi\|_{l+2,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + |k| + \|w\|_{1,\mathrm{I}} \le C(\|g\|_{l,\mathrm{I}} + \|f\|_{l,\mathrm{I}}).$$

In fact, given $f \in H^l(\mathrm{I})$, $l = 0, 1$, from Lax–Milgram's Lemma we have that there exists a unique $\psi \in H_0^1(\mathrm{I})$ solution of $(2.12)_1$ and moreover it satisfies $\|\psi\|_{l+2,\mathrm{I}} \le C\|f\|_{l,\mathrm{I}}$, $l = 0, 1$. Now, for all $t \in (0, t_{\max}]$ we can apply Theorem 5.1 of [9] to obtain that there exists a unique solution $(\beta, k) \in H_0^1(\mathrm{I}) \times \mathbb{R}$ of problem $(2.12)_{2-3}$ and it satisfies

$$\|\beta\|_{1,\mathrm{I}} + |k| \le C(\|g\|_{0,\mathrm{I}} + \|\psi'\|_{0,\mathrm{I}}) \le C(\|g\|_{0,\mathrm{I}} + \|f\|_{0,\mathrm{I}}),$$

where the constant C is independent of t. Finally, we obtain again from Lax–Milgram's Lemma that there exists a unique solution $w \in H_0^1(\mathrm{I})$ of problem $(2.12)_4$ and, taking $\delta = w$, we obtain

$$\|w\|_{1,\mathrm{I}} \le C(\|\beta\|_{0,\mathrm{I}} + |k| + \|\psi'\|_{0,\mathrm{I}}) \le C(\|g\|_{0,\mathrm{I}} + \|f\|_{0,\mathrm{I}}).$$

To end the proof, we derive the estimates for $\|V\|_{l+1,\mathrm{I}}$ and $\|M\|_{l+1,\mathrm{I}}$ from Theorem 2.3 and $(2.4)_{2-3}$. Thus, we conclude that there exists $C > 0$ independent of t, f and g such that, for $l = 0, 1$,

$$\|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|V\|_{l+1,\mathrm{I}} + \|M\|_{l+1,\mathrm{I}} \le C(\|g\|_{l,\mathrm{I}} + \|f\|_{l,\mathrm{I}}). \quad \square$$

Remark 2.1. It is possible to refine the proposition above by considering piecewise smooth loads. Suppose that there exists a partition, $0 = s_0 < \cdots < s_n = L$, of the interval I. We denote $S_i := (s_{i-1}, s_i)$ and assume that $f, g \in H^1(S_i)$, $i = 1, \ldots, n$. Then, repeating the arguments used in the proof above, we obtain the following result for the solution of problem (2.6)–(2.7): there exists a constant $C > 0$, independent of t, g and f, such that

$$\begin{aligned} &\|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|V\|_{1,\mathrm{I}} + \left(\sum_{i=1}^n \|V''\|_{0,S_i}^2\right)^{1/2} + \|M\|_{1,\mathrm{I}} + \left(\sum_{i=1}^n \|M''\|_{0,S_i}^2\right)^{1/2} \\ &\quad \le C\left(\|g\|_{0,\mathrm{I}}^2 + \|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^n \left(\|g'\|_{0,S_i}^2 + \|f'\|_{0,S_i}^2\right)\right)^{1/2}. \end{aligned}$$

3. The mixed finite element scheme

In this section, we present our discrete methods for the Timoshenko beam problem. With this aim, first we consider a family of partitions of the interval I:

$$\mathcal{T}_h: \ 0 = x_0 < \cdots < x_N = L.$$

We denote $\mathrm{I}_j = (x_{j-1}, x_j)$, with length $h_j = x_j - x_{j-1}$, $j = 1, \ldots, N$, and the maximum subinterval length is denoted $h := \max_{1\leq j\leq N} h_j$.

To approximate the shear stress and the bending moment, we consider the space of piecewise linear continuous finite elements:

$$W_h := \{\xi_h \in H^1(\mathrm{I}) : \xi_h|_{\mathrm{I}_j} \in \mathbb{P}_1(\mathrm{I}_j),\ j = 1, \ldots, N\}.$$

Let $\mathcal{L}(\xi) \in W_h$ be the Lagrange interpolant of $\xi \in H^1(\mathrm{I})$, we recall that

$$\|\xi - \mathcal{L}(\xi)\|_{1,\mathrm{I}} \leq Ch|\xi|_{2,\mathrm{I}} \quad \forall \xi \in H^2(\mathrm{I}). \tag{3.1}$$

To approximate the transverse displacement and the rotation, we will use the space of piecewise constant functions:

$$Z_h := \{v_h \in L^2(\mathrm{I}) : v_h|_{\mathrm{I}_j} \in \mathbb{P}_0(\mathrm{I}_j),\ j = 1, \ldots, N\}.$$

We also consider the L^2-projector onto Z_h:

$$\mathcal{P}: \ L^2(\mathrm{I}) \rightarrow Z_h,$$
$$v \mapsto \mathcal{P}(v) := \bar{v} \in Z_h \ : \ \int_{\mathrm{I}} (v - \bar{v}) q_h = 0 \quad \forall q_h \in Z_h.$$

Then, we have

$$\|v - \mathcal{P}(v)\|_{0,\mathrm{I}} \leq Ch|v|_{1,\mathrm{I}} \quad \forall v \in H^1(\mathrm{I}). \tag{3.2}$$

