SIAM J. NUMER. ANAL.
Vol. 48, No. 2, pp. 603–632

APPROXIMATION OF THE BUCKLING PROBLEM FOR REISSNER–MINDLIN PLATES*

CARLO LOVADINA†, DAVID MORA‡, AND RODOLFO RODRÍGUEZ§

Abstract. This paper deals with the approximation of the buckling coefficients and modes of a clamped plate modeled by the Reissner–Mindlin equations. These coefficients are the reciprocals of the eigenvalues of a noncompact operator. We give a spectral characterization of this operator and show that the relevant buckling coefficients correspond to isolated nondefective eigenvalues. Then we consider the numerical computation of these coefficients and their corresponding modes. For the finite element approximation of Reissner–Mindlin equations, it is well known that some kind of reduced integration or mixed interpolation has to be used to avoid locking. In particular we consider Durán–Liberman elements, which have been already proved to be locking-free for load and vibration problems. We adapt the classical approximation theory for noncompact operators to obtain optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. These estimates are valid with constants independent of the plate thickness. We report some numerical experiments confirming the theoretical results. Finally, we refine the analysis in the case of a uniformly compressed plate.

Key words. buckling, Reissner–Mindlin plates, finite elements, noncompact spectral problems

AMS subject classifications. 65N25, 65N30, 74S05, 74K20

DOI. 10.1137/090747336

1. Introduction. This paper deals with the analysis of the elastic stability of plates, in particular the so-called *buckling problem*. This problem has attracted much interest since it is frequently encountered in engineering applications such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the limit of elastic stability of the plate (i.e., eigenvalues-buckling coefficients and eigenfunctions-buckling modes).

The buckling problem has been studied for years by many researchers, with the Kirchhoff–Love and the Reissner–Mindlin plate theories the most used. For the Kirchhoff–Love theory, there exists a thorough mathematical analysis; let us mention, for instance, [5, 13, 16, 17, 18]. This is not the case for the Reissner–Mindlin theory for which only numerical experiments (cf. [15, 22]) or analytical solutions in particular cases (cf. [24]) have been reported so far. Recently, Dauge and Suri introduced in [7] the mathematical spectral analysis of a problem of this kind based on three-dimensional elasticity. In the present paper, we will perform a similar analysis for Reissner–Mindlin plates.

The Reissner–Mindlin theory is the most-used model to approximate the deformation of a thin or moderately thick elastic plate. It is very well understood that standard finite elements applied to this model lead to wrong results when the thick-

*Received by the editors January 21, 2009; accepted for publication (in revised form) April 2, 2010; published electronically May 21, 2010.
http://www.siam.org/journals/sinum/48-2/74733.html

†Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, I-27100 Pavia, Italy (carlo.lovadina@unipv.it). This author's work was supported by FONDAP and BASAL projects CMM, Universidad de Chile (Chile).

‡Departamento de Matemática, Facultad de Ciencias, Universidad del Bío Bío, Casilla 5-C, Concepción, Chile (dmora@ubiobio.cl). This author's work was supported by a CONICYT fellowship (Chile) and MECESUP UCO0406 (Chile).

§CI^{2}MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile (rodolfo@ing-mat.udec.cl). This author's work was partially supported by FONDAP and BASAL projects CMM, Universidad de Chile (Chile).

ness is small with respect to the other dimensions of the plate due to the *locking* phenomenon. Several families of methods have been rigorously shown to be free of locking and optimally convergent. We mention the recent monograph by Falk [12] for a thorough description of the state of the art and further references.

The aim of this paper is to analyze one of these methods applied to compute the buckling coefficients and buckling modes of a clamped plate. We choose the low-order, nonconforming finite elements introduced by Durán and Liberman in [11] (see also [10] for the analysis of this method applied to the plate vibration problem). However, the developed framework could be useful to analyze other methods as well.

One drawback of the Reissner–Mindlin formulation for plate buckling is the fact that the corresponding solution operator is noncompact. This is the reason why the essential spectrum no longer reduces to zero (as is the case for compact operators). This means that the spectrum may now contain nonzero eigenvalues of infinite multiplicity, accumulation points, continuous spectrum, etc. Thus, our first task is to prove that the eigenvalue corresponding to the limit of elastic stability (i.e., the smallest buckling coefficient) can be isolated from the essential spectrum, at least for sufficiently thin plates.

On the other hand, the abstract spectral theory for noncompact operators introduced by Descloux, Nassif, and Rappaz in [8, 9] cannot be directly applied to analyze the numerical method because we look for error estimates valid uniformly in the plate thickness. However, using optimal order convergence results for the Durán–Liberman elements (cf. [10, 11]) and the theoretical framework used to prove additional regularity for Reissner–Mindlin equations (cf. [1]), under the assumption that the family of meshes is *quasi-uniform*, we can adapt the theory from [8, 9] to obtain optimal order error estimates for the approximation of the buckling modes, including a double order for the buckling coefficients. Moreover, these estimates are shown to be valid with constants independent of the plate thickness, which allows us to conclude that the proposed method is locking-free.

An outline of the paper is as follows. In the next section we derive the buckling problem and introduce a noncompact linear operator whose spectrum is related with the solution of this problem. In section 3 we provide a thorough spectral characterization of this operator. In section 4 we introduce a finite element discretization of the problem based on Durán–Liberman elements and prove some auxiliary results. In section 5 we prove that the proposed numerical scheme is free of spurious modes and that optimal order error estimates hold true. In section 6 we report some numerical tests which confirm the theoretical results. We include in this section a benchmark with a known analytical solution for a simply supported plate, which shows the efficiency of the method under other kind of boundary conditions as well. Finally, in an appendix, we show that the results of sections 3, 4, and 5 can be refined when considering the particular case of a uniformly compressed plate.

Throughout the paper we will use standard notations for Sobolev spaces, norms, and seminorms. Moreover, we will denote with C a generic constant independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

2. The buckling problem. The first step will be to derive the equations for the Reissner–Mindlin plate buckling problem. With this aim, we will begin by considering the plate as a three-dimensional elastic solid, and we will write the corresponding equations for the buckling in this case. Then we will perform the dimensional reduction by means of the usual Reissner–Mindlin assumptions.

Consider a (three-dimensional) elastic plate of thickness $t > 0$ with reference configuration $\widetilde{\Omega} := \Omega \times (-\frac{t}{2}, \frac{t}{2})$, where Ω is a convex polygonal domain of $\mathbb{R}^2$ occupied by the midsection of the plate. We assume that the plate is clamped on its lateral boundary $\partial\Omega \times (-\frac{t}{2}, \frac{t}{2})$. In what follows, we summarize the arguments given in [7] to obtain the equations for the corresponding buckling problem (see this reference and also [23] for further details). We will use tildes on the quantities corresponding to the three-dimensional elastic model (as in $\widetilde{\Omega}$, for instance) to help distinguishing them form the corresponding ones in the Reissner–Mindlin model.

Suppose that $\widetilde{\boldsymbol{\sigma}}^0 := (\widetilde{\sigma}^0_{ij})_{1\le i,j\le 3}$ is a preexisting stress state in the plate. This stress $\widetilde{\boldsymbol{\sigma}}^0$ is already present in the reference configuration. It satisfies the equations of equilibrium, and it is assumed to be independent of any subsequent displacements that the reference configuration may undergo.

Let $\widetilde{V} := \{\widetilde{v} \in \mathrm{H}^1(\widetilde{\Omega})^3 : \widetilde{v} = 0 \text{ on } \partial\Omega \times (-\frac{t}{2}, \frac{t}{2})\}$ be the space of admissible displacements of the three-dimensional plate. If the reference configuration is now perturbed by a small change $\widetilde{f} \in \widetilde{V}'$ (which could be a change in loading for instance), then the work done by $\widetilde{\boldsymbol{\sigma}}^0$ cannot be neglected. The corresponding displacement $\widetilde{u} = (\widetilde{u}_i)_{1\le i\le 3}$ may be expressed as the solution of the following problem (see [7]):

Given $\widetilde{f} \in \widetilde{V}'$, find $\widetilde{u} \in \widetilde{V}$ such that

$$\int_{\widetilde{\Omega}} \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \partial_j \widetilde{u}_i\, \partial_l \widetilde{v}_k + \int_{\widetilde{\Omega}} \sum_{i,j,m=1}^{3} \widetilde{\sigma}^0_{ij}\, \partial_i \widetilde{u}_m\, \partial_j \widetilde{v}_m = \langle \widetilde{f}, \widetilde{v} \rangle \qquad \forall \widetilde{v} \in \widetilde{V}.$$

Above, $(\widetilde{C}_{ijkl})_{1\le i,j,k,l\le 3}$ is the tensor of elastic constants of the material and $\langle \cdot, \cdot \rangle$ denotes the duality between $\widetilde{V}'$ and $\widetilde{V}$. The second term in the left-hand side is the work done by $\widetilde{\boldsymbol{\sigma}}^0$.

We restrict our attention to multiples of a fixed *prebuckling stress* $\widetilde{\boldsymbol{\sigma}}$, namely,

$$\widetilde{\boldsymbol{\sigma}}^0 = -\widetilde{\lambda} \widetilde{\boldsymbol{\sigma}}.$$

Then the equation above reads

$$\int_{\widetilde{\Omega}} \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \partial_j \widetilde{u}_i\, \partial_l \widetilde{v}_k - \widetilde{\lambda} \int_{\widetilde{\Omega}} \sum_{i,j,m=1}^{3} \widetilde{\sigma}_{ij}\, \partial_i \widetilde{u}_m\, \partial_j \widetilde{v}_m = \langle \widetilde{f}, \widetilde{v} \rangle \qquad \forall \widetilde{v} \in \widetilde{V}.$$

According to [7], we will say that this problem is *stably solvable* if it has a unique solution for every $\widetilde{f} \in \widetilde{V}'$ and there exists a constant C, independent of $\widetilde{f}$, such that

$$\|\widetilde{u}\|_{\widetilde{V}} \le C \|\widetilde{f}\|_{\widetilde{V}'}.$$

Our goal will be to find the smallest value of $\widetilde{\lambda}$ for which this problem is not stably solvable. This value, which we will denote $\widetilde{\lambda}_{\mathrm{b}}$, is called the *limit of elastic stability*. Physically, it represents the smallest multiple of the prebuckling stress $\widetilde{\boldsymbol{\sigma}}$ for which a small perturbation in external conditions on the plate may cause it to buckle. As shown in [7], this can be formulated as finding the minimum positive spectral value of the following problem:

Find $\widetilde{\lambda}_{\mathrm{b}} \in \mathbb{R}$ and $0 \ne \widetilde{u} \in \widetilde{V}$ such that

$$\int_{\widetilde{\Omega}} \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \partial_j \widetilde{u}_i\, \partial_l \widetilde{v}_k = \widetilde{\lambda}_{\mathrm{b}} \int_{\widetilde{\Omega}} \sum_{i,j,m=1}^{3} \widetilde{\sigma}_{ij}\, \partial_i \widetilde{u}_m\, \partial_j \widetilde{v}_m \qquad \forall \widetilde{v} \in \widetilde{V}. \tag{2.1}$$

The eigenvalues of this problem are called the *buckling coefficients* and the eigenfunctions the *buckling modes*.

The above analysis is valid for any three-dimensional solid. In what follows we use it to derive the equations for the corresponding Reissner–Mindlin plate model. In such a case, the deformation of the plate is described by means of the rotations $\beta = (\beta_1, \beta_2)$ of the fibers initially normal to the plate midsurface and the transverse displacement w, as follows:

$$\widetilde{u}(x, y, z) = \begin{bmatrix} -z\beta_1(x, y) \\ -z\beta_2(x, y) \\ w(x, y) \end{bmatrix}. \tag{2.2}$$

The prebuckling stress $\widetilde{\boldsymbol{\sigma}}$ is assumed to arise from an elastic plane strain problem so that

$$\widetilde{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix},$$

with $\boldsymbol{\sigma}(x, y) \in \mathbb{R}^{2\times 2}$ a symmetric tensor. For the remaining arguments of this section, it is enough to consider $\boldsymbol{\sigma} \in \mathrm{L}^\infty(\Omega)^{2\times 2}$. However, we will assume some additional regularity which will be used in the forthcoming sections, namely,

$$\boldsymbol{\sigma} \in \mathrm{W}^{1,\infty}(\Omega)^{2\times 2}. \tag{2.3}$$

Notice that we do not assume $\boldsymbol{\sigma}$ to be positive definite. Avoiding such an assumption allows us to apply this approach, for instance, to shear loaded plates (cf. section 6.3). Therefore, the buckling coefficients can be in principle positive or negative, the limit of elastic stability being that of smallest absolute value.

Next we use Hooke's law with the plane stress assumption and the kinematically admissible displacements from the Reissner–Mindlin model. Thus, by substituting $\widetilde{u}$ and $\widetilde{v}$ in (2.1) by means of (2.2), using the appropriate elastic constants $\widetilde{C}_{ijkl}$, and integrating over the thickness, we obtain the following variational spectral problem (see [22] for an alternative derivation):

Find $\lambda_{\mathrm{b}} \in \mathbb{R}$ *and* $0 \neq (\beta, w) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$ *such that*

$$\begin{aligned} t^3 a(\beta, \eta) + \kappa t\,(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} \\ = \lambda_{\mathrm{b}} \Big[t\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} + t^3\,(\boldsymbol{\sigma}\nabla\beta_1, \nabla\eta_1)_{0,\Omega} + t^3\,(\boldsymbol{\sigma}\nabla\beta_2, \nabla\eta_2)_{0,\Omega} \Big] \\ \forall (\eta, v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega). \end{aligned} \tag{2.4}$$

Above, $\kappa := Ek/(2\,(1+\nu))$ is the shear modulus, with E being the Young modulus, ν the Poisson ratio, and k a correction factor (usually taken as 5/6 for clamped plates); $a(\cdot,\cdot)$ is the $\mathrm{H}^1_0(\Omega)^2$ elliptic bilinear form defined by

$$a(\beta, \eta) := \frac{E}{12\,(1-\nu^2)} \int_\Omega \left[(1-\nu)\,\boldsymbol{\varepsilon}(\beta) : \boldsymbol{\varepsilon}(\eta) + \nu \operatorname{div}\beta \operatorname{div}\eta \right],$$

where $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1\le i,j\le 2}$ is the standard strain tensor with components $\varepsilon_{ij}(\beta) := \frac{1}{2}\,(\partial_i\beta_j + \partial_j\beta_i)$, $1 \le i, j \le 2$. Finally, $(\cdot,\cdot)_{0,\Omega}$ denotes the usual L^2 inner product.

