

# A fully-discrete virtual element method for the nonstationary Boussinesq equations in stream-function form

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## Abstract

In the present work we propose and analyze a fully-coupled virtual element method of high order for solving the two dimensional nonstationary Boussinesq system in terms of the stream-function and temperature fields. The discretization for the spatial variables is based on the coupling  $C^1$ - and  $C^0$ -conforming virtual element approaches, while a backward Euler scheme is employed for the temporal variable. Well-posedness and unconditional stability of the fully-discrete problem are provided. Moreover, error estimates in  $H^2$ - and  $H^1$ -norms are derived for the stream-function and temperature, respectively. Finally, a set of benchmark tests are reported to confirm the theoretical error bounds and illustrate the behavior of the fully-discrete scheme. © 2023 Elsevier B.V. All rights reserved.

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## 1. Introduction

The Boussinesq system is typically used to describe the natural convection in a viscous incompressible fluid, which consists of coupling between the Navier–Stokes equations with a convection–diffusion equation. Such coupling is done by means of a buoyancy term (in the momentum equation of the Navier–Stokes system) and convective heat transfer (in the energy equation). Applications of this fluid–thermal system appears in several engineering processes, such as, industrial ovens, cooling procedures (cooling of steel industries, electronic and electric equipments, nuclear reactors, etc.). Moreover, this physical phenomena appears in oceanography and geophysics when studying oceanic flows and climate predictions.

Due its relevance and presence in different applications, several works have been devoted to study these equations (and some variants). For the analysis of existence, uniqueness and regularity of the solution, we refer to [1,2]. Besides, over the last decades several discretizations have been employed to solve this system; see for instance

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[3–12] and the references therein, where the steady and unsteady regimens, temperature-dependent parameters problems have been studied, considering the classical velocity–pressure–temperature and pseudostress–velocity–temperature formulations.

Typically, in the existing literature, the majority of the discretizations for the fluid part involve the standard velocity–pressure formulation for the Boussinesq system. However, some researchers have developed numerical methods by using the stream-function–vorticity and pure stream-function approaches to approximate this system. For instance, in [13] a finite element discretization is considered to solve the problem in stream-function–vorticity–temperature form, numerical solutions are obtained for the natural convection in a square cavity and compared with some results available in the literature. In [14] a fourth-order compact finite difference scheme is formulated for solving the steady regimen, by using also the stream-function–vorticity–temperature formulation. Numerical experiments are also presented. More recently, in [15,16], the authors present an analysis of stability and convergence for a fourth-order finite difference method for the unsteady regimen of Boussinesq equations with the stream-function–vorticity–temperature approach. Numerical results are provided in [15]. On the other hand, in [17], the authors employed a  $C^1$  finite element method to approximate the stream-function variable. Numerical solution for the 2D natural convection in a square cavity are presented and compared with benchmark results [18].

For two dimensional fluid problems, the formulation in terms of the stream-function presents several attractive features, among these we can mention: the velocity vector and pressure fields are not present in the formulation, instead only one scalar variable (the stream-function) is the main unknown to approximate. By construction the incompressibility constraint is automatically satisfied. Moreover, the resulting trilinear form in the momentum equation is naturally skew-symmetric, which allows more direct stability and convergence arguments. On the other hand, in comparison with the stream-function–vorticity form, our approach avoid the difficulties related with the definition of the boundary values for the vorticity field, present in such formulation.

Nevertheless, the construction of subspaces of  $H^2$  (space where the stream-function belongs) by using finite element method involve high order polynomials and a large number of degrees of freedom, which are considered a difficult task principally from the computational viewpoint, even for triangular decompositions. As an alternative to avoid the aforementioned drawback, we consider the approach presented in [19,20] to introduce  $C^1$ -virtual element schemes of arbitrary order  $k \geq 2$ , to approximate the stream-function variable of the Boussinesq system.

The Virtual Element Method (VEM) were introduced in the seminal work [21] as an extension of Finite Elements Method (FEM) to polygonal or polyhedral decompositions. In this first work the Poisson equation is used to illustrate the main ideas of VEM approach. The virtual element spaces are constituted by polynomial and nonpolynomial functions, the degrees of freedom must be chosen appropriately so that the stiffness matrix and load term can be computed without computing these nonpolynomial functions. Later on, in [19] is introduced a new family of  $C^1$ -virtual element of high order  $k \geq 2$ , to solve Kirchhoff–Love plate problems, which in the lowest order polynomial degree employed only 3 degrees of freedom per mesh vertex (the function and its gradient values vertex). This fact represents a very significant advantage over  $C^1$  schemes based on FEM. Moreover, in [22,23], the authors discuss the application of VEM to construct finite dimensional spaces of arbitrarily regular  $C^\alpha$ , with  $\alpha \geq 1$ , where promising results have been observed to solve equations involving high order PDEs. In the last year a wide variety of second- and fourth-order problems have been discretized by using VEM. Due to the large number of papers available in the literature, we here limit ourselves in citing some representative articles within the area of fluid mechanics, where several models have been addressed with the *conforming* VEM approach: the Stokes equations [24–27], the Brinkman model [28,29], Navier–Stokes and incompressible flows [30–35], the Quasi-Geostrophic equations of the ocean [36] and Boussinesq system [37,38], where different formulations have been considered.