Defining $\mathbb{H}_h := W_h \times W_h$ and $\mathbb{Q}_h := Z_h \times Z_h$, the discretization of problem (2.6)–(2.7) reads as follows:
Find $((M_h, V_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ *such that*

$$a((M_h, V_h), (\tau_h, \xi_h)) + b((\tau_h, \xi_h), (\beta_h, w_h)) = 0 \quad \forall (\tau_h, \xi_h) \in \mathbb{H}_h, \tag{3.3}$$

$$b((M_h, V_h), (\eta_h, v_h)) = F(\eta_h, v_h) \quad \forall (\eta_h, v_h) \in \mathbb{Q}_h. \tag{3.4}$$

Our next goal is to show the corresponding discrete versions of Lemmas 2.1 and 2.2 to conclude the solvability and stability of problem (3.3)–(3.4). With this aim, we note that the discrete null space of the bilinear form b reduces to:

$$K_h := \{(\tau_h, \xi_h) \in \mathbb{H}_h : b((\tau_h, \xi_h), (\eta_h, v_h)) = 0 \ \forall (\eta_h, v_h) \in \mathbb{Q}_h\}.$$

Let $(\tau_h, \xi_h) \in K_h$, taking $(0, v_h) \in \mathbb{Q}_h$ and using that $\xi_h'|_{\mathrm{I}_j}$ is a constant, since $v_h|_{\mathrm{I}_j}$ is also a constant, we conclude that $\xi_h' = 0$ in I.

Now, taking $(\eta_h, 0) \in \mathbb{Q}_h$, since $\tau_h'|_{\mathrm{I}_j}$ is a constant and $\xi_h|_{\mathrm{I}_j}$ is also a constant, we conclude that $(\tau_h' - \xi_h) = 0$ in I_j and hence $\tau_h' = \xi_h$ in I. Thus, we obtain

$$K_h = \left\{(\tau_h, \xi_h) \in \mathbb{H}_h : (\tau_h' - \xi_h) = 0 \text{ and } \xi_h' = 0 \text{ in I}\right\} = \left\{(\tau_h, \tau_h') : \tau_h \in \mathbb{P}_1(\mathrm{I})\right\}.$$

Hence, we have proved that K_h coincides with K, so that we have from Lemma 2.1.

Lemma 3.1. *There exists* $\alpha > 0$ *independent of* h *and* t *such that*

$$a((\tau_h, \xi_h), (\tau_h, \xi_h)) \geq \alpha \|(\tau_h, \xi_h)\|_{\mathbb{H}}^2 \quad \forall (\tau_h, \xi_h) \in K_h.$$

We continue with the following discrete analogue to Lemma 2.2.

Lemma 3.2. *There exists* $C > 0$*, independent of* h *and* t*, such that*

$$\sup_{\substack{(\tau_h, \xi_h) \in \mathbb{H}_h \\ (\tau_h, \xi_h) \neq 0}} \frac{b((\tau_h, \xi_h), (\eta_h, v_h))}{\|(\tau_h, \xi_h)\|_{\mathbb{H}}} \geq C\|(\eta_h, v_h)\|_{\mathbb{Q}} \quad \forall (\eta_h, v_h) \in \mathbb{Q}_h.$$

Proof. Let $(\eta_h, v_h) \in \mathbb{Q}_h$. The arguments used in the proof of Lemma 2.2 can be applied. In fact, $\widetilde{\tau}(r) := \int_0^r \eta_h(s)ds$ lies in W_h, and the same happens with $\widetilde{\xi}(r) := \int_0^r v_h(s)ds$. $\square$

We are now in a position to establish the unique solvability, the stability, and the convergence properties of the discrete problem (3.3)–(3.4).

Theorem 3.3. *There exists a unique* $((M_h, V_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ *solution of the discrete problem (3.3)–(3.4). Moreover, there exist constants* $\widetilde{C}, C > 0$*, independent of* h *and* t*, such that*

$$\|((M_h, V_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \le \widetilde{C}(\|g\|_{0,\mathrm{I}} + \|f\|_{0,\mathrm{I}}),$$

and

$$\begin{aligned} &\|((M, V), (\beta, w)) - ((M_h, V_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \\ &\quad \le C \inf_{((\tau_h,\xi_h),(\eta_h,v_h))\in\mathbb{H}_h\times\mathbb{Q}_h} \|((M, V), (\beta, w)) - ((\tau_h, \xi_h), (\eta_h, v_h))\|_{\mathbb{H}\times\mathbb{Q}}, \end{aligned} \tag{3.5}$$

where $((M, V), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ *is the unique solution of the mixed variational formulation (2.6)–(2.7).*

Proof. Existence and uniqueness of solutions to problem (3.3)–(3.4) and the error bound (3.5) follow from the abstract theory for saddle point problems [23, Theorem II.2.1] and Lemmas 3.1 and 3.2. □

The following theorem provides the rate of convergence of our mixed finite element scheme (3.3)–(3.4).