Since the terms involving the rotations β in the right-hand size of (2.4) are $O(t^3)$, they are typically negligible (see, for instance, [15, 24]). Thus, neglecting these terms, scaling the problem, and defining $\lambda := \lambda_{\mathrm{b}}/t^2$, we obtain

$$a(\beta, \eta) + \frac{\kappa}{t^2}\,(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} \qquad \forall (\eta, v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega).$$

Finally, introducing the shear stress $\gamma := \frac{\kappa}{t^2}(\nabla w - \beta)$, we arrive at the following problem.

PROBLEM 2.1. *Find* $\lambda \in \mathbb{R}$ *and* $0 \neq (\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{cases} a(\beta,\eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). \end{cases}$$

The goal of this paper is to propose and analyze a finite element method to solve Problem 2.1. In particular, our aim is to obtain accurate approximations of the smallest (in absolute value) eigenvalues λ, which correspond to the buckling coefficients $\lambda_{\mathrm{b}} = t^2\lambda$, and the associated eigenfunctions or buckling modes. For the analysis of this problem and its finite element approximation, we will rewrite it in several different forms and will consider other auxiliary problems. However, Problem 2.1 is the only one to be discretized for the numerical computations.

The first step is to obtain a thorough spectral characterization of Problem 2.1, which will be the goal of the following section. With this end we introduce the so-called *solution operator* whose spectrum is related with that of Problem 2.1. Let

$$\begin{aligned} T_t :\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega), \\ f &\mapsto w, \end{aligned} \tag{2.5}$$

where w is the second component of the solution to the following *source problem*:

Given $f \in \mathrm{H}_0^1(\Omega)$, *find* $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{cases} a(\beta,\eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). \end{cases} \tag{2.6}$$

The operator T_t is linear and bounded, and it is easy to see that (μ, w), with $\mu \neq 0$, is an eigenpair of T_t (i.e., $T_t w = \mu w$, $w \neq 0$) if and only if (λ, β, w) is a solution of Problem 2.1 with $\lambda = 1/\mu$ and a suitable $\beta \in \mathrm{H}_0^1(\Omega)^2$. Let us recall that our aim is to approximate the smallest eigenvalues of Problem 2.1, which correspond to the largest eigenvalues of the operator T_t.

To end this section we prove an additional regularity result for the solution to problem (2.6) which will be used in what follows. To do this, first we rewrite problem (2.6) in a convenient way (see [1]). Using the following Helmholtz decomposition,

$$\gamma = \nabla\psi + \operatorname{curl} p, \qquad \psi \in \mathrm{H}_0^1(\Omega),\ p \in \mathrm{H}^1(\Omega)/\mathbb{R}, \tag{2.7}$$

we have that problem (2.6) is equivalent to the following one:

Given $f \in \mathrm{H}_0^1(\Omega)$, *find* $(\psi, \beta, p, w) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}^1(\Omega)/\mathbb{R} \times \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{cases} (\nabla\psi, \nabla v)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} & \forall v \in \mathrm{H}_0^1(\Omega), \\ a(\beta,\eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla\psi, \eta)_{0,\Omega} & \forall \eta \in \mathrm{H}_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2(\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla\xi)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} + \kappa^{-1}t^2(\nabla\psi, \nabla\xi)_{0,\Omega} & \forall \xi \in \mathrm{H}_0^1(\Omega). \end{cases} \tag{2.8}$$

We recall the following result for the solution of problem (2.8) (see [1]).

THEOREM 2.1. *Let Ω be a convex polygon or a smoothly bounded domain in the plane. For any $t>0$, $\boldsymbol{\sigma}\in \mathrm{L}^{\infty}(\Omega)^{2\times 2}$, and $f\in \mathrm{H}_0^1(\Omega)$, there exists a unique solution of problem* (2.8). *Moreover, $\beta\in \mathrm{H}^2(\Omega)^2$, $p\in \mathrm{H}^2(\Omega)$, and there exists a constant C, independent of t and f, such that*

$$\|\psi\|_{1,\Omega}+\|\beta\|_{2,\Omega}+\|p\|_{1,\Omega}+t\,\|p\|_{2,\Omega}+\|w\|_{1,\Omega}\leq C\,\|f\|_{1,\Omega}\,.$$

As a consequence of Theorem 2.1 and by virtue of (2.7) and the equivalence between problems (2.6) and (2.8), we have that problem (2.6) is well-posed and there exists a constant C, independent of t and f, such that

$$\|\beta\|_{2,\Omega}+\|w\|_{1,\Omega}+\|\gamma\|_{0,\Omega}\leq C\,\|f\|_{1,\Omega}\,. \tag{2.9}$$

3. Spectral properties. The aim of this section is threefold: (i) to prove a spectral characterization for the operator T_t defined above, (ii) to study the convergence of T_t and the behavior of its spectrum as t goes to zero, and (iii) to prove additional regularity for the eigenfunctions of T_t.

3.1. Spectral characterization. As stated above, we are only interested in approximating the largest eigenvalues of T_t. However, we will show that the spectrum of this operator does not reduce to eigenvalues. In fact, T_t is not compact, and it has a nontrivial *essential spectrum*. Such essential spectrum is not relevant from the physical viewpoint, but its presence is a potential source of spectral pollution in the numerical methods (see for instance [8]).

This will not be the case for the numerical method that we will propose, thanks to the results that will be proved in this subsection, which can be summarized as follows: Although T_t has a nontrivial essential spectrum, this is confined within a small ball around the origin, which is well separated from the largest eigenvalues of T_t (that is the goal of our numerical computation). To prove this, first we recall some basic definitions from spectral theory.

Given a generic linear bounded operator $T: X\to X$, defined on a Hilbert space X, the *spectrum* of T is the set $\mathrm{Sp}(T):=\{z\in\mathbb{C}:\,(zI-T)$ is not invertible$\}$ and the *resolvent set* of T is its complement: $\rho(T):=\mathbb{C}\setminus\mathrm{Sp}(T)$. For any $z\in\rho(T)$, $R_z(T):=(zI-T)^{-1}: X\to X$ is the *resolvent operator* of T corresponding to z.

We recall the definitions of the following components of the spectrum.

- *Discrete spectrum*:
$$\mathrm{Sp_d}(T):=\{z\in\mathbb{C}:\,\mathrm{Ker}(zI-T)\neq\{0\}\text{ and }(zI-T):X\to X\text{ is Fredholm}\}\,.$$
- *Essential spectrum*:
$$\mathrm{Sp_e}(T):=\{z\in\mathbb{C}:\,(zI-T):\,X\to X\text{ is not Fredholm}\}\,.$$

The main result of this subsection is the following theorem, which provides a suitable spectral characterization for the operator T_t defined in (2.5).

THEOREM 3.1. *The spectrum of T_t decomposes as follows:* $\mathrm{Sp}(T_t)=\mathrm{Sp_d}(T_t)\cup\mathrm{Sp_e}(T_t)$ *with*

- $\mathrm{Sp_d}(T_t)$*, the discrete spectrum, which consists of real isolated eigenvalues of finite multiplicity and ascent one,*
- $\mathrm{Sp_e}(T_t)$*, the essential spectrum.*

Moreover, $\mathrm{Sp_e}(T_t)\subset\{z\in\mathbb{C}:\,|z|\leq\kappa^{-1}t^2\,\|\boldsymbol{\sigma}\|_{\infty,\Omega}\}$.

The proof of this theorem will be given at the end of this subsection. Here and thereafter, we denote $\|\boldsymbol{\sigma}\|_{\infty,\Omega} := \max_{x\in\bar{\Omega}} |\boldsymbol{\sigma}(x)|$, with $|\cdot|$ being the matrix norm induced by the standard Euclidean norm in $\mathbb{R}^2$. Notice that the maximum above is well defined because of (2.3) and the fact that $\mathrm{W}^{1,\infty}(\Omega) \subset \mathcal{C}(\bar{\Omega})$.

As a consequence of this theorem we know that, although T_t may have essential spectrum, all the points of $\mathrm{Sp}(T_t)$ outside a ball centered at the origin of the complex plane are nondefective isolated eigenvalues. Moreover, the thinner the plate, the smaller the ball containing the essential spectrum.

The proof of Theorem 3.1 will be an immediate consequence of the results that follow. Consider the following continuous bilinear forms defined in $\mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$:

$$A((\beta, w), (\eta, v)) := a(\beta, \eta) + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega}\,, \tag{3.1}$$

$$B((g, f), (\eta, v)) := (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega}\,. \tag{3.2}$$

We notice that $A(\cdot,\cdot)$ is symmetric and elliptic (cf. [4]). Moreover, from the symmetry of $\boldsymbol{\sigma}$, it follows that $B(\cdot,\cdot)$ is symmetric too. Consider the bounded linear operator

$$\begin{aligned} \widetilde{T}_t :\ \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega) &\to \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega), \\ (g, f) &\mapsto (\beta, w), \end{aligned} \tag{3.3}$$

where $(\beta, w) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$ is the solution of

$$A((\beta, w), (\eta, v)) = B((g, f), (\eta, v)) \qquad \forall (\eta, v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega).$$

We will prove in Lemma 3.4 below that the spectra of T_t and $\widetilde{T}_t$ coincide.

By virtue of the symmetry of $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$, we have

$$A(\widetilde{T}_t(g, f), (\eta, v)) = B((g, f), (\eta, v)) = B((\eta, v), (g, f)) = A((g, f), \widetilde{T}_t(\eta, v))$$

for every $(g, f), (\eta, v) \in \mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$. Therefore, $\widetilde{T}_t$ is self-adjoint with respect to the inner product $A(\cdot,\cdot)$. As a consequence, we have the following theorem (see, for instance, [7, Theorem 3.3]).

THEOREM 3.2. *The spectrum of $\widetilde{T}_t$ is real (i.e., $\mathrm{Sp}(\widetilde{T}_t) \subset \mathbb{R}$), and it decomposes as follows: $\mathrm{Sp}(\widetilde{T}_t) = \mathrm{Sp}_{\mathrm{d}}(\widetilde{T}_t) \cup \mathrm{Sp}_{\mathrm{e}}(\widetilde{T}_t)$. Finally, if $\mu \in \mathrm{Sp}_{\mathrm{d}}(\widetilde{T}_t)$, then μ is an isolated eigenvalue of finite multiplicity.*

The following result shows that the essential spectrum of $\widetilde{T}_t$ is confined in a neighborhood of the origin of diameter proportional to t^2.

PROPOSITION 3.3. *Let $\mu \in \mathrm{Sp}(\widetilde{T}_t)$ be such that $|\mu| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$. Then $\mu \in \mathrm{Sp}_{\mathrm{d}}(\widetilde{T}_t)$.*

Proof. Let $\mu \in \mathrm{Sp}(\widetilde{T}_t)$ be such that $|\mu| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$. By virtue of Theorem 3.2, we have only to prove that $(\mu\widetilde{I} - \widetilde{T}_t)$ is a Fredholm operator. To this end, it is enough to show that there exists a compact operator $\widetilde{G}$ such that $(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible. Let us introduce the operator S as follows:

$$\begin{aligned} S :\ \mathrm{H}^1_0(\Omega) &\to \mathrm{H}^1_0(\Omega)^2, \\ f &\mapsto \beta, \end{aligned}$$

where β is the first component of the the unique solution (β, w) of problem (2.6). Notice that

$$\widetilde{T}_t(g, f) = (Sf, T_t f). \tag{3.4}$$

According to (2.9), we have that $\beta \in \mathrm{H}^2(\Omega)^2$, and hence S is compact. Let us now define the operator G as follows:

$$G: \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega), \qquad f \mapsto u, \tag{3.5}$$

where $u \in \mathrm{H}_0^1(\Omega)$ is the unique solution of

$$(\nabla u, \nabla \xi)_{0,\Omega} = (Sf, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).$$

The operator G is compact as a consequence of the compactness of S. Next, we define $\widetilde{G}$ as follows:

$$\widetilde{G}: \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \qquad (g, f) \mapsto (Sf, Gf).$$

Since S and G are compact, $\widetilde{G}$ is compact, too. In addition,

$$(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})(g, f) = ((\mu g - Sf + Sf), (\mu I - T_t + G) f) = (\mu g, (\mu I - T_t + G) f).$$

Therefore, $(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible if and only if $(\mu I - T_t + G)$ is invertible.