Theorem 3.4. *Let* $((M, V), (\beta, w)) \in \mathbb{H}\times\mathbb{Q}$ *and* $((M_h, V_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ *be the solutions of the continuous and discrete problems (2.6)–(2.7) and (3.3)–(3.4), respectively. If* $f, g \in H^1(\mathrm{I})$*, then*

$$\|((M, V), (\beta, w)) - ((M_h, V_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \le Ch(\|g\|_{1,\mathrm{I}} + \|f\|_{1,\mathrm{I}}),$$

where the constant $C > 0$ *is independent of* h *and* t*.*

Proof. From Theorem 3.3 we have

$$\begin{aligned} &\|((M, V), (\beta, w)) - ((M_h, V_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \\ &\quad \le C\|((M, V), (\beta, w)) - ((\mathcal{L}(M), \mathcal{L}(V)), (\mathcal{P}(\beta), \mathcal{P}(w)))\|_{\mathbb{H}\times\mathbb{Q}}. \end{aligned} \tag{3.6}$$

Then, the proof follows from the term above by using the error estimates for $\mathcal{P}$ in (3.2), and $\mathcal{L}$ in (3.1), and Proposition 2.1. □

The theorem above implies that $\|M - M_h\|_{0,\mathrm{I}} + \|V - V_h\|_{0,\mathrm{I}} = O(h)$. However, it is possible to improve this estimate as shown in the following result.

Theorem 3.5. *Under the assumptions of Theorem 3.4*

$$\|(M - M_h, V - V_h)\|_{\mathbb{Q}} := \left(\|M - M_h\|_{0,\mathrm{I}}^2 + \|V - V_h\|_{0,\mathrm{I}}^2\right)^{1/2} \le Ch^2(\|g\|_{1,\mathrm{I}} + \|f\|_{1,\mathrm{I}}),$$

where $C > 0$ *is independent of* h *and* t*.*

Proof. We resort to a duality argument. First, we consider the following well posed problem: find $((\phi, \rho), (\chi, u)) \in \mathbb{H} \times \mathbb{Q}$ such that

$$a((\tau, \xi), (\phi, \rho)) + b((\tau, \xi), (\chi, u)) = (z, (\tau, \xi))_{0,\mathrm{I}} \quad \forall(\tau, \xi) \in \mathbb{H}, \tag{3.7}$$

$$b((\phi, \rho), (\eta, v)) = 0 \quad \forall(\eta, v) \in \mathbb{Q}. \tag{3.8}$$

The unique solution of this problem satisfies the following additional regularity result: there exists a constant $C > 0$, independent of t and q such that

$$\|u\|_{1,\mathrm{I}} + \|\chi\|_{1,\mathrm{I}} + \|\rho\|_{2,\mathrm{I}} + \|\phi\|_{2,\mathrm{I}} \le C\|z\|_{0,\mathrm{I}}. \tag{3.9}$$

In fact, considering a decomposition similar to (2.11) for the variable ρ (namely, writing $\rho = \lambda' + r$, with $\lambda \in H_0^1(\mathrm{I})$ and $r := (\frac{1}{L}\int_{\mathrm{I}} \rho) \in \mathbb{R}$), we have that problem (3.7)–(3.8) and the following are equivalent: find $(\lambda, \chi, r, u) \in H_0^1(\mathrm{I})\times H_0^1(\mathrm{I})\times\mathbb{R}\times H_0^1(\mathrm{I})$ such that

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \lambda' v' = 0 \quad \forall v \in H_0^1(\mathrm{I}), \\ \displaystyle\int_{\mathrm{I}} \mathsf{E}\chi'\eta' + \int_{\mathrm{I}} r\eta = -\int_{\mathrm{I}} \mathsf{E}z_1\eta' - \int_{\mathrm{I}} \lambda'\eta \quad \forall\eta \in H_0^1(\mathrm{I}), \\ \displaystyle\int_{\mathrm{I}} \chi q - t^2 \int_{\mathrm{I}} \frac{rq}{\kappa} = -\int_{\mathrm{I}} z_2 q + t^2 \int_{\mathrm{I}} \frac{\lambda' q}{\kappa} \quad \forall q \in \mathbb{R}, \\ \displaystyle\int_{\mathrm{I}} u'\delta' = \int_{\mathrm{I}} \chi\delta' + \int_{\mathrm{I}} z_2\delta' - t^2 \int_{\mathrm{I}} \frac{\delta' r}{\kappa} - t^2 \int_{\mathrm{I}} \frac{\lambda'\delta'}{\kappa} \quad \forall\delta \in H_0^1(\mathrm{I}). \end{cases} \tag{3.10}$$

From $(3.10)_1$, we have that $\lambda = 0$. Then, repeating the arguments used to prove Proposition 2.1, we conclude the claimed estimate (3.9).

On the other hand, by choosing $(\tau, \xi) = (M - M_h, V - V_h)$ in problem (3.7)–(3.8), we obtain

$$(z, (M - M_h, V - V_h))_{0,\mathrm{I}} = a((M - M_h, V - V_h), (\phi, \rho)) + b((M - M_h, V - V_h), (\chi, u)). \tag{3.11}$$

Subtracting (2.6) and (3.3) and using (3.8), we have

$$\begin{aligned} a((M - M_h, V - V_h), (\phi_h, \rho_h)) &= -b((\phi_h, \rho_h), (\beta - \beta_h, w - w_h)) \\ &= b((\phi - \phi_h, \rho - \rho_h), (\beta - \beta_h, w - w_h)) \quad \forall (\phi_h, \rho_h) \in \mathbb{H}_h. \end{aligned}$$

Moreover, from (2.7) and (3.4), we also have

$$b((M - M_h, V - V_h), (\chi_h, u_h)) = 0 \quad \forall (\chi_h, u_h) \in \mathbb{Q}_h.$$

Substituting the last two terms into (3.11) we obtain:

$$\begin{aligned} (z, (M - M_h, V - V_h))_{0,\mathrm{I}} &= a((M - M_h, V - V_h), (\phi - \phi_h, \rho - \rho_h)) \\ &\quad + b((M - M_h, V - V_h), (\chi - \chi_h, u - u_h)) \\ &\quad + b((\phi - \phi_h, \rho - \rho_h), (\beta - \beta_h, w - w_h)) \end{aligned}$$

for all $(\phi_h, \rho_h) \in \mathbb{H}_h$ and all $(\chi_h, u_h) \in \mathbb{Q}_h$. Hence,

$$\begin{aligned} |(z, (M - M_h, V - V_h))_{0,\mathrm{I}}| &\le C\,(\|(M - M_h, V - V_h)\|_{\mathbb{H}} \|(\phi - \phi_h, \rho - \rho_h)\|_{\mathbb{H}} \\ &\quad + \|(M - M_h, V - V_h)\|_{\mathbb{H}} \|(\chi - \chi_h, u - u_h)\|_{\mathbb{Q}} \\ &\quad + \|(\phi - \phi_h, \rho - \rho_h)\|_{\mathbb{H}} \|(\beta - \beta_h, w - w_h)\|_{\mathbb{Q}}\big) \\ &\le Ch \left(\|g\|_{1,\mathrm{I}} + \|f\|_{1,\mathrm{I}}\right) \left(\|(\phi - \phi_h, \rho - \rho_h)\|_{\mathbb{H}} + \|(\chi - \chi_h, u - u_h)\|_{\mathbb{Q}}\right) \end{aligned}$$

for all $(\phi_h, \rho_h) \in \mathbb{H}_h$ and $(\chi_h, u_h) \in \mathbb{Q}_h$, where in the last inequality we have used Theorem 3.4. Taking in particular $(\phi_h, \rho_h) := (\mathcal{L}(\phi), \mathcal{L}(\rho))$ and $(\chi_h, u_h) := (\mathcal{P}(\chi), \mathcal{P}(u))$ in the above estimate we obtain

$$\begin{aligned} |(z, (M - M_h, V - V_h))_{0,\mathrm{I}}| &\le Ch \left(\|g\|_{1,\mathrm{I}} + \|f\|_{1,\mathrm{I}}\right) \left(\|(\phi - \mathcal{L}(\phi), \rho - \mathcal{L}(\rho))\|_{\mathbb{H}} + \|(\chi - \mathcal{P}(\chi), u - \mathcal{P}(u))\|_{\mathbb{Q}}\right) \\ &\le Ch^2 \left(\|g\|_{1,\mathrm{I}} + \|f\|_{1,\mathrm{I}}\right) \left(\|\phi\|_{2,\mathrm{I}} + \|\rho\|_{2,\mathrm{I}} + \|\chi\|_{1,\mathrm{I}} + \|u\|_{1,\mathrm{I}}\right) \\ &\le Ch^2 \left(\|g\|_{1,\mathrm{I}} + \|f\|_{1,\mathrm{I}}\right) \|z\|_{0,\mathrm{I}}, \end{aligned}$$

where in the last inequality, we have used first the error estimates for $\mathcal{P}$ in (3.2) and $\mathcal{L}$ in (3.1) and then the additional regularity result (3.9).

Thus, the result follows by taking $z = (M - M_h, V - V_h)$. $\square$

Remark 3.1. It is possible to extend Theorems 3.4 and 3.5 to cover piecewise smooth loads, provided all the meshes

$$\widetilde{\mathcal{T}}_h : \; 0 = x_0 < \cdots < x_N = L$$

are refinements of the initial partition $0 = s_0 < \cdots < s_n = L$ (see Remark 2.1). We denote $\mathrm{I}_j := (x_{j-1}, x_j),\ j = 1, \ldots, N$, and note that for any mesh $\widetilde{\mathcal{T}}_h$, each I_j is contained in some subinterval S_i. Hence, if $f, g \in H^1(S_i),\ i = 1, \ldots, n$, then

$$\|((M, V), (\beta, w)) - ((M_h, V_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \le Ch \left(\|g\|_{0,\mathrm{I}}^2 + \|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \left(\|g'\|_{0,S_i}^2 + \|f'\|_{0,S_i}^2\right) \right)^{1/2}$$

and

$$\|(M - M_h, V - V_h)\|_{\mathbb{Q}} \le \hat{C} h^2 \left(\|g\|_{0,\mathrm{I}}^2 + \|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \left(\|g'\|_{0,S_i}^2 + \|f'\|_{0,S_i}^2\right) \right)^{1/2},$$

where the constants $C, \hat{C} > 0$ are independent of h and t. In fact, the first estimate follows from inequality (3.6), the local character of the Lagrange interpolant of V and the additional regularity result given in Remark 2.1. In its turn, the second one follows from the first one and Theorem 3.5.

4. Numerical results

The numerical method analyzed above has been implemented in a MATLAB code. We report in this section some numerical examples which confirm the theoretical results.

Let us remark that the direct solution of the linear system arising from (3.3)–(3.4) would be much more expensive than computing the bending of the Timoshenko beam with a classical locking-free lowest-order discretization of displacements and rotations (see [9], for instance). However, this cost can be significantly reduced by means of a hybridization process, similar to that introduced in [19, Section III.D] for the bending moment formulation of Reissner–Mindlin plates.