From the fourth equation in (2.8), we notice that $v := (\mu I - T_t + G) f$ satisfies

$$\begin{aligned}(\nabla v, \nabla \xi)_{0,\Omega} &= \mu (\nabla f, \nabla \xi)_{0,\Omega} - (\nabla w, \nabla \xi)_{0,\Omega} + (\beta, \nabla \xi)_{0,\Omega} \\ &= \left(\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) \nabla f, \nabla \xi\right)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).\end{aligned}$$

Consequently, the operator $(\mu I - T_t + G)$ will be invertible if and only if, given $v \in \mathrm{H}_0^1(\Omega)$, there exists a unique $f \in \mathrm{H}_0^1(\Omega)$ solution of

$$\left(\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) \nabla f, \nabla \xi\right)_{0,\Omega} = (\nabla v, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega). \tag{3.6}$$

Now, because of the symmetry of $\boldsymbol{\sigma}(x)$, there exists an orthogonal matrix $\boldsymbol{P}(x)$ such that $\boldsymbol{\sigma}(x) = \boldsymbol{P}(x)\boldsymbol{D}(x)\boldsymbol{P}^{\mathrm{t}}(x)$, where

$$\boldsymbol{D}(x) := \begin{bmatrix} \overline{\omega}(x) & 0 \\ 0 & \underline{\omega}(x) \end{bmatrix},$$

with $\underline{\omega}(x) \leq \overline{\omega}(x)$ being the two real eigenvalues of $\boldsymbol{\sigma}(x)$. Hence, we write

$$\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) = \boldsymbol{P}(x) \begin{bmatrix} \mu - \kappa^{-1} t^2 \overline{\omega}(x) & 0 \\ 0 & \mu - \kappa^{-1} t^2 \underline{\omega}(x) \end{bmatrix} \boldsymbol{P}(x)^{\mathrm{t}}.$$

Let us denote $\omega_{\max} := \max_{x \in \bar{\Omega}} \overline{\omega}(x)$ and $\omega_{\min} := \min_{x \in \bar{\Omega}} \underline{\omega}(x)$. Since $\|\boldsymbol{\sigma}\|_{\infty,\Omega} = \max_{x \in \bar{\Omega}} |\boldsymbol{\sigma}(x)| = \max\{|\omega_{\max}|, |\omega_{\min}|\}$, for $|\mu| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$, there holds either $\mu > \kappa^{-1} t^2 \omega_{\max}$ or $\mu < \kappa^{-1} t^2 \omega_{\min}$. Hence, $\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right)$ is uniformly positive definite in the first case or uniformly negative definite in the second one. Therefore, in both cases, there exists a unique solution $f \in \mathrm{H}_0^1(\Omega)$ of (3.6). Consequently, $(\mu I - T_t + G)$ is invertible, and hence $(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible, too. Thus, we have that $(\mu\widetilde{I} - \widetilde{T}_t)$ is Fredholm, and we conclude the proof. ∎

The following result shows that T_t and $\widetilde{T}_t$ have the same spectrum.

LEMMA 3.4. *If T_t and $\widetilde{T}_t$ are the operators defined in* (2.5) *and* (3.3)*, respectively, then* $\mathrm{Sp}(\widetilde{T}_t) = \mathrm{Sp}(T_t)$.

Proof. We will prove that $\rho(\widetilde{T}_t) = \rho(T_t)$. Let z be such that $(z\widetilde{I} - \widetilde{T}_t)$ is invertible. We will prove that $(zI - T_t)$ is invertible too. By hypothesis, for every $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ there exists a unique $(g, f) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ such that

$$(z\widetilde{I} - \widetilde{T}_t)(g, f) = (\beta, w). \tag{3.7}$$

Recalling (3.4), we infer that there is a unique (g, f) such that $zg - Sf = \beta$ and $(zI - T_t)\, f = w$. Hence, we deduce that the operator $(zI - T_t) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ is surjective. Now, let us assume that there exists another $\hat{f}$ such that $(zI - T_t)\,\hat{f} = w$. Taking $\hat{g} = \frac{1}{z}(S\hat{f} + \beta)$, we have that $(z\widetilde{I} - \widetilde{T}_t)(\hat{g}, \hat{f}) = (\beta, w)$. Since by hypothesis $(z\widetilde{I} - \widetilde{T}_t)$ is invertible, from (3.7) it follows that $f = \hat{f}$. Therefore, $(zI - T_t)$ is also one-to-one and thus invertible.

Conversely, let z be such that $(zI - T_t)$ is invertible. We will prove that $(z\widetilde{I} - \widetilde{T}_t)$ is invertible too. Recalling (3.4) again, we have to show that for every $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$, there exists a unique $(g, f) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} zg - Sf = \beta, \\ zf - T_t f = w. \end{cases}$$

Let $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ be given. There exists a unique $f \in \mathrm{H}_0^1(\Omega)$ such that $(zI - T_t)\, f = w$. Therefore, taking $g := \frac{1}{z}\,(Sf + \beta)$, we obtain $(z\widetilde{I} - \widetilde{T}_t)(g, f) = (\beta, w)$. The uniqueness of g follows immediately from the uniqueness of f and the first equation of the system above. The proof is complete. $\square$

The following result shows that the eigenvalues of T_t are nondefective.

LEMMA 3.5. *Suppose that $\mu \neq 0$ is an isolated eigenvalue of T_t. Then its ascent is one.*

Proof. We prove Lemma 3.5 by contradiction. Let (μ, w) be an eigenpair of T_t, $\mu \neq 0$, and let us assume that T_t has a corresponding generalized eigenfunction, namely, there exists $\hat{w} \neq 0$ such that $T_t\hat{w} = \mu\hat{w} + w$. Since (μ, w) is an eigenpair of T_t, there exists $\beta \in \mathrm{H}_0^1(\Omega)^2$ such that (cf. (2.5) and Problem 2.1)

$$a(\beta, \eta) + \frac{\kappa}{t^2}\,(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \frac{1}{\mu}\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} \qquad \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega). \tag{3.8}$$

On the other hand, since $T_t\hat{w} = \mu\hat{w} + w$, the definition of T_t implies the existence of $\hat{\beta} \in \mathrm{H}_0^1(\Omega)^2$ such that

$$\begin{aligned} a(\hat{\beta}, \eta) + &\frac{\kappa}{t^2}(\nabla\,(w + \mu\hat{w}) - \hat{\beta}, \nabla v - \eta)_{0,\Omega} \\ &= (\boldsymbol{\sigma}\nabla\hat{w}, \nabla v)_{0,\Omega}\,\forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega). \end{aligned}$$

Defining $\bar{\beta} := (\hat{\beta} - \beta)/\mu$, the equation above can be written as follows:

$$\mu a(\bar{\beta}, \eta) + a(\beta, \eta) + \frac{\kappa\mu}{t^2}(\nabla\hat{w} - \bar{\beta}, \nabla v - \eta)_{0,\Omega} + \frac{\kappa}{t^2}(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla\hat{w}, \nabla v)_{0,\Omega}.$$

We now take $(\eta, v) = \mu(\bar{\beta}, \hat{w})$ in (3.8) and $(\eta, v) = (\beta, w)$ in the equation above and subtract the resulting equations. Using also the symmetry of $a(\cdot, \cdot)$ and $\boldsymbol{\sigma}$, we obtain

$$a(\beta, \beta) + \frac{\kappa}{t^2}\,\|\nabla w - \beta\|_{0,\Omega}^2 = 0.$$

Thus, from the ellipticity of $a(\cdot,\cdot)$, we infer $\beta = 0$ and hence $w = 0$, which is a contradiction since w is an eigenfunction of T_t. The proof is complete. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof follows easily by combining Lemma 3.4 with Theorem 3.2, Proposition 3.3, and Lemma 3.5. $\square$

3.2. Limit problem. In this subsection we study the convergence properties of the operator T_t as t goes to zero. First, let us recall that it is well known (see [4]) that, when t goes to zero, the solution (β, w, γ) of problem (2.6) converges to the solution of the following problem:

Given $f \in \mathrm{H}_0^1(\Omega)$, *find* $(\beta_0, w_0, \gamma_0) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0(\mathrm{rot};\Omega)'$ *such that*

$$\begin{cases} a(\beta_0, \eta) + \langle \gamma_0, \nabla v - \eta \rangle = (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \nabla w_0 - \beta_0 = 0. \end{cases} \tag{3.9}$$

Above, $\langle \cdot, \cdot \rangle$ stands now for the duality pairing in $\mathrm{H}_0(\mathrm{rot};\Omega)$. Problem (3.9) is a mixed formulation for the following well-posed problem, which corresponds to the buckling of a Kirchhoff plate:

Given $f \in \mathrm{H}_0^1(\Omega)$, *find* $w_0 \in \mathrm{H}_0^2(\Omega)$ *such that*

$$\frac{E}{12\,(1-\nu^2)}\,(\Delta w_0, \Delta v)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega} \qquad \forall v \in \mathrm{H}_0^2(\Omega). \tag{3.10}$$

Let T_0 be the bounded linear operator defined by

$$\begin{aligned} T_0 :\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega), \\ f &\mapsto w_0, \end{aligned}$$

where w_0 is the second component of the solution of problem (3.9). Since $w_0 \in \mathrm{H}_0^2(\Omega)$, the operator T_0 is compact. Hence, apart from $\mu_0 = 0$, the spectrum of T_0 consists of a sequence of finite multiplicity isolated eigenvalues converging to zero. The following lemma, which yields the convergence in norm of T_t to T_0 has been essentially proved in [10, Lemma 3.1].

LEMMA 3.6. *There exists a constant C, independent of t, such that*

$$\|(T_t - T_0)\, f\|_{1,\Omega} \le Ct\, \|f\|_{1,\Omega} \qquad \forall f \in \mathrm{H}_0^1(\Omega).$$

As a consequence of this lemma, standard properties about the separation of isolated parts of the spectrum (see [14] for instance) yield the following result.

LEMMA 3.7. *Let $\mu_0 \neq 0$ be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0. Then there exists $t_0 > 0$ such that, $\forall t < t_0$, D contains exactly m isolated eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each nonzero eigenvalue μ_0 of T_0 is a limit of isolated eigenvalues μ_t of T_t as t goes to zero.*

Our next goal is to show that the largest eigenvalues of T_t converge to the largest eigenvalues of T_0 as t goes to zero. With this aim, we prove first the following lemma. Here and thereafter, we will use $\|\cdot\|$ to denote the operator norm induced by the $\mathrm{H}^1(\Omega)$ norm.

LEMMA 3.8. *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. Then there exist strictly positive constants t_0 and C such that, $\forall t < t_0$, $F \cap \mathrm{Sp}(T_t) = \emptyset$ and*

$$\|R_z(T_t)\| := \sup_{\substack{w \in \mathrm{H}_0^1(\Omega) \\ w \neq 0}} \frac{\|R_z(T_t) w\|_{1,\Omega}}{\|w\|_{1,\Omega}} \le C \qquad \forall z \in F.$$

Proof. The mapping $z \mapsto \| (zI - T_0)^{-1} \|$ is continuous for all $z \in \rho(T_0)$ and goes to zero as $|z| \to \infty$. Consequently, it attains its maximum on any closed subset $F \subset \rho(T_0)$. Let $C_1 := 1/\max_{z\in F} \| (zI - T_0)^{-1} \|$; there holds

$$\|(zI - T_0)\, w\|_{1,\Omega} \geq \frac{1}{C_1} \|w\|_{1,\Omega} \qquad \forall w \in \mathrm{H}_0^1(\Omega) \quad \forall z \in F.$$

Now, according to Lemma 3.6, there exists $t_1 > 0$ such that, for all $t < t_1$,

$$\|(T_t - T_0)\, w\|_{1,\Omega} \leq \frac{1}{2C_1} \|w\|_{1,\Omega} \qquad \forall w \in \mathrm{H}_0^1(\Omega).$$

Therefore, for all $w \in \mathrm{H}_0^1(\Omega)$, for all $z \in F$, and for all $t < t_1$,

$$\|(zI - T_t)\, w\|_{1,\Omega} \geq \|(zI - T_0)\, w\|_{1,\Omega} - \|(T_t - T_0)\, w\|_{1,\Omega} \geq \frac{1}{2C_1} \|w\|_{1,\Omega} \tag{3.11}$$

and, consequently, $z \notin \mathrm{Sp_d}(T_t)$.

On the other hand, $d := \min_{z\in F} |z|$ is strictly positive because $\mathrm{Sp}(T_0) \ni 0$, $F \cap \mathrm{Sp}(T_0) = \emptyset$, and F is closed. Let $t_2 > 0$ be such that $\kappa^{-1} t_2^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega} < d$. Hence, for all $z \in F$ and for all $t < t_2$, we have $|z| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$ and, consequently, by virtue of Theorem 3.1, either $z \in \mathrm{Sp_d}(T_t)$ or $z \notin \mathrm{Sp}(T_t)$.

Altogether, if $t_0 := \min\{t_1, t_2\}$, then $(zI - T_t)$ is invertible for all $t < t_0$ and all $z \in F$. Moreover, because of (3.11),

$$\|R_z(T_t)\| = \| (zI - T_t)^{-1} \| \leq 2C_1,$$

and we conclude the proof. $\square$

It is easy to show that the spectrum of T_0 is real; in fact, this follows readily from the symmetric formulation (3.10). Since T_0 is compact, its nonzero eigenvalues are isolated and of finite multiplicity so that we can order the positive ones as follows:

$$\mu_0^{(1)} \geq \mu_0^{(2)} \geq \cdots \geq \mu_0^{(k)} \geq \cdots ,$$

where each eigenvalue is repeated as many times as its corresponding multiplicity. A similar ordering holds for the negative eigenvalues, too, if they exist.

According to Lemma 3.7, for t sufficiently small there exist eigenvalues of T_t close to each $\mu_0^{(k)}$. On the other hand, according to Theorem 3.1, the essential spectrum of T_t is confined within a ball centered at the origin of the complex plane with radius proportional to t^2. Therefore, at least for t sufficiently small, the points of the spectrum of T_t largest in modulus have to be isolated eigenvalues of finite multiplicity. Since the spectrum of T_t is also real, we order the positive eigenvalues as we did with those of T_0:

$$\mu_t^{(1)} \geq \mu_t^{(2)} \geq \cdots \geq \mu_t^{(k)} \geq \cdots .$$

Once more, a similar ordering holds for the negative eigenvalues of T_t if they exist.

The following theorem shows that the kth positive eigenvalue of T_t converges to the kth positive eigenvalue of T_0 as t goes to zero. A similar result holds for the negative eigenvalues as well.

THEOREM 3.9. *Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \geq 0$, be as defined above. For all $k \in \mathbb{N}$, $\mu_t^{(k)} \to \mu_0^{(k)}$ as $t \to 0$.*

Proof. We will prove the result for the largest eigenvalue $\mu_t^{(1)}$. The proof for the others is a straightforward modification of this one.

Let D be an open disk in the complex plane centered at $\mu_0^{(1)}$ with radius $r < [\mu_0^{(1)} - \mu_0^{(k)}]/2$, where $\mu_0^{(k)}$ is the largest eigenvalue of T_0 satisfying $\mu_0^{(k)} < \mu_0^{(1)}$. Therefore, $D \cap \operatorname{Sp}(T_0) = \{\mu_0^{(1)}\}$.