Table 1
Example 1. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for M and V.

d.o.f.	h	$\mathbf{e}_0(M)$	$\mathrm{rc}_0(M)$	$\mathbf{e}_1(M)$	$\mathrm{rc}_1(M)$	$\mathbf{e}_0(V)$	$\mathrm{rc}_0(V)$
34	0.125	1.6539e−3	–	6.4508e−2	–	2.6697e−3	–
66	0.0625	4.1355e−4	1.98	3.2249e−2	0.99	6.6905e−4	2.00
130	0.03125	1.0339e−4	1.99	1.6124e−2	1.00	1.6736e−4	2.00
258	0.015625	2.5848e−5	2.00	8.0617e−3	1.00	4.1847e−5	2.00
514	0.0078125	6.4621e−6	2.00	4.0309e−3	1.00	1.0462e−5	2.00

Table 2
Example 1. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for V, β and w.

d.o.f.	h	$\mathbf{e}_1(V)$	$\mathrm{rc}_1(V)$	$\mathbf{e}_0(\beta)$	$\mathrm{rc}_0(\beta)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$
34	0.125	6.4474e−2	–	1.3298e−3	–	2.1197e−4	–
66	0.0625	3.2245e−2	0.99	6.7574e−4	0.98	1.0523e−4	1.01
130	0.03125	1.6123e−2	1.00	3.3923e−4	1.00	5.2507e−5	1.00
258	0.015625	8.0618e−3	1.00	1.6978e−4	1.00	2.6240e−5	1.00
514	0.0078125	4.0310e−3	1.00	8.4912e−5	1.00	1.3118e−5	1.00

Such a hybridization process consists in using discontinuous $\mathcal{P}_1$ finite elements to discretize M and V (instead of standard continuous elements) and introducing Lagrange multipliers to enforce the continuity of these quantities. Each of these Lagrange multipliers reduces to a set of point values at each inner node of the mesh. The advantage of this procedure is that most variables of the resulting system can be eliminated by means of a static condensation, which in a matrix–vector language like MATLAB turns out particularly simple to implement. Such static condensation involves solving local 2×2 problems on each subinterval and, thus, is quite inexpensive.

By so doing, we arrive at a final symmetric positive definite sparse linear system in which the only unknowns are the $2(N-1)$ Lagrange multipliers, which constitute the main bulk of the computation. Therefore, the cost of solving the final system is almost the same as that of using a classical lowest-order discretization as that from [9]. We refer to [19, Section III.D] for further details on this hybridization process.

In what follows d.o.f. denotes the number of degrees of freedom, namely, d.o.f. $:= \dim(\mathbb{H}_h \times \mathbb{Q}_h)$. Moreover, we define the individual errors by:

$$\mathbf{e}_0(M) := \|M - M_h\|_{0,\mathrm{I}}, \qquad \mathbf{e}_0(V) := \|V - V_h\|_{0,\mathrm{I}}, \qquad \mathbf{e}_0(\beta) := \|\beta - \beta_h\|_{0,\mathrm{I}},$$
$$\mathbf{e}_1(M) := |M - M_h|_{1,\mathrm{I}}, \qquad \mathbf{e}_1(V) := |V - V_h|_{1,\mathrm{I}}, \qquad \mathbf{e}_0(w) := \|w - w_h\|_{0,\mathrm{I}},$$

where $((M, V), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ and $((M_h, V_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ are the solutions of problems (2.6)–(2.7) and (3.3)–(3.4), respectively.

We define the experimental rates of convergence (rc_i) for the errors $\mathbf{e}_i(\cdot)$ by

$$\mathrm{rc}_i(\cdot) := \frac{\log(\mathbf{e}_i(\cdot)/\mathbf{e}_i'(\cdot))}{\log(h/h')}, \quad i = 0, 1,$$

where h and h' denote two consecutive mesh-sizes and $\mathbf{e}$ and $\mathbf{e}'$ the corresponding errors.

4.1. Example 1

In this case we take $\mathrm{I} := (0, 1)$ and solve Eqs. (2.4) with $g(x) = 0$, $f(x) = e^x$, $\mathsf{E}(x) = e^x$, and $\kappa(x) = e^{-x}$ (see [10–12]). Thus, we obtain the following exact solution:

$$\begin{aligned}
V(x) &= e^x + c_1,\\
M(x) &= e^x + c_1 x + c_2,\\
\beta(x) &= x - c_1((x+1)e^x - 1) - c_2(e^{-x} - 1),\\
w(x) &= \frac{x^2}{2} + c_1((x+2)e^{-x} + x + t^2(1 - e^x) - 2) + c_2(e^{-x} + x - 1) + \frac{t^2}{2}(1 - e^{2x}),
\end{aligned}$$

where

$$c_1 = \frac{t^2(e^2 - 1) - \frac{2}{1-e} - 1}{6e^{-1} + 2t^2(1-e) - \frac{2(e^{-1}-1)}{1-e}} \quad \text{and} \quad c_2 = \frac{1 - c_1(2e^{-1} - 1)}{e^{-1} - 1}.$$

First, we analyze the convergence keeping the thickness fixed: $t = 0.01$. Then, we also show an analysis for various thickness in order to assess the locking free nature of the proposed method.

Tables 1 and 2 show the convergence history of the mixed finite element scheme (3.3)–(3.4) applied to this example.

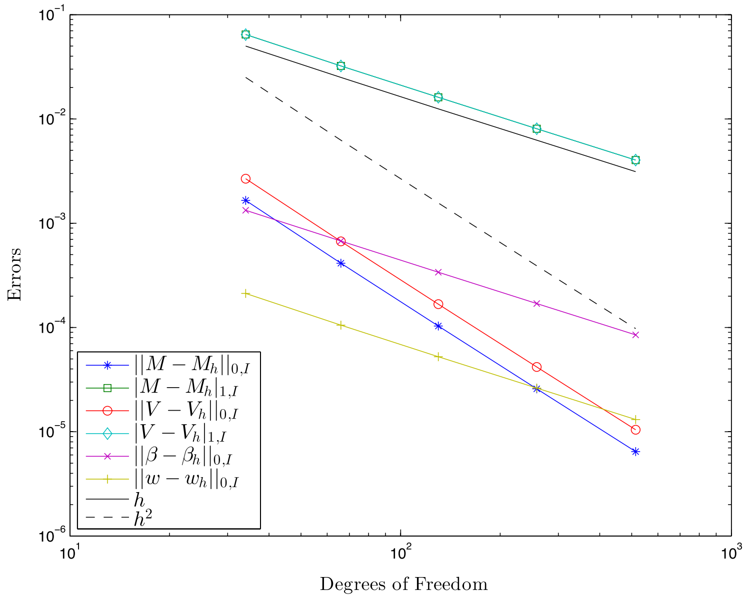

Fig. 1. Example 1. Error curves: log–log plot of errors vs. number of degrees of freedom (d.o.f.).

Table 3
Example 1. Locking-free analysis for the transverse displacement w.

d.o.f.	h	$t = 1.0e{-3}$		$t = 1.0e{-4}$		$t = 1.0e{-5}$	
		$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$
34	0.125	2.0928e−4	–	2.0929e−4	–	2.0925e−4	–
66	0.0625	1.0388e−4	1.01	1.0387e−4	1.01	1.0386e−4	1.01
130	0.03125	5.1831e−5	1.00	5.1824e−5	1.00	5.1824e−5	1.00
258	0.015625	2.5902e−5	1.00	2.5898e−5	1.00	2.5898e−5	1.00
514	0.0078125	1.2949e−5	1.00	1.2947e−5	1.00	1.2947e−5	1.00

We observe from Tables 1 and 2 that the rates of convergence $O(h)$ and $O(h^2)$ predicted by Theorems 3.4 and 3.5 are attained for all the variables. These experimental rates of convergence can also be checked from Fig. 1, where we display the error curves for all these quantities (namely, a log–log plot of errors vs. d.o.f.).

Next, we solve the same problem with a varying thickness to assess the locking-free character of the method. We report in Table 3 the errors and the rates of convergence for the transverse displacement.

We observe from Table 3 that our method does not deteriorate when the thickness parameter becomes small. The same happens with all the other variables, so that we can assert that the method is thoroughly locking free.

4.2. Example 2

As a second numerical example, we take $\mathrm{I} := (0, 1)$ and solve Eqs. (2.4) with $g(x) = 0$,

$$f(x) = \begin{cases} x, & 0 \le x \le 0.5, \\ e^{-x}, & 0.5 < x \le 1, \end{cases} \qquad \mathrm{E}(x) = \begin{cases} 1, & 0 \le x \le 0.5, \\ e^{-x}, & 0.5 < x \le 1, \end{cases}$$

and

$$\kappa(x) = \begin{cases} e^{x}, & 0 \le x \le 0.5, \\ 1, & 0.5 < x \le 1. \end{cases}$$

In this case, we have considered piecewise smooth data. As required by the theory (see Remark 3.1), we have used meshes where the point $x = 0.5$ is always a node.

The analytical solution of this particular example can be obtained by solving the corresponding problems in $(0, 0.5)$ and $(0.5, 1)$ in terms of the unknown values at $x = 0.5$ and matching the solutions at this point.

Tables 4 and 5 show the convergence history of the mixed finite element scheme (3.3)–(3.4) applied to a thin beam $(t = 0.01)$.

Once more, we observe from Tables 4 and 5 that the rates of convergence predicted by Remark 3.1 are attained for all the quantities. The experimental rates of convergence can also be checked in Fig. 2, where we display the corresponding error curves.

Table 4
Example 2. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for M and V.

d.o.f.	h	$\mathbf{e}_0(M)$	$\mathrm{rc}_0(M)$	$\mathbf{e}_1(M)$	$\mathrm{rc}_1(M)$	$\mathbf{e}_0(V)$	$\mathrm{rc}_0(V)$
34	0.125	4.6083e−4	–	1.4357e−2	–	1.2335e−3	–
66	0.0625	1.1605e−4	1.99	7.1723e−3	0.99	3.0852e−4	1.99
130	0.03125	2.9063e−5	2.00	3.5854e−3	1.00	7.7138e−5	2.00
258	0.015625	7.2691e−6	2.00	1.7926e−3	1.00	1.9285e−5	2.00
514	0.0078125	1.8175e−6	2.00	8.9628e−4	1.00	4.8213e−6	2.00

Table 5
Example 2. Convergence analysis for $t = 0.01$. Errors and experimental rates of convergence for V, β and w.

d.o.f.	h	$\mathbf{e}_1(V)$	$\mathrm{rc}_1(V)$	$\mathbf{e}_0(\beta)$	$\mathrm{rc}_0(\beta)$	$\mathbf{e}_0(w)$	$\mathrm{rc}_0(w)$
34	0.125	2.8322e−2	–	9.6598e−4	–	1.5040e−4	–
66	0.0625	1.4163e−2	0.99	4.9102e−4	0.98	7.3356e−5	1.02
130	0.03125	7.0817e−3	1.00	2.4652e−4	0.99	3.6417e−5	1.00
258	0.015625	3.5409e−3	1.00	1.2338e−4	1.00	1.8175e−5	1.00
514	0.0078125	1.7705e−3	1.00	6.1707e−5	1.00	9.0830e−6	1.00