Let H be the half-plane $\{z \in \mathbb{C} : \operatorname{Re}(z) < [\mu_0^{(k)} + \mu_0^{(1)}]/2\}$. Hence $\operatorname{Sp}(T_0) \subset D \cup H$. Let $F := \mathbb{C} \setminus (D \cup H)$. The set F is closed, and $F \cap \operatorname{Sp}(T_0) = \emptyset$. Hence, according to Lemma 3.8, there exists $t_0 > 0$ such that, for all $t < t_0$, $F \cap \operatorname{Sp}(T_t) = \emptyset$, too, and hence $\operatorname{Sp}(T_t) \subset D \cup H$ as well.

On the other hand, because of Lemma 3.7, there exists $t_1 > 0$ such that, for all $t < t_1$, D contains as many eigenvalues of T_t as the multiplicity of $\mu_0^{(1)}$. Therefore, for all $t < \min\{t_0, t_1\}$, the largest eigenvalue of T_t, $\mu_t^{(1)}$, has to lie in D. Since D can be taken arbitrarily small, we conclude that $\mu_t^{(1)}$ converges to $\mu_0^{(1)}$ as t goes to zero. Thus, we conclude the proof. ∎

3.3. Additional regularity of the eigenfunctions. The aim of this subsection is to prove a regularity result for the eigenfunctions of Problem 2.1. More precisely, we have the following proposition.

PROPOSITION 3.10. *Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \geq 0$, be as in Theorem* 3.9. *Let $(\lambda, \beta, w, \gamma)$ be a solution of Problem* 2.1 *with $\lambda = 1/\mu_t^{(k)}$. Then there exists $t_0 > 0$ such that, for all $t < t_0$, $\beta \in \mathrm{H}^2(\Omega)^2$, $w \in \mathrm{H}^2(\Omega)$, $\operatorname{div}\gamma \in \mathrm{L}^2(\Omega)$, and there holds*

$$\|\beta\|_{2,\Omega} \leq C\,|\lambda|\,\|w\|_{1,\Omega}\,, \tag{3.12}$$

$$\|w\|_{2,\Omega} \leq C\,|\lambda|\,\|w\|_{1,\Omega}\,, \tag{3.13}$$

$$\|\operatorname{div}\gamma\|_{0,\Omega} \leq C\,|\lambda|\,\|w\|_{2,\Omega}\,, \tag{3.14}$$

with C a positive constant independent of t.

Proof. Using the Helmholtz decomposition (2.7), Problem 2.1 is equivalent to finding $\lambda \in \mathbb{R}$ and $0 \neq (\psi, \beta, p, w) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}^1(\Omega)/\mathbb{R} \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} (\nabla\psi, \nabla v)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} & \forall v \in \mathrm{H}_0^1(\Omega), \\ a(\beta, \eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla\psi, \eta)_{0,\Omega} & \forall \eta \in \mathrm{H}_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2\,(\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla\xi)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} + \kappa^{-1}t^2\,(\nabla\psi, \nabla\xi)_{0,\Omega} & \forall \xi \in \mathrm{H}_0^1(\Omega). \end{cases}$$

From Theorem 2.1 applied to the problem above, we immediately obtain that $\beta \in \mathrm{H}^2(\Omega)^2$ and the estimate (3.12).

On the other hand, the first and the last equations of the system above lead to

$$\left(\left(\boldsymbol{I} - \lambda\kappa^{-1}t^2\boldsymbol{\sigma}\right)\nabla w, \nabla\xi\right)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).$$

Since $\mu_t^{(k)} \to \mu_0^{(k)} > 0$ as $t \to 0$, there exists $t_1 > 0$ such that $\mu_t^{(k)} > \mu_0^{(k)}/2\ \ \forall t < t_1$. Hence $\lambda = 1/\mu_t^{(k)} < 2/\mu_0^{(k)}$. We take $t_0 < t_1$ such that $\kappa^{-1}t_0^2\,\|\boldsymbol{\sigma}\|_{\infty,\Omega} < \mu_0^{(k)}/2$. Therefore, for all $t < t_0$, $\left(\boldsymbol{I} - \lambda\kappa^{-1}t^2\boldsymbol{\sigma}\right)$ is uniformly positive definite. Thus, since w is the solution of the problem

$$\begin{cases} \operatorname{div}\left[\left(\boldsymbol{I} - \lambda\kappa^{-1}t^2\boldsymbol{\sigma}\right)\nabla w\right] = \operatorname{div}\beta & \text{in } \Omega, \\ w = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

using a standard regularity result (see [21]), we have that $w \in \mathrm{H}^2(\Omega)$ and

$$\|w\|_{2,\Omega} \leq C \|\operatorname{div} \beta\|_{0,\Omega} \leq C \|\beta\|_{1,\Omega} \leq C |\lambda| \|w\|_{1,\Omega},$$

the last inequality because of (3.12).

Furthermore, taking $\eta = 0$ in Problem 2.1, using the estimate above, and (2.3), it follows that

$$\operatorname{div} \gamma = \lambda \operatorname{div}(\boldsymbol{\sigma} \nabla w) \in \mathrm{L}^2(\Omega)$$

and

$$\|\operatorname{div} \gamma\|_{0,\Omega} \leq C |\lambda| \|w\|_{2,\Omega}.$$

The proof is complete. $\square$

Once more a similar result holds for negative eigenvalues $\mu_t^{(k)} \to \mu_0^{(k)} < 0$.

4. Spectral approximation. For the numerical approximation, we focus on the finite element method proposed and studied in [11]. In what follows we introduce briefly this method (see this reference for further details). Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangular meshes of $\bar{\Omega}$. We will define finite element spaces H_h, W_h, and Γ_h for the rotations, the transverse displacements, and the shear stress, respectively.

For $K \in \mathcal{T}_h$, let α_1, α_2, α_3 be its barycentric coordinates. We denote by τ_i a unit vector tangent to the edge $\alpha_i = 0$ and define

$$p_1^K = \alpha_2 \alpha_3 \tau_1, \qquad p_2^K = \alpha_1 \alpha_3 \tau_2, \qquad p_3^K = \alpha_1 \alpha_2 \tau_3.$$

The finite element space for the rotations is defined by

$$H_h := \left\{ \eta_h \in \mathrm{H}_0^1(\Omega)^2 : \eta_h|_K \in \mathbb{P}_1^2 \oplus \langle p_1^K, p_2^K, p_3^K \rangle \quad \forall K \in \mathcal{T}_h \right\}.$$

To approximate the transverse displacements, we use the usual piecewise-linear continuous finite element space

$$W_h := \left\{ v_h \in \mathrm{H}_0^1(\Omega) : v_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\}.$$

Finally, for the shear stress, we use the lowest-order rotated Raviart–Thomas space

$$\Gamma_h := \left\{ \phi \in \mathrm{H}_0(\mathrm{rot}; \Omega) : \phi|_K \in \mathbb{P}_0^2 \oplus (x_2, -x_1)\mathbb{P}_0 \quad \forall K \in \mathcal{T}_h \right\}.$$

We consider as reduction operator the rotated Raviart–Thomas interpolant

$$R : \mathrm{H}^1(\Omega)^2 \cap \mathrm{H}_0(\mathrm{rot}; \Omega) \to \Gamma_h,$$

which is uniquely determined by

$$\int_\ell R\phi \cdot \tau_\ell = \int_\ell \phi \cdot \tau_\ell$$

for every edge ℓ of the triangulation, τ_ℓ being a unit vector tangent to ℓ. It is well known that

$$\|R\phi\|_{0,\Omega} \leq C \|\phi\|_{1,\Omega} \qquad \forall \phi \in \mathrm{H}^1(\Omega)^2, \tag{4.1}$$

$$\|\phi - R\phi\|_{0,\Omega} \leq Ch \|\phi\|_{1,\Omega} \qquad \forall \phi \in \mathrm{H}^1(\Omega)^2. \tag{4.2}$$

Moreover, the operator R can be extended continuously to $\mathrm{H}^s(\Omega)^2 \cap \mathrm{H}_0(\mathrm{rot};\Omega)$ for any $s > 0$, and it is also well known that, for all $v \in \mathrm{H}^{1+s}(\Omega) \cap \mathrm{H}_0^1(\Omega)$,

$$R(\nabla v) = \nabla v_{\mathrm{I}}, \tag{4.3}$$

where $v_{\mathrm{I}} \in W_h$ is the standard piecewise-linear Lagrange interpolant of v (which is well defined because $\mathrm{H}^{1+s}(\Omega) \subset \mathcal{C}(\bar{\Omega})\ \forall s > 0$).

The discretization of Problem 2.1 reads as follows.

PROBLEM 4.1. *Find $\lambda_h \in \mathbb{R}$ and $0 \neq (\beta_h, w_h) \in H_h \times W_h$ such that*

$$\begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda_h \left(\boldsymbol{\sigma}\nabla w_h, \nabla v_h\right)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2}\left(\nabla w_h - R\beta_h\right). \end{cases}$$

Notice that this leads to a nonconforming method, since consistency terms arise because of the reduction operator R. The final goal of this paper is to prove that the smallest (in absolute value) eigenvalues λ_h converge to the smallest (in absolute value) eigenvalues λ of Problem 2.1. We will also prove convergence of the corresponding eigenfunctions and error estimates.

Our first step is to obtain a characterization of the solutions to Problem 4.1.

LEMMA 4.1. *Let $Y_h := \{w_h \in W_h : (\boldsymbol{\sigma}\nabla w_h, \nabla v_h)_{0,\Omega} = 0 \quad \forall v_h \in W_h\}$. Then Problem 4.1 has exactly $\dim W_h - \dim Y_h$ eigenvalues, repeated according to their respective multiplicities. All of them are real and nonzero.*

Proof. We eliminate γ_h in Problem 4.1 to write it as follows:

$$\begin{aligned} a(\beta_h, \eta_h) + &\frac{\kappa}{t^2}\left(\nabla w_h - R\beta_h, \nabla v_h - R\eta_h\right)_{0,\Omega} \\ &= \lambda_h \left(\boldsymbol{\sigma}\nabla w_h, \nabla v_h\right)_{0,\Omega} \forall (\eta_h, v_h) \in H_h \times W_h. \end{aligned} \tag{4.4}$$

Taking particular bases of H_h and W_h, this problem can be written in matrix form as follows:

$$\boldsymbol{\mathcal{A}} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix} = \lambda_h \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix}, \tag{4.5}$$

where $\boldsymbol{\beta}_h$ and $\boldsymbol{w}_h$ denote the vectors whose entries are the components in those basis of β_h and w_h, respectively. The matrix $\boldsymbol{\mathcal{A}}$ is symmetric and positive definite because the bilinear form on the left-hand side of (4.4) is elliptic in $\mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ (cf. [11]). Consequently, $\lambda_h \neq 0$ and, since $\boldsymbol{E}$ is also symmetric, $\lambda_h \in \mathbb{R}$. Now, (4.5) holds true if and only if

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix} = \mu_h \boldsymbol{\mathcal{A}} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix}$$

with $\lambda_h = 1/\mu_h$ and $\mu_h \neq 0$. The latter is a well-posed generalized eigenvalue problem with $\dim W_h - \dim \mathrm{Ker}(\boldsymbol{E})$ nonzero eigenvalues. Thus, we conclude the lemma by noting that $\boldsymbol{E}\boldsymbol{w}_h = \mathbf{0}$ if and only if $w_h \in Y_h$. $\square$

Remark 4.1. If $(\lambda_h, \beta_h, w_h)$ is a solution of Problem 4.1, then

$$\boldsymbol{w}_h^{\mathrm{t}} \boldsymbol{E} \boldsymbol{w}_h = \left(\boldsymbol{\sigma}\nabla w_h, \nabla w_h\right)_{0,\Omega} \neq 0.$$

In fact, this follows by left multiplying both sides of (4.5) by $(\boldsymbol{\beta}_h^{\mathrm{t}}, \boldsymbol{w}_h^{\mathrm{t}})$ and using the positive definiteness of $\boldsymbol{\mathcal{A}}$.

As in the continuous case, we introduce for the analysis the *discrete solution operator*

$$\begin{aligned} T_{th} : \mathrm{H}_0^1(\Omega) &\to W_h \hookrightarrow \mathrm{H}_0^1(\Omega), \\ f &\mapsto w_h, \end{aligned}$$

where w_h is the second component of the solution (β_h, w_h) to the corresponding *discrete source problem*:

Given $f \in \mathrm{H}_0^1(\Omega)$, *find* $(\beta_h, w_h) \in H_h \times W_h$ *such that*

(4.6)

$$\begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2}\left(\nabla w_h - R\beta_h\right). \end{cases}$$

The existence and the uniqueness of the solution to problem (4.6) follow easily (see [11]). Moreover, the nonzero eigenvalues of T_{th} are given by $\mu_h := 1/\lambda_h$, with λ_h being the eigenvalues of Problem 4.1, and the corresponding eigenfunctions coincide.

Remark 4.2. The solution to (4.6) is a finite element approximation of the solution to (2.6). However, given a generic $f \in \mathrm{H}_0^1(\Omega)$, the usual convergence rate in terms of positive powers of the mesh-size h does not hold in this case because the solution to (2.6) is not sufficiently smooth. Indeed, the right-hand side is not regular enough, since $\operatorname{div}(\boldsymbol{\sigma}\nabla f) \notin \mathrm{L}^2(\Omega)$. Now, whenever f is more regular, for instance assuming $f \in \mathrm{H}^2(\Omega)$, by taking into account the regularity of $\boldsymbol{\sigma}$ (cf. (2.3)), the convergence results of [11] can be applied to obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t\,\|\gamma - \gamma_h\|_{0,\Omega} + \|w - w_h\|_{1,\Omega} \le Ch\,\|f\|_{2,\Omega}.$$

4.1. Auxiliary results. In what follows we will prove several auxiliary results which will be used in the following section to prove convergence and error estimates for our spectral approximation. The first of them is the following lemma, which shows that the operator T_{th} defined above is bounded uniformly in t and h.

LEMMA 4.2. *There exists* $C > 0$ *such that* $\|T_{th}\| \le C$ $\forall t > 0$ *and all* $h > 0$.