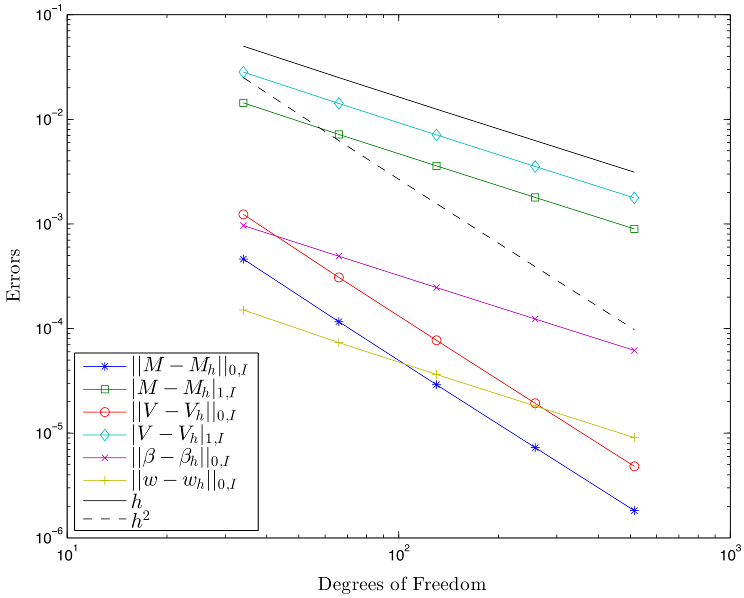

Fig. 2. Example 2. Error curves: log–log plot of errors vs. number of degrees of freedom (d.o.f.).

4.3. Example 3

Our method can also be used with boundary conditions different to those of the theoretical analysis. In what follows, we apply it to a problem studied in [8,24], where a homogeneous clamped–free beam subjected to a sinusoidally distributed moment loading is considered. We have used the same data as in the references above: length of the beam $L = 10$, width $b = 1$, Young's modulus $E = 10\,000$ and Poisson's ratio $\nu = 0.0$. The distributed moment loading has been taken $m(x) = 5t^3 \sin(\frac{1}{5}x)$, where t is the thickness of the beam.

As is claimed in [8,24], this type of loading imposes a state of pure bending with zero transverse shear strains and stresses and thus implies a pronounced tendency to shear locking. The load has been scaled with t^3 in order to be proportional to the bending stiffness, so that displacements and rotations are independent of the thickness, the exact solution of this problem can be found in [24, Section 3.2].

In Fig. 3, we present the error $\mathbf{e}_0(w)$ in dependence of the slenderness. We have considered five and ten elements (i.e., d.o.f. $= 20$ and d.o.f. $= 40$). We observe that the solution becomes slenderness independent.

In Fig. 4, we show the percentage relative error of the bending moment for a thick beam ($\frac{L}{t} = 10$) and a thin beam ($\frac{L}{t} = 100$). We observe that the two curves are superimposed, which indicates that the rate of convergence of the method is independent of the thickness of the beam. The same behavior is observed for the transverse displacement and rotation.

In Figs. 5 and 6, we report the errors of the different quantities for a thick beam ($\frac{L}{t} = 10$) and a thin beam ($\frac{L}{t} = 100$), respectively. The same orders of convergence are observed in both cases. As a consequence, we can assert once more that the method is locking free.

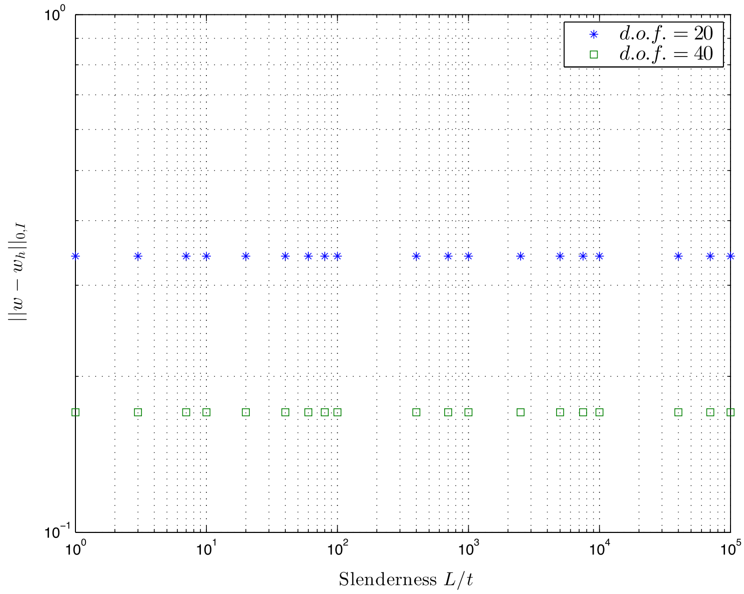

Fig. 3. Example 3. Dependence of the solution on the slenderness of the beam.