Proof. Let $f \in \mathrm{H}_0^1(\Omega)$ and (β_h, w_h) be the solution to problem (4.6). Taking $(\eta_h, v_h) = (\beta_h, w_h)$ as test function in (4.6), we obtain

$$a(\beta_h, \beta_h) + \kappa^{-1}t^2\,\|\gamma_h\|_{0,\Omega}^2 \le \|\boldsymbol{\sigma}\|_{\infty,\Omega}\,\|\nabla f\|_{0,\Omega}\,\|\nabla w_h\|_{0,\Omega}.$$

Hence, from the ellipticity of $a(\cdot,\cdot)$,

$$\|\beta_h\|_{1,\Omega}^2 + \kappa^{-1}t^2\,\|\gamma_h\|_{0,\Omega}^2 \le C\,\|\boldsymbol{\sigma}\|_{\infty,\Omega}\,\|\nabla f\|_{0,\Omega}\,\|\nabla w_h\|_{0,\Omega}.$$

Therefore, using the definition of γ_h (cf. (4.6)) and (4.1),

$$\|\nabla w_h\|_{0,\Omega}^2 = \left\|\kappa^{-1}t^2\gamma_h + R\beta_h\right\|_{0,\Omega}^2 \le C\,\|\boldsymbol{\sigma}\|_{\infty,\Omega}\,\|\nabla f\|_{0,\Omega}\,\|\nabla w_h\|_{0,\Omega},$$

which allows us to conclude the proof. ∎

Next, we will adapt the theory developed in [8, 9] for noncompact operators to our case. With this aim, we will prove the following properties:

P1. $\displaystyle \|T_0 - T_{th}\|_h := \sup_{\substack{f_h \in W_h \\ f_h \neq 0}} \frac{\|(T_0 - T_{th})\,f_h\|_{1,\Omega}}{\|f_h\|_{1,\Omega}} \to 0 \qquad \text{as } (h,t) \to (0,0);$

P2. $\displaystyle \forall u \in \mathrm{H}_0^1(\Omega) \qquad \inf_{v_h \in W_h} \|u - v_h\|_{1,\Omega} \to 0 \qquad \text{as } h \to 0.$

From now on, we will use the operator norm $\|\cdot\|_h$ as defined in property P1.

We focus on property P1, since property P2 follows from standard approximation results. We notice first that

$$\|T_0 - T_{th}\|_h \leq \|T_0 - T_t\|_h + \|T_t - T_{th}\|_h \,, \tag{4.7}$$

where T_t is the operator defined in (2.5). Since $W_h \subset \mathrm{H}_0^1(\Omega)$, from Lemma 3.6 we deduce that for all $h > 0$

$$\|T_0 - T_t\|_h \leq Ct. \tag{4.8}$$

Regarding the other term in the right-hand side of (4.7), we aim at proving the following result.

PROPOSITION 4.3. *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Then we have*

$$\|T_t - T_{th}\|_h \leq C\,(h+t)\,.$$

The proof of Proposition 4.3 will be given at the end of this section. With this aim, we consider problems (2.6) and (4.6) with source term in W_h:

Given $f_h \in W_h$, find $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} a(\beta,\eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f_h, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}\,(\nabla w - \beta)\,. \end{cases} \tag{4.9}$$

Given $f_h \in W_h$, find $(\beta_h, w_h) \in H_h \times W_h$ such that

$$\begin{cases} a(\beta_h,\eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f_h, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2}\,(\nabla w_h - R\beta_h)\,. \end{cases} \tag{4.10}$$

We need some results concerning the solutions of these problems. First, we apply the Helmholtz decomposition (2.7) to the term γ from (4.9):

$$\gamma = \nabla\psi + \operatorname{curl} p, \qquad \psi \in \mathrm{H}_0^1(\Omega),\ p \in \mathrm{H}^1(\Omega)/\mathbb{R}. \tag{4.11}$$

Then we apply Theorem 2.1 and (2.9) to obtain the following a priori estimate for the solution to problem (4.9):

$$\|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|p\|_{1,\Omega} + t\,\|p\|_{2,\Omega} + \|\gamma\|_{0,\Omega} \leq C\,\|f_h\|_{1,\Omega}\,. \tag{4.12}$$

The following result shows that, for $f_h \in W_h$, w and ψ are actually smoother. Furthermore, we establish an inverse estimate which will be used to prove Proposition 4.3.

LEMMA 4.4. *Let w be defined by problem (4.9) and ψ as in (4.11). Then $w, \psi \in \mathrm{H}^{1+s}(\Omega)$ $\forall s \in (0, \frac{1}{2})$. Moreover, if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, then*

$$\|\psi\|_{1+s,\Omega} \leq Ch^{-s}\,\|f_h\|_{1,\Omega}\,.$$

Proof. Recall the equivalence between problems (4.9) and (2.8), the latter with source term f_h instead of f. From the first equation of (2.8) we have that ψ is the weak solution of

$$\begin{cases} \Delta\psi = \operatorname{div}(\boldsymbol{\sigma}\nabla f_h) \in \mathrm{H}^{-1}(\Omega), \\ \psi = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

Since f_h is a continuous piecewise linear function, we have that $f_h \in \mathrm{H}^{1+s}(\Omega)\ \forall s \in (0, \frac{1}{2})$. Therefore, the assumption (2.3) implies $\boldsymbol{\sigma}\nabla f_h \in \mathrm{H}^s(\Omega)^2$. Hence, $\mathrm{div}(\boldsymbol{\sigma}\nabla f_h) \in \mathrm{H}^{s-1}(\Omega)$. Then, from standard regularity results for problem (4.13), $\psi \in \mathrm{H}^{1+s}(\Omega)$ $\forall s \in (0, \frac{1}{2})$ and

$$\|\psi\|_{1+s,\Omega} \leq C \|\mathrm{div}(\boldsymbol{\sigma}\nabla f_h)\|_{s-1,\Omega} \leq C \|f_h\|_{1+s,\Omega} .$$

If the family of meshes is quasi-uniform, then the inverse inequality $\|f_h\|_{1+s,\Omega} \leq Ch^{-s} \|f_h\|_{1,\Omega}$ holds true, and from this and the estimate above we obtain

$$\|\psi\|_{1+s,\Omega} \leq Ch^{-s} \|f_h\|_{1,\Omega} .$$

On the other hand, from the last equation of (2.8) we have that

$$\left(\nabla \left(w - \kappa^{-1}t^2\psi\right), \nabla\xi\right)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} \qquad \forall\xi \in \mathrm{H}_0^1(\Omega).$$

Therefore, $\left(w - \kappa^{-1}t^2\psi\right)$ is the weak solution to the problem

$$\begin{cases} \Delta \left(w - \kappa^{-1}t^2\psi\right) = \mathrm{div}\,\beta \in \mathrm{L}^2(\Omega), \\ \left(w - \kappa^{-1}t^2\psi\right) = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Hence, $\left(w - \kappa^{-1}t^2\psi\right) \in \mathrm{H}^2(\Omega)$ (recall Ω is convex) and $w = \left(w - \kappa^{-1}t^2\psi\right) + \kappa^{-1}t^2\psi \in \mathrm{H}^{1+s}(\Omega)$ $\forall s \in (0, \frac{1}{2})$. Thus the proof is complete. ◻

The following lemma is the key point to prove Proposition 4.3.

LEMMA 4.5. *If* (β, w, γ) *and* (β_h, w_h, γ_h) *as in* (4.9) *and* (4.10), *respectively, then*

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \leq C (h + t) \|f_h\|_{1,\Omega} .$$

Proof. It has been proved in [11] (see Example 4.1 from this reference) that there exists $\widetilde{\beta} \in H_h$ satisfying

$$R\widetilde{\beta} = R\beta,$$
$$\|\beta - \widetilde{\beta}\|_{1,\Omega} \leq Ch \|\beta\|_{2,\Omega} .$$

Let

$$\widetilde{\gamma} := \frac{\kappa}{t^2}(\nabla w_{\mathrm{I}} - R\widetilde{\beta}),$$

where $w_{\mathrm{I}} \in W_h$ is the Lagrange interpolant of w, which is well defined because of Lemma 4.4. Notice that by virtue of (4.3) and the equation above,

$$\widetilde{\gamma} = R\gamma.$$

It has also been proved in [11] that

$$\|\widetilde{\beta} - \beta_h\|_{1,\Omega} + t \|\widetilde{\gamma} - \gamma_h\|_{0,\Omega} \leq C \left(\|\widetilde{\beta} - \beta\|_{1,\Omega} + t \|\widetilde{\gamma} - \gamma\|_{0,\Omega} + h \|\gamma\|_{0,\Omega}\right).$$

Hence, by adding and subtracting $\widetilde{\beta}$ and $\widetilde{\gamma} = R\gamma$, we obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \leq C \left(\|\beta - \widetilde{\beta}\|_{1,\Omega} + t \|\gamma - R\gamma\|_{0,\Omega} + h \|\gamma\|_{0,\Omega}\right).$$

The first and last term in the right-hand side above are already bounded. To estimate the second one, we use (4.11), Lemma 4.4, and (4.3) to obtain

$$\|\gamma - R\gamma\|_{0,\Omega} \leq \|\nabla\psi - \nabla\psi_{\mathrm{I}}\|_{0,\Omega} + \|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega}\,. \tag{4.14}$$

Next, from standard error estimates for the Lagrange interpolant, we have that

$$\|\nabla\psi - \nabla\psi_{\mathrm{I}}\|_{0,\Omega} \leq Ch^s \|\psi\|_{1+s,\Omega}\,,$$

whereas from (4.2) and the fact that $p \in \mathrm{H}^2(\Omega)$ (cf. (4.12))

$$\|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega} \leq Ch \|p\|_{2,\Omega}\,.$$

Thus, by using Lemma 4.4, we conclude that

$$\begin{aligned}\|\beta - \beta_h\|_{1,\Omega} + t\|\gamma - \gamma_h\|_{0,\Omega} &\leq C\left(h\|\beta\|_{2,\Omega} + t\|f_h\|_{1,\Omega} + th\|p\|_{2,\Omega} + h\|\gamma\|_{0,\Omega}\right)\\ &\leq C(h+t)\|f_h\|_{1,\Omega}\,,\end{aligned}$$

where we have used (4.12) for the last inequality. The proof is complete. ∎

We are now in a position to prove Proposition 4.3.

Proof of Proposition 4.3. Let (β, w, γ) and (β_h, w_h, γ_h) be as in (4.9) and (4.10), respectively. We need to prove that

$$\|w - w_h\|_{1,\Omega} \leq C(h+t)\|f_h\|_{1,\Omega}\,.$$

Since

$$\nabla w - \nabla w_h = \kappa^{-1}t^2(\gamma - \gamma_h) + (\beta - R\beta_h)\,,$$

adding and subtracting $R\beta$, we obtain

$$\|\nabla w - \nabla w_h\|_{0,\Omega} \leq \kappa^{-1}t^2\|\gamma - \gamma_h\|_{0,\Omega} + \|\beta - R\beta\|_{0,\Omega} + \|R(\beta - \beta_h)\|_{0,\Omega}\,. \tag{4.15}$$

Hence, using the Poincaré inequality, (4.1), Lemma 4.5, (4.2), and (4.12), we have

$$\|w - w_h\|_{1,\Omega} \leq C(h+t)\|f_h\|_{1,\Omega}\,.$$

The proof is complete. ∎

We end this section by proving property P1.

LEMMA 4.6. *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Then we have*

$$\|T_0 - T_{th}\|_h \leq C(h+t)\,.$$

Proof. The assertion follows immediately from estimate (4.7), by using (4.8) and Proposition 4.3. ∎

5. Convergence and error estimates. In this section we will adapt the arguments from [9] to prove error estimates for the approximate eigenvalues and eigenfunctions. Throughout this section, we will assume that the family of meshes $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform so that property P1 holds true, although such an assumption is not actually necessary in some particular cases (see the appendix below).

Our first goal is to prove that, provided the plate is sufficiently thin, the numerical method does not introduce spurious modes with eigenvalues interspersed among the relevant ones of T_t (namely, around $\mu_t^{(k)}$ for small k). Let us remark that such a spectral pollution could be in principle expected from the fact that T_t has a nontrivial essential spectrum. However, that this is not the case is an immediate consequence of the following theorem, which is essentially identical to Lemma 1 from [8].

THEOREM 5.1. *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. There exist strictly positive constants h_0, t_0, and C such that, $\forall h < h_0$ and $\forall t < t_0$, there holds $F \cap \mathrm{Sp}(T_{th}) = \emptyset$ and*

$$\|R_z(T_{th})\|_h \le C \qquad \forall z \in F.$$

Proof. The same arguments used to prove Lemma 3.8 (but using Lemma 4.6 instead of Lemma 3.6) allow us to show an estimate analogous to (3.11), namely, for all $w_h \in W_h$ and all $z \in F$,

$$\|(zI - T_{th})\, w_h\|_{1,\Omega} \ge \|(zI - T_0)\, w_h\|_{1,\Omega} - \|(T_0 - T_{th})\, w_h\|_{1,\Omega} \ge \frac{1}{2C_1}\, \|w_h\|_{1,\Omega}\,,$$

provided h and t are small enough. Since W_h is finite dimensional, the inequality above implies that $(zI - T_{th})\,|_{W_h}$ is invertible and, hence, $z \notin \mathrm{Sp}(T_{th}|_{W_h})$. Now, $\mathrm{Sp}(T_{th}) = \mathrm{Sp}(T_{th}|_{W_h}) \cup \{0\}$ (see, for instance, [3, Lemma 4.1]) and, for $z \in F$, $z \neq 0$. Thus, $z \notin \mathrm{Sp}(T_{th})$ either. Then $(zI - T_{th})$ is invertible too and

$$\|R_z(T_{th})\|_h = \|\, (zI - T_{th})^{-1}\, \|_h \le 2C_1 \qquad \forall z \in F.$$

The proof is complete. ∎

We have already proved in Theorem 3.1 that the essential spectrum of T_t is confined to the real interval $\left(-\kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega}, \kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega}\right)$. The spectrum of T_t outside this interval consists of finite multiplicity isolated eigenvalues of ascent one, which converge to eigenvalues of T_0 as t goes to zero (cf. Theorem 3.9). The eigenvalue of T_t with physical significance is the largest in modulus, $\mu_t^{(1)}$, which corresponds to the limit of elastic stability that leads to buckling effects. This eigenvalue is typically simple and converges to a simple eigenvalue of T_0 as t tends to zero. Because of this, for simplicity, from now on we restrict our analysis to simple eigenvalues.

Let $\mu_0 \neq 0$ be an eigenvalue of T_0 with multiplicity $m = 1$. Let D be a closed disk centered at μ_0 with boundary Γ such that $0 \notin D$ and $D \cap \mathrm{Sp}(T_0) = \{\mu_0\}$. Let $t_0 > 0$ be small enough so that for all $t < t_0$

- D contains only one eigenvalue μ_t of T_t, which we already know is simple (cf. Lemma 3.7), and
- D does not intersect the real interval $(-\kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega}, \kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega})$, which contains the essential spectrum of T_t.

According to Theorem 5.1 there exist $t_0 > 0$ and $h_0 > 0$ such that $\forall t < t_0$ and $\forall h < h_0$, $\Gamma \subset \rho(T_{th})$. Moreover, proceeding as in [8, section 2], from properties P1 and P2 it follows that, for h small enough, T_{th} has exactly one eigenvalue $\mu_{th} \in D$. The theory in [9] could be adapted too, to prove error estimates for the eigenvalues and eigenfunctions of T_{th} to those of T_0 as h and t go to zero. However, our goal is not this one, but to prove that μ_{th} converges to μ_t as h goes to zero, with $t < t_0$ fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [9].

Let $\Pi_h : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ be the projector with range W_h defined for all $u \in \mathrm{H}_0^1(\Omega)$ by

$$\left(\nabla\left(\Pi_h u - u\right), \nabla v_h\right)_{0,\Omega} = 0 \qquad \forall v_h \in W_h.$$

The projector Π_h is bounded uniformly on h, namely, $\|\Pi_h u\|_{1,\Omega} \leq \|u\|_{1,\Omega}$, and the following error estimate is well known:

$$\|\Pi_h u - u\|_{1,\Omega} \leq Ch\,\|u\|_{2,\Omega} \qquad \forall u \in \mathrm{H}^2(\Omega). \tag{5.1}$$

Let us define

$$B_{th} := T_{th}\Pi_h : \mathrm{H}_0^1(\Omega) \to W_h \hookrightarrow \mathrm{H}_0^1(\Omega).$$

It is clear that T_{th} and B_{th} have the same nonzero eigenvalues and corresponding eigenfunctions. Furthermore, we have the following result (cf. [9, Lemma 1]).

LEMMA 5.2. *There exist h_0, t_0, and C such that*

$$\|R_z(B_{th})\| \leq C \qquad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

Proof. Since B_{th} is compact, it suffices to verify that $\|(zI - B_{th})u\|_{1,\Omega} \geq C\|u\|_{1,\Omega}$ $\forall u \in \mathrm{H}_0^1(\Omega)$ and $z \in \Gamma$. Taking into account that $0 \notin \Gamma$ and using Theorem 5.1, we have

$$\|u\|_{1,\Omega} \leq \|\Pi_h u\|_{1,\Omega} + \|u - \Pi_h u\|_{1,\Omega} \leq C\,\|(zI - T_{th})\,\Pi_h u\|_{1,\Omega} + |z|^{-1}\,\|z\,(u - \Pi_h u)\|_{1,\Omega}\,.$$

By using properties of the projector Π_h, we obtain

$$\begin{aligned}\|u\|_{1,\Omega} &\leq C\,\|(zI - B_{th})\,\Pi_h u\|_{1,\Omega} + |z|^{-1}\,\|z\,(u - \Pi_h u) - B_{th}(u - \Pi_h u)\|_{1,\Omega}\\ &= C\,\|\Pi_h(zI - B_{th})u\|_{1,\Omega} + |z|^{-1}\,\|(I - \Pi_h)\,(zI - B_{th})\,u\|_{1,\Omega}\\ &\leq C\,\|(zI - B_{th})\,u\|_{1,\Omega}\,.\end{aligned}$$

Thus we end the proof. ∎

Next, we introduce

- $E_t : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$, the spectral projector of T_t corresponding to the isolated eigenvalue μ_t, namely,

$$E_t := \frac{1}{2\pi i}\int_\Gamma R_z(T_t)\,dz;$$

- $F_{th} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$, the spectral projector of B_{th} corresponding to the eigenvalue μ_{th}, namely,

$$F_{th} := \frac{1}{2\pi i}\int_\Gamma R_z(B_{th})\,dz.$$

As a consequence of Lemma 5.2, the spectral projectors F_{th} are bounded uniformly in h and t for h and t small enough. Notice that $E_t(\mathrm{H}_0^1(\Omega))$ is the eigenspace of T_t associated to μ_t and $F_{th}(\mathrm{H}_0^1(\Omega))$ the eigenspace of B_{th} (and hence of T_{th}, too) associated to μ_{th}. According to our assumptions, $E_t(\mathrm{H}_0^1(\Omega))$ and $F_{th}(\mathrm{H}_0^1(\Omega))$ are both one dimensional. The following estimate (cf. [9, Lemma 3]) will be used to prove convergence of the eigenspaces.

LEMMA 5.3. *There exist positive constants h_0, t_1, and C such that for all $h < h_0$ and for all $t < t_1$,*

$$\|\,(E_t - F_{th})\,|_{E_t(\mathrm{H}_0^1(\Omega))}\| \leq C\|\,(T_t - B_{th})\,|_{E_t(\mathrm{H}_0^1(\Omega))}\| \leq Ch.$$

Proof. The first inequality is proved using the same arguments of [9, Lemma 3] and Lemmas 3.8 and 5.2. For the other estimate, fix $w \in E_t(\mathrm{H}^1_0(\Omega))$. From Proposition 3.10, Remark 4.2, Lemma 4.2, and (5.1) we have

$$\begin{aligned}\|(T_t - B_{th})\, w\|_{1,\Omega} &\leq \|(T_t - T_{th})w\|_{1,\Omega} + \|(T_{th} - B_{th})w\|_{1,\Omega}\\ &\leq \|(T_t - T_{th})w\|_{1,\Omega} + \|T_{th}\|\, \|(I - \Pi_h)\, w\|_{1,\Omega}\\ &\leq Ch\, \|w\|_{2,\Omega}.\end{aligned}$$

Therefore, by using (3.13), we conclude the proof. ◻

To prove an error estimate for the eigenspaces, we also need the following result.

LEMMA 5.4. *Let*

$$\Lambda_{th} := F_{th}|_{E_t(\mathrm{H}^1_0(\Omega))} : E_t(\mathrm{H}^1_0(\Omega)) \to F_{th}(\mathrm{H}^1_0(\Omega)).$$

For h and t small enough, the operator Λ_{th} is invertible and

$$\left\|\Lambda_{th}^{-1}\right\| \leq C,$$

with C independent of h and t.

Proof. See the proof of Theorem 1 in [9]. ◻

We recall the definition of the gap $\hat{\delta}$ between two closed subspaces Y and Z of $\mathrm{H}^1_0(\Omega)$:

$$\hat{\delta}(Y, Z) := \max\left\{\delta(Y, Z), \delta(Z, Y)\right\},$$

where

$$\delta(Y, Z) := \sup_{\substack{y \in Y\\ \|y\|_{1,\Omega}=1}} \left(\inf_{z \in Z} \|y - z\|_{1,\Omega}\right).$$

The following theorem shows that the eigenspace of T_{th} (which coincides with that of B_{th}) approximate the eigenspace of T_t with optimal order.

THEOREM 5.5. *There exist constants h_0, t_1, and C such that, for all $h < h_0$ and for all $t < t_1$, there holds*

$$\hat{\delta}\left(F_{th}(\mathrm{H}^1_0(\Omega)), E_t(\mathrm{H}^1_0(\Omega))\right) \leq Ch.$$

Proof. It follows by arguing exactly as in the proof of Theorem 1 from [9] and using Lemmas 5.3 and 5.4. ◻

Next, we prove a preliminary suboptimal error estimate for $|\mu_t - \mu_{th}|$, which will be improved below (cf. Theorem 5.8).

LEMMA 5.6. *There exists a positive constant C such that, for h and t small enough,*

$$|\mu_t - \mu_{th}| \leq Ch.$$

Proof. We define the following operators:

$$\begin{aligned}\widehat{T}_t &:= T_t|_{E_t(\mathrm{H}^1_0(\Omega))} : E_t(\mathrm{H}^1_0(\Omega)) \to E_t(\mathrm{H}^1_0(\Omega)),\\ \widehat{B}_{th} &:= \Lambda_{th}^{-1} B_{th} \Lambda_{th} : E_t(\mathrm{H}^1_0(\Omega)) \to E_t(\mathrm{H}^1_0(\Omega)).\end{aligned}$$

The operator $\widehat{T}_t$ has a unique eigenvalue μ_t of multiplicity $m = 1$, while the unique eigenvalue of $\widehat{B}_{th}$ is μ_{th}.

Let $v \in E_t(\mathrm{H}^1_0(\Omega))$. Since $(\Lambda_{th}^{-1}F_{th} - I)T_t|_{E_t(\mathrm{H}^1_0(\Omega))} = 0$ and B_{th} commutes with its spectral projector F_{th}, we have

$$(\widehat{T}_t - \widehat{B}_{th})v = (T_t - B_{th})\,v + \left(\Lambda_{th}^{-1}F_{th} - I\right)(T_t - B_{th})\,v.$$

Therefore, using Lemmas 5.3 and 5.4 and the fact that $\|F_{th}\|$ is bounded uniformly in h and t, for h and t small enough, we obtain

$$\|(\widehat{T}_t - \widehat{B}_{th})v\|_{1,\Omega} \leq \|(T_t - B_{th})\,v\|_{1,\Omega} + \left\|\left(\Lambda_{th}^{-1}F_{th} - I\right)(T_t - B_{th})\,v\right\|_{1,\Omega} \leq Ch\,\|v\|_{1,\Omega}\,.$$

Hence, the lemma follows from the fact that $\widehat{T}_t = \mu_t I$ and $\widehat{B}_{th} = \mu_{th} I$. ∎

Since the eigenvalue $\mu_t \neq 0$ of T_t corresponds to an eigenvalue $\lambda = 1/\mu_t$ of Problem 2.1, Lemma 5.6 leads to an error estimate for the approximation of λ as well. However, the order of convergence is $O(h)$ as in this lemma. We now aim at improving this result. Let $\lambda_h := 1/\mu_{th}$, w_h, β_h, and γ_h be such that $(\lambda_h, w_h, \beta_h, \gamma_h)$ is a solution of Problem 4.1, with $\|w_h\|_{1,\Omega} = 1$. According to Theorem 5.5, there exists a solution $(\lambda, w, \beta, \gamma)$ to Problem 2.1, with $\|w\|_{1,\Omega} = 1$, such that

$$\|w - w_h\|_{1,\Omega} \leq Ch.$$

The following lemma will be used to prove a double order of convergence for the corresponding eigenvalues, but it is interesting by itself, too. In fact, it shows optimal order convergence for the rotations of the vibration modes.

LEMMA 5.7. *Let (λ, w, β) be a solution of Problem* 2.1, *with $\|w\|_{1,\Omega} = 1$, and $(\lambda_h, w_h, \beta_h)$ a solution of Problem* 4.1, *with $\|w_h\|_{1,\Omega} = 1$, such that*

$$\|w - w_h\|_{1,\Omega} \leq Ch. \tag{5.2}$$

Let γ and γ_h be as defined in Problems 2.1 *and* 4.1, *respectively. Then for h and t small enough there holds*

$$\|\beta - \beta_h\|_{1,\Omega} + t\,\|\gamma - \gamma_h\|_{0,\Omega} \leq Ch. \tag{5.3}$$

Proof. Let $\hat{w}_h \in W_h$, $\hat{\beta}_h \in H_h$, and $\hat{\gamma}_h$ be the solution of the auxiliary problem

$$\begin{cases} a(\hat{\beta}_h, \eta_h) + (\hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v_h)_{0,\Omega} \qquad \forall(\eta_h, v_h) \in H_h \times W_h, \\ \hat{\gamma}_h = \dfrac{\kappa}{t^2}(\nabla \hat{w}_h - R\hat{\beta}_h). \end{cases}$$

This problem is the finite element discretization of Problem 2.1, with source term $f = \lambda w \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$. Then, from Remark 4.2, (3.13), and the fact that $\|w_h\|_{1,\Omega} = 1$, we obtain the following error estimate:

$$\|\beta - \hat{\beta}_h\|_{1,\Omega} + t\,\|\gamma - \hat{\gamma}_h\|_{0,\Omega} + \|w - \hat{w}_h\|_{1,\Omega} \leq Ch\,|\lambda|\,\|w\|_{2,\Omega} \leq Ch\,|\lambda|\,. \tag{5.4}$$

On the other hand, from Problem 4.1, we have that $(\beta_h - \hat{\beta}_h, w_h - \hat{w}_h) \in H_h \times W_h$ satisfies

$$\begin{cases} a(\beta_h - \hat{\beta}_h, \eta_h) + (\gamma_h - \hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma}\nabla\left(\lambda_h w_h - \lambda w\right), \nabla v_h)_{0,\Omega} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall(\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h - \hat{\gamma}_h = \dfrac{\kappa}{t^2}(\nabla\left(w_h - \hat{w}_h\right) - R(\beta_h - \hat{\beta}_h)). \end{cases}$$

Taking $\eta_h = \beta_h - \hat{\beta}_h$ and $v_h = w_h - \hat{w}_h$ in the system above, from the ellipticity of $a(\cdot,\cdot)$, we obtain

$$\begin{aligned}
&\|\beta_h - \hat{\beta}_h\|_{1,\Omega}^2 + \kappa^{-1}t^2 \|\gamma_h - \hat{\gamma}_h\|_{0,\Omega}^2 \\
&\qquad \le C \|\lambda_h w_h - \lambda w\|_{1,\Omega} \|w_h - \hat{w}_h\|_{1,\Omega} \\
&\qquad \le C \left(|\lambda_h| \|w - w_h\|_{1,\Omega} + |\lambda - \lambda_h| \|w\|_{1,\Omega} \right) \left(\|w - w_h\|_{1,\Omega} + \|w - \hat{w}_h\|_{1,\Omega} \right) \\
&\qquad \le Ch^2,
\end{aligned}$$

where we have used Lemma 5.6 and estimates (5.2) and (5.4) for the last inequality. Therefore, we have

$$\|\beta_h - \hat{\beta}_h\|_{1,\Omega} + t \|\gamma_h - \hat{\gamma}_h\|_{0,\Omega} \le Ch.$$

Thus, the lemma follows from this estimate and (5.4). ∎

We are now in a position to prove an optimal double order error estimate for the eigenvalues.