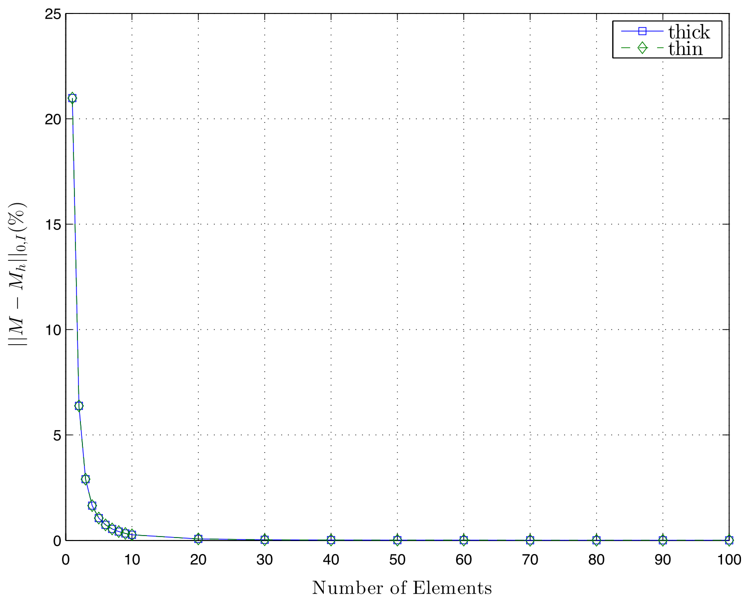

Fig. 4. Example 3. Convergence of the solution with respect to the thickness of the beam.

5. Conclusions

In the present paper we analyzed a mixed finite element method to approximate the bending of a non-homogeneous Timoshenko beam. We proposed a variational formulation in terms of the bending moment, the shear stress, the rotation and the transverse displacement, which was shown to be well posed by using the Babuška–Brezzi theory. The proofs cover the cases of non-homogeneous beams with varying geometry and physical parameters. The formulation was discretized by continuous piecewise linear (P_1) and discontinuous piecewise constant (P_0) finite elements, the former for the bending moment and the shear stress and the latter for the rotation and the transverse displacement. We proved linear convergence with respect to the mesh size in the natural norm, as well as a quadratic order for the bending moment and the shear stress in L^2, all the estimates being uniform in the beam thickness. Finally, we reported numerical results that confirm the numerical analysis of the proposed method.

A distinctive feature of this low-order method is that it allows computing directly two quantities of interest, the bending moment and the shear stress, both with quadratic order and without the need of any post-process which might produce

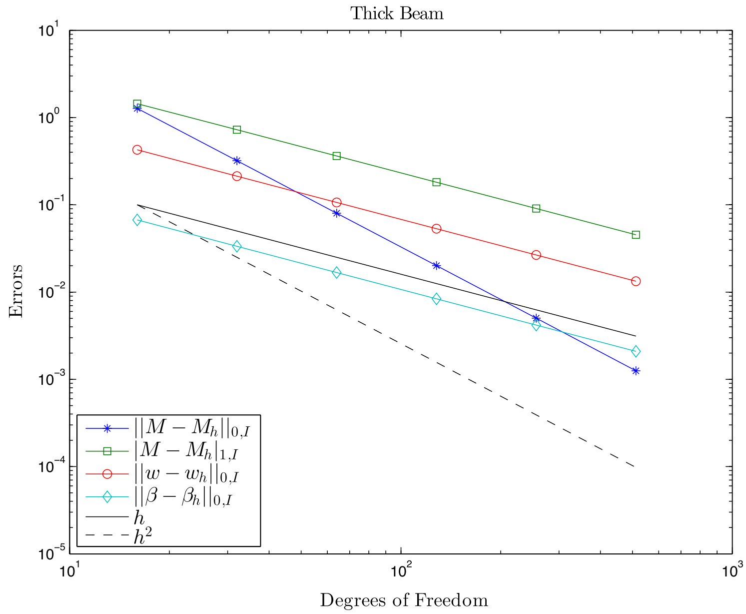

Fig. 5. Example 3. Convergence analysis for a thick beam: $\frac{L}{t} = 10$.

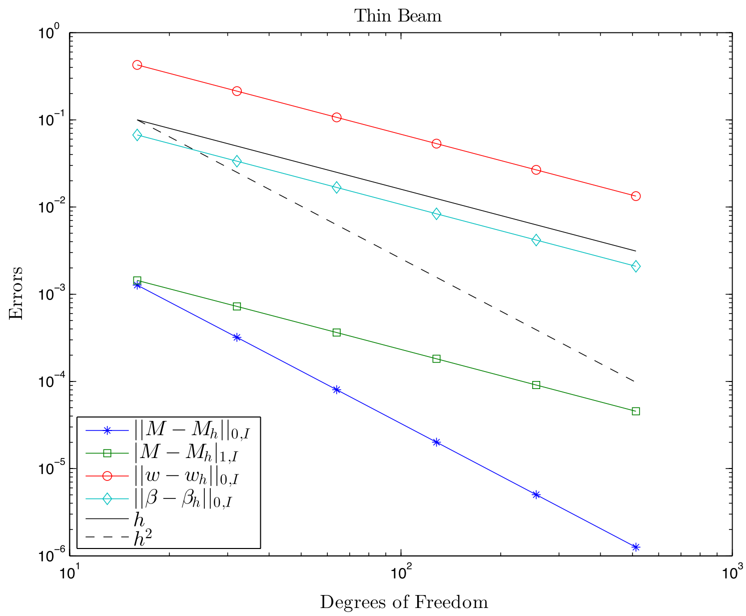

Fig. 6. Example 3. Convergence analysis for a thin beam: $\frac{L}{t} = 100$.

loss of accuracy. Moreover, the fact that these two quantities appear explicitly in the formulation could be useful to apply it to coupled problems in which the coupling involve these quantities. Furthermore, in spite of the fact that the number of unknowns doubles that of a classical lowest-order discretization, a hybridization process combined with an inexpensive static condensation lead to solving a system of similar size and sparseness as those of the classical lowest-order methods.

Finally, let us remark that, in principle, the method can be applied to Timoshenko rods in a straightforward manner. However, the corresponding theoretical analysis of convergence needs of further research.

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