THEOREM 5.8. *There exist positive constants h_0, t_1, and C such that, $\forall h < h_0$ and $\forall t < t_1$,*

$$|\lambda - \lambda_h| \le Ch^2.$$

Proof. We adapt to our case a standard argument for eigenvalue problems (see [2, Lemma 9.1]). Let $(\lambda, \beta, w, \gamma)$ and $(\lambda_h, \beta_h, w_h, \gamma_h)$ be as in Lemma 5.7. We will use the bilinear forms A and B defined in (3.1) and (3.2), respectively, as well as the bilinear form A_h defined in $H_h \times W_h$ as follows:

$$A_h((\beta_h, w_h), (\eta_h, v_h)) := a(\beta_h, \eta_h) + \frac{\kappa}{t^2} (\nabla w_h - R\beta_h, \nabla v_h - R\eta_h)_{0,\Omega}.$$

With this notation, Problems 2.1 and 4.1 can be written as follows:

$$\begin{aligned}
A((\beta, w), (\eta, v)) &= \lambda B((\beta, w), (\eta, v)), \\
A_h((\beta_h, w_h), (\eta_h, v_h)) &= \lambda_h B((\beta_h, w_h), (\eta_h, v_h)).
\end{aligned}$$

From these equations, straightforward computations lead to

$$\begin{aligned}
(\lambda_h - \lambda) B((\beta_h, w_h), (\beta_h, w_h)) &= A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\
&\quad - \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\
&\quad + [A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h))]\,.
\end{aligned} \tag{5.5}$$

Next, we define $\bar{\gamma}_h := \frac{\kappa}{t^2} (\nabla w_h - \beta_h)$. Recalling that $R\nabla w_h = \nabla w_h$ (cf. (4.3)), from the definition of γ_h (cf. Problem 4.1) we have that $\gamma_h = R\bar{\gamma}_h$. On the other hand, from the definition of A and A_h we write

$$\begin{aligned}
&A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) = a(\beta - \beta_h, \beta - \beta_h) + \kappa^{-1}t^2 \|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2, \\
&A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h)) = \kappa^{-1}t^2 \left(\|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}(\lambda_h - \lambda)\, B((\beta_h, w_h), (\beta_h, w_h)) = {} & a(\beta - \beta_h, \beta - \beta_h) \\ & + \kappa^{-1} t^2 \left(\|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right) \\ & - \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)).\end{aligned}$$

The first and the third term in the right-hand side above are easily bounded by virtue of (5.2) and (5.3). For the second term, we write

$$\begin{aligned}\|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 &= \|\gamma - R\bar{\gamma}_h\|_{0,\Omega}^2 - 2\,(\gamma, \bar{\gamma}_h - R\bar{\gamma}_h)_{0,\Omega} \\ &= \|\gamma - \gamma_h\|_{0,\Omega}^2 + \frac{2\kappa}{t^2}\,(\gamma, \beta_h - R\beta_h)_{0,\Omega}.\end{aligned} \tag{5.6}$$

For $\beta \in \mathrm{H}^2(\Omega)^2 \cap \mathrm{H}_0^1(\Omega)$, we denote by $\beta^{\mathrm{I}} \in \mathrm{H}_h$ the standard Clément interpolant of β, which satisfies

$$\left\|\beta^{\mathrm{I}}\right\|_{1,\Omega} \leq C\, \|\beta\|_{1,\Omega} \qquad \text{and} \qquad \left\|\beta - \beta^{\mathrm{I}}\right\|_{1,\Omega} \leq Ch\, \|\beta\|_{2,\Omega}. \tag{5.7}$$

It follows that

$$\begin{aligned}(\gamma, \beta_h - R\beta_h)_{0,\Omega} &= \left(\gamma, (\beta_h - \beta^{\mathrm{I}}) - R(\beta_h - \beta^{\mathrm{I}})\right)_{0,\Omega} + \left(\gamma, \beta^{\mathrm{I}} - R\beta^{\mathrm{I}}\right)_{0,\Omega} \\ &\leq \|\gamma\|_{0,\Omega} \left\|(\beta_h - \beta^{\mathrm{I}}) - R(\beta_h - \beta^{\mathrm{I}})\right\|_{0,\Omega} + \left(\gamma, \beta^{\mathrm{I}} - R\beta^{\mathrm{I}}\right)_{0,\Omega}.\end{aligned}$$

Thus, using (4.2) and Lemma 3.3 from [10], we obtain

$$\begin{aligned}(\gamma, \beta_h - R\beta_h)_{0,\Omega} &\leq Ch\, \|\gamma\|_{0,\Omega} \left\|\beta_h - \beta^{\mathrm{I}}\right\|_{1,\Omega} + Ch^2\, \|\mathrm{div}\, \gamma\|_{0,\Omega}\, \|\beta\|_{1,\Omega} \\ &\leq Ch\, \|\gamma\|_{0,\Omega} \left(\|\beta - \beta_h\|_{1,\Omega} + \left\|\beta - \beta^{\mathrm{I}}\right\|_{1,\Omega} \right) + Ch^2\, \|\mathrm{div}\, \gamma\|_{0,\Omega}\, \|\beta\|_{1,\Omega},\end{aligned}$$

and from Lemma 5.7, (5.7), and Proposition 3.10, we have

$$(\gamma, \beta_h - R\beta_h)_{0,\Omega} \leq Ch\, \|\gamma\|_{0,\Omega} \left(Ch + Ch\, \|\beta\|_{2,\Omega} \right) + Ch^2\, |\lambda|\, \|w\|_{2,\Omega}\, \|\beta\|_{1,\Omega} \leq Ch^2\, |\lambda|.$$

Finally, we use this estimate, (5.5), (5.6), and the fact that $B((\beta_h, w_h), (\beta_h, w_h)) = (\boldsymbol{\sigma} \nabla w_h, \nabla w_h)_{0,\Omega} \neq 0$ (cf. Remark 4.1) to obtain

$$|\lambda - \lambda_h| \leq C\, \frac{\|\beta - \beta_h\|_{1,\Omega}^2 + \|w - w_h\|_{1,\Omega}^2 + \kappa^{-1} t^2\, \|\gamma - \gamma_h\|_{0,\Omega}^2 + Ch^2\, |\lambda|}{|B((\beta_h, w_h), (\beta_h, w_h))|}.$$

Consequently, from Lemma 5.7,

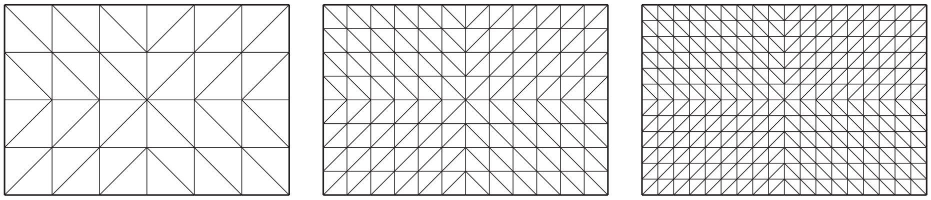
$$|\lambda - \lambda_h| \leq Ch^2,$$

and we conclude the proof. $\square$

6. Numerical results. In this section we report some numerical experiments carried out with our method applied to Problem 2.1. We recall that the buckling coefficients can be directly computed from the eigenvalues of Problem 2.1: $\lambda_{\mathrm{b}} = \lambda t^2$.

For all of the computations we have taken $\Omega := (0, 6) \times (0, 4)$ (all of the lengths are measured in meters) and typical parameters of steel: the Young modulus has been chosen $E = 1.44 \times 10^{11}$ Pa and the Poisson ratio $\nu = 0.30$. The shear correction factor has been taken $k = 5/6$.

We have used uniform meshes as those shown in Figure 6.1; the meaning of the refinement parameter N can be easily deduced from this figure. Notice that $h \sim N^{-1}$.



$N = 1$ $N = 2$ $N = 3$

FIG. 6.1. *Rectangular plate. Uniform meshes.*

TABLE 6.1
Lowest eigenvalues λ_i (multiplied by 10^{-10}) of a uniformly compressed simply supported plate with thickness $t = 0.001$.

Eigenvalue	$N=2$	$N=4$	$N=8$	$N=16$	Order	Extrapolated	Exact
λ_1	1.1793	1.1759	1.1752	1.1750	2.14	1.1750	1.1749
λ_2	2.2638	2.2602	2.2596	2.2595	2.68	2.2595	2.2595
λ_3	3.7293	3.6441	3.6224	3.6170	1.98	3.6151	3.6152
λ_4	4.1573	4.0892	4.0726	4.0685	2.03	4.0672	4.0671

6.1. Uniformly compressed rectangular plate. For this test we have used $\boldsymbol{\sigma} = \boldsymbol{I}$, which corresponds to a uniformly compressed plate.

6.1.1. Simply supported plate. First, we have considered a simply supported plate, because analytical solutions are available in this case (see [19, 20]). Even though our theoretical analysis has been developed only for clamped plates, we think that the results of sections 4 and 5 should hold true for more general boundary conditions as well. The results that follow give some numerical evidence of this.

In Table 6.1 we report the four lowest eigenvalues (λ_i, $i = 1, 2, 3, 4$) computed by our method with four different meshes ($N = 2, 4, 8, 16$) for a a simply supported plate with thickness $t = 0.001$. The table includes computed orders of convergence, as well as more accurate values extrapolated by means of a least-squares fitting. The last column shows the exact eigenvalues.

It can be seen from Table 6.1 that the method converges to the exact values with an optimal quadratic order.

Figure 6.2 shows the transverse displacements for the principal buckling mode computed with the finest mesh ($N = 16$).

6.1.2. Clamped plate. In Table 6.2 we present the results for the lowest eigenvalue of a uniformly compressed clamped rectangular plate with varying thickness. We have used the same meshes as in the previous test. Again, we have computed the orders of convergence and more accurate values obtained by a least-squares fitting. In the last row we report for each mesh the limit values as t goes to zero obtained by extrapolation.

Figure 6.3 shows the transverse displacements for the principal buckling mode, for $t = 0.1$, and the finest mesh ($N = 16$).

According to Lemma 3.7, the values on the last row of Table 6.2 should correspond to the lowest eigenvalues of a Kirchhoff–Love uniformly compressed clamped plate with thickness $t = 1$. As a further test, we have also computed the latter by using the methods analyzed in [6] and [17]. We show the obtained results in Table 6.3, where an excellent agreement with the last row of Table 6.2 can be appreciated.

It is clear that the results from the Reissner–Mindlin model do not deteriorate as the plate thickness become smaller, which confirms that our method is *locking-free.*

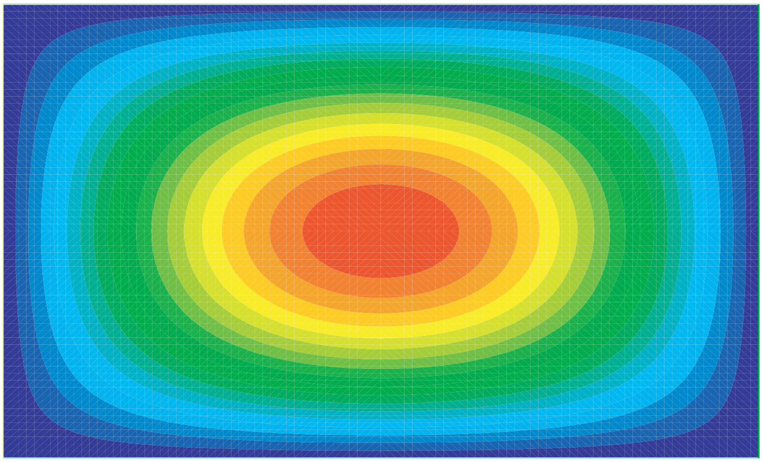


FIG. 6.2. *Uniformly compressed simply supported plate; principal buckling mode.*

TABLE 6.2
Lowest eigenvalue λ_1 *(multiplied by* 10^{-10}*) of uniformly compressed clamped plates with varying thickness.*

Thickness	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated
$t = 0.1$	3.4031	3.3440	3.3293	3.3258	2.02	3.3246
$t = 0.01$	3.4324	3.3723	3.3571	3.3533	1.99	3.3520
$t = 0.001$	3.4327	3.3726	3.3574	3.3536	1.99	3.3522
$t = 0.0001$	3.4327	3.3726	3.3574	3.3536	1.98	3.3522
$t = 0$ (extrap.)	3.4327	3.3726	3.3574	3.3536	1.99	3.3523

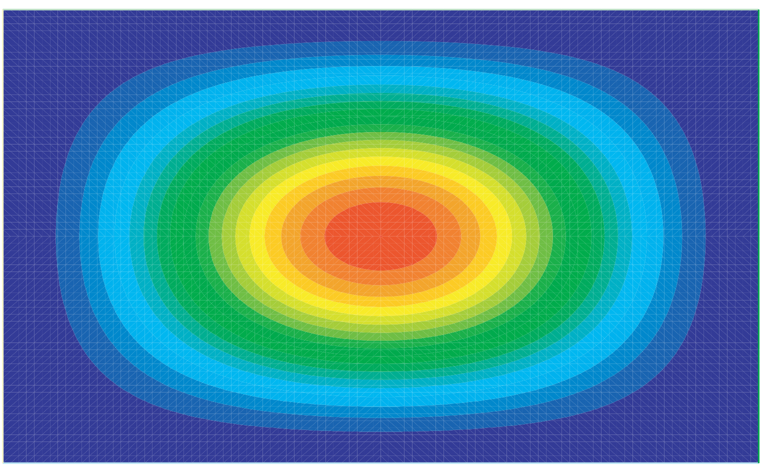


FIG. 6.3. *Uniformly compressed clamped plate; principal buckling mode.*

TABLE 6.3
Lowest eigenvalue λ_1 *(multiplied by* 10^{-10}*) of a uniformly compressed clamped thin plate (Kirchhoff–Love model) computed with the methods from* [6] *and* [17].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[6]	3.3718	3.3611	3.3573	3.3555	1.97	3.3523
[17]	3.3514	3.3519	3.3521	3.3522	1.95	3.3523

TABLE 6.4
Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of clamped plates with varying thickness, uniformly compressed in one direction.

Thickness	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated
$t = 0.1$	6.7969	6.7274	6.7104	6.7066	2.05	6.7052
$t = 0.01$	6.8825	6.8143	6.7971	6.7930	2.00	6.7915
$t = 0.001$	6.8834	6.8151	6.7980	6.7939	2.00	6.7924
$t = 0.0001$	6.8834	6.8152	6.7980	6.7939	2.00	6.7924
$t = 0$ (extrap.)	6.8834	6.8152	6.7980	6.7939	2.00	6.7924

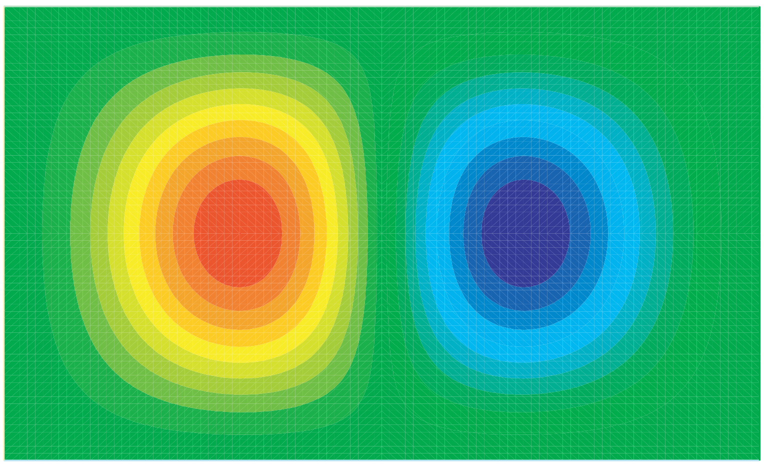


FIG. 6.4. *Clamped plate uniformly compressed in one direction; principal buckling mode.*

TABLE 6.5
Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a clamped thin plate (Kirchhoff–Love model) uniformly compressed in one direction, computed with the methods from [6] *and* [17].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[6]	6.8450	6.8158	6.8056	6.8009	2.00	6.7925
[17]	6.7904	6.7913	6.7917	6.7920	1.92	6.7926

6.2. Clamped plate uniformly compressed in one direction. We have used for this test

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which corresponds to a plate uniformly compressed in one direction. Notice that in this test $\boldsymbol{\sigma}$ is only positive semidefinite. Table 6.4 shows the same quantities as Table 6.2 in this case.

Figure 6.4 shows the principal buckling mode for $t = 0.1$ and the finest mesh $(N = 16)$.

Finally, Table 6.5 shows the same quantities as Table 6.3 in this case. Once more, an excellent agreement with the values extrapolated from the Reissner–Mindlin model (last row of Table 6.4) can be clearly appreciated.

6.3. Shear loaded clamped plate. In this case we have used

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

TABLE 6.6
Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of shear loaded clamped plates with varying thickness.

Thickness	$N = 4$	$N = 8$	$N = 12$	$N = 16$	Order	Extrapolated
$t = 0.1$	9.4306	9.2179	9.1783	9.1645	1.99	9.1464
$t = 0.01$	9.6098	9.3923	9.3514	9.3371	1.98	9.3184
$t = 0.001$	9.6116	9.3942	9.3533	9.3389	1.98	9.3202
$t = 0.0001$	9.6117	9.3942	9.3533	9.3389	1.98	9.3202
$t = 0$ (extrap.)	9.6117	9.3942	9.3533	9.3389	1.98	9.3202

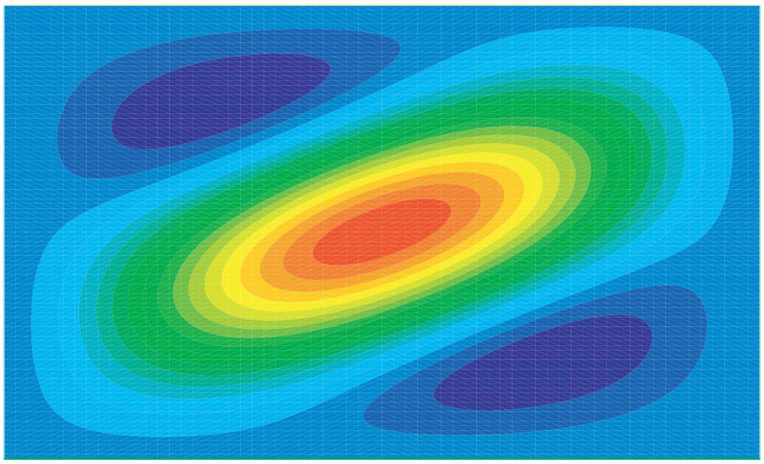


FIG. 6.5. *Shear loaded clamped plate; principal buckling mode.*

TABLE 6.7
Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a shear loaded clamped thin plate (Kirchhoff–Love model) computed with the methods from [6] *and* [17].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[6]	9.4625	9.3840	9.3563	9.3435	1.98	9.3203
[17]	9.3660	9.3408	9.3319	9.3278	1.99	9.3204

which corresponds to a uniform shear load. Notice that $\boldsymbol{\sigma}$ is indefinite in this test. The numerical results are reported in Table 6.6, Figure 6.5, and Table 6.7, using the same pattern as the previous tests.

In all cases, an excellent agreement between the numerical experiments and the theoretical results detailed in section 5 can be noticed, and the method appears thoroughly locking-free.

Appendix. Uniformly compressed plates. The aim of this appendix is to show that the results of sections 3, 4, and 5 can be refined when $\boldsymbol{\sigma} = \boldsymbol{I}$, which corresponds to a uniformly compressed plate. In this case, we are able to give a better characterization of the spectrum of T_t and to prove the spectral approximation without assuming that the family of meshes is *quasi-uniform*.

A.1. Spectral characterization. We have the following counterpart of Theorem 3.1, showing that the spectrum of T_t is simply a shift of the spectrum of a compact operator.

THEOREM A.1. *Suppose that $\boldsymbol{\sigma} = \boldsymbol{I}$. For all $t > 0$, the spectrum of T_t satisfies*

$$\mathrm{Sp}(T_t) = \mathrm{Sp}(G) + \kappa^{-1}t^2,$$

where G is the compact operator defined in (3.5).

Proof. The first equation of (2.8) leads in this case to $\psi = f$, due to the fact that $\boldsymbol{\sigma} = \boldsymbol{I}$. Therefore, (2.8) reduces to

$$\text{(A.1)} \qquad \begin{cases} a(\beta,\eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla f, \eta)_{0,\Omega} \qquad \forall \eta \in \mathrm{H}_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2 (\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 \qquad \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} + \kappa^{-1}t^2 (\nabla f, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega). \end{cases}$$

Next, recall that G is defined in (3.5) as the operator mapping $f \mapsto u$, with $u \in \mathrm{H}_0^1(\Omega)$ such that

$$(\nabla u, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega),$$

where $\beta \in \mathrm{H}_0^1(\Omega)^2$ is determined in this case by the first two equations from (A.1). Therefore, the third equation from (A.1) yields $T_t = G + \kappa^{-1}t^2 I$. Since G has been already shown to be compact, this allows us to conclude the theorem. ∎

As a consequence of this theorem, $\mathrm{Sp}(T_t) = \left\{\kappa^{-1}t^2\right\} \cup \{\mu_n : \ n \in \mathbb{N}\}$, with μ_n being a sequence of finite-multiplicity eigenvalues converging to $\kappa^{-1}t^2$. Therefore, in this particular case, the *essential spectrum* of T_t reduces to a unique point: $\kappa^{-1}t^2$.

A.2. Spectral approximation. In this particular case, we will improve the error estimate shown in section 4 in that we will not need to assume quasi uniformity of the meshes. Indeed, this property was used above only to prove Proposition 4.3. Instead, we have now the following result.

PROPOSITION A.2. *Suppose that $\boldsymbol{\sigma} = \boldsymbol{I}$. Then, for any regular family of triangular meshes $\{\mathcal{T}_h\}_{h>0}$, there exists $C > 0$ such that, for all $t > 0$,*

$$\|T_t - T_{th}\|_h \le Ch.$$

Proof. We will simply sketch the proof, since it follows exactly the same steps as that of Proposition 4.3. First, we notice that in the decomposition (4.11) we have $\psi = f_h \in W_h$ (cf. problem (4.13) with $\boldsymbol{\sigma} = \boldsymbol{I}$).

As a consequence, we infer that the term $\|\nabla\psi - \nabla\psi_{\mathrm{I}}\|_{0,\Omega}$ in (4.14) vanishes. Hence, the last estimate in the proof of Lemma 4.5 changes into

$$\|\beta - \beta_h\|_{1,\Omega} + t\|\gamma - \gamma_h\|_{0,\Omega} \le C\left(h\|\beta\|_{2,\Omega} + th\|p\|_{2,\Omega} + h\|\gamma\|_{0,\Omega}\right) \le Ch\|f_h\|_{1,\Omega}.$$

By using the above estimate in the proof of Proposition 4.3 (in particular in (4.15)), we obtain

$$\|(T_t - T_{th}) f_h\|_{1,\Omega} = \|w - w_h\|_{1,\Omega} \le Ch\|f_h\|_{1,\Omega},$$

from which we conclude the proof. ∎

As a consequence of Proposition A.2, we can improve Lemma 4.6. In fact, now for t small enough there holds directly

$$\|T_t - T_{th}\|_h \le Ch,$$

with a constant C independent of h and t. By using this instead of property P1, we could give somewhat simpler proofs for the error estimates from section 5. However, the final results, Theorems 5.1, 5.5, and 5.8, are the same, although now valid for any regular family of triangular meshes, without the need of being quasi-uniform.

REFERENCES

[1] D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner–Mindlin plate*, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.

[2] I. Babuška and J. Osborn, *Eigenvalue Problems*, Handb. Numer. Anal. 2, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 641–787.

[3] A. Bermúdez, R. Durán, M. A. Muschietti, R. Rodríguez, and J. Solomin, *Finite element vibration analysis of fluid-solid systems without spurious modes*, SIAM J. Numer. Anal., 32 (1995), pp. 1280–1295.

[4] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[5] C. Canuto, *Eigenvalue approximations by mixed methods*, RAIRO Anal. Numér., 12 (1978), pp. 27–50.

[6] P. G. Ciarlet and P. A. Raviart, *A mixed finite element method for the biharmonic equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., Academic Press, New York, 1974, pp. 125–145.

[7] M. Dauge and M. Suri, *Numerical approximation of the spectra of non-compact operators arising in buckling problems*, J. Numer. Math., 10 (2002), pp. 193–219.

[8] J. Descloux, N. Nassif, and J. Rappaz, *On spectral approximation. Part* 1: *The problem of convergence*, RAIRO Anal. Numér., 12 (1978), pp. 97–112.

[9] J. Descloux, N. Nassif, and J. Rappaz, *On spectral approximation. Part* 2: *Error estimates for the Galerkin method*, RAIRO Anal. Numér., 12 (1978), pp. 113–119.

[10] R. Durán, L. Hervella-Nieto, E. Liberman, R. Rodríguez, and J. Solomin, *Approximation of the vibration modes of a plate by Reissner-Mindlin equations*, Math. Comp., 68 (1999), pp. 1447–1463.

[11] R. Durán and E. Liberman, *On mixed finite elements methods for the Reissner-Mindlin plate model*, Math. Comp., 58 (1992), pp. 561–573.

[12] R. Falk, *Finite Elements for the Reissner-Mindlin Plate*, in Mixed Finite Elements, Compatibility Conditions, and Applications, D. Boffi and L. Gastaldi, eds., Springer-Verlag, Berlin, 2008, pp. 195–230.

[13] K. Ishihara, *On the mixed finite element approximation for the buckling of plates*, Numer. Math., 33 (1979), pp. 195–210.

[14] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.

[15] S. Kitipornchai, Y. Xiang, C. M. Wang, and K. M. Liew, *Buckling of thick skew plates*, Internat. J. Numer. Methods Engrg., 36, (1993), pp. 1299–1310.

[16] B. Mercier, J. Osborn, J. Rappaz, and P. A. Raviart, *Eigenvalue approximation by mixed and hybrid methods*, Math. Comp., 36 (1981), pp. 427–453.

[17] D. Mora and R. Rodríguez, *A piecewise linear finite element method for the buckling and the vibration problems of thin plates*, Math. Comp., 78 (2009), pp. 1891–1917.

[18] R. Rannacher, *Nonconforming finite element methods for eigenvalue problems in linear plate theory*, Numer. Math., 33 (1979), pp. 23–42.

[19] J. N. Reddy, *Energy and Variational Methods in Applied Mechanics*, Wiley, New York, 1984.

[20] J. N. Reddy, *Mechanics of Laminated Composite Plates—Theory and Analysis*, CRC Press, Boca Raton, 1997.

[21] G. Savaré, *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal., 152 (1998), pp. 176–201.

[22] J. P. Smith, *Buckling of shear deformable plates using the p-version of the finite element method*, Comput. & Structures, 57 (1995), pp. 527–532.

[23] B. Szabo and G. Kiralyfalvi, *Linear models of buckling and stress-stiffening*, Comput. Methods Appl. Mech. Engrg., 171 (1999), pp. 43–59.

[24] H. Zhong and C. Gu, *Buckling of simply supported rectangular Reissner–Mindlin plates subjected to linearly varying in-plane loading*, J. Engrg. Mech., 132 (2006), pp. 578–581.