According to the previously discussed, in the present contribution, we are interested in further exploring the ability of VEM to approximate coupled nonlinear fluid flow problems considering the stream-function approach. More precisely, we develop and analyze a fully-discrete VE scheme for solving the nonstationary Boussinesq system. Under assumption that the domain is simply connected and by using the incompressibility condition of the velocity field, we write a equivalent variational formulation in terms of the stream-function and temperature unknowns. The discretization for the spatial variables is based on the *coupling* of  $C^1$ - and  $C^0$ - conforming virtual element approaches [19,21], for the stream-function and temperature fields, respectively, and we handle the time derivatives with a classical backward Euler implicit method. Employing the discretizations mentioned above, we propose a fully-discrete scheme of high order, which is fully-coupled, implicit in the nonlinear terms and unconditionally stable. By using the fixed point theory, we establish the corresponding existence of a discrete solution and, under

a small time step assumption, we prove that such discrete solution is also unique. Moreover, employing the natural skew-symmetry property of the resulting discrete trilinear form (in the momentum equation) we provide optimal error estimates in  $H^2$ - and  $H^1$ -norms for the stream-function and temperature, respectively.

The remainder of this paper has been organized as follows: In Section 2 we provide preliminaries notations and recall the unsteady Boussinesq equations in its standard velocity–pressure–temperature formulation. Moreover, we write a weak form of the system in terms of the stream-function and temperature variables. We finish this section by recalling the corresponding stability and well-posedness results for the continuous problem. In Section 3 we present the VE discretization, introducing the polygonal decomposition and mesh notations, the construction of stream-function and temperature VE spaces along with their corresponding degrees of freedom, the polynomial projections and the construction of the multilinear forms. In Section 4 we present the fully-discrete VE formulation and provide its stability and well-posedness. In Section 5 we derive error estimates for the stream-function and temperature fields. Finally, three numerical experiments, including the solution of the 2D natural convection benchmark problem, are presented in Section 6, to illustrate the good performance of the scheme and confirm our theoretical predictions.

## 2. Preliminaries and the continuous problem

We start this section introducing some preliminary notations that will be used throughout this work. Thenceforth,  $\Omega$  will denote a simply connected bounded domain of  $\mathbb{R}^2$ , with Lipschitz-continuous boundary  $\Gamma := \partial\Omega$  and  $\mathbf{n} = (n_i)_{1 \leq i \leq 2}$  is the outward unit normal vector to the boundary  $\Gamma$  and  $\mathbf{t} = (t_i)_{i=1,2} := (-n_2, n_1)$  is the unit tangent vector to  $\Gamma$ . Moreover, we denote by  $\partial_n$  to the normal derivative. According to [39], for any open measurable bounded domain  $\mathcal{D} \subseteq \Omega$ , we will employ the usual notation for the Banach spaces  $L^p(\mathcal{D})$  and the Sobolev spaces  $\mathbf{W}_p^s(\mathcal{D})$ , with  $s \geq 0$  and  $p \in [1, +\infty]$ , with the corresponding seminorms and norms are denoted by  $|\cdot|_{\mathbf{W}_p^s(\mathcal{D})}$  and  $\|\cdot\|_{\mathbf{W}_p^s(\mathcal{D})}$ , respectively. We adopt the convention  $\mathbf{W}_p^0(\mathcal{D}) := L^p(\mathcal{D})$  and in particular when  $p = 2$ , we write  $H^s(\mathcal{D})$  instead to  $\mathbf{W}_2^s(\mathcal{D})$ , the corresponding seminorm and norm of these space will be denoted by  $|\cdot|_{s,\mathcal{D}}$  and  $\|\cdot\|_{s,\mathcal{D}}$ , respectively. Furthermore, we denote by  $\mathbf{S}$  the corresponding vectorial version of a generic scalar  $S$ , examples of this are:  $\mathbf{L}^p(\mathcal{D}) := [L^p(\mathcal{D})]^2$  and  $\mathbf{W}_p^s(\mathcal{D}) := [\mathbf{W}_p^s(\mathcal{D})]^2$ .

We denote by  $t$  the temporal variable with values in the interval  $I := (0, T]$ , where  $T > 0$  is a given final time. Moreover, given a Banach space  $V$  endowed with the norm  $\|\cdot\|_V$ , we define the space  $L^p(0, T; V)$  as the space of classes of functions  $\phi : (0, T) \rightarrow V$  that are Bochner measurable and such that  $\|\phi\|_{L^p(0,T;V)} < \infty$ , with

$$\|\phi\|_{L^p(0,T;V)} := \left( \int_0^T \|\phi(t)\|_V^p dt \right)^{1/p} \quad \text{and} \quad \|\phi\|_{L^\infty(0,T;V)} := \operatorname{ess\,sup}_{t \in [0,T]} \|\phi(t)\|_V.$$

### 2.1. The time dependent Boussinesq system

In this work we are interested in approximating the solution of the nonstationary Boussinesq system, modeling incompressible nonisothermal fluid flows. The system consists of a coupling between the Navier–Stokes equations with a convection–diffusion equation for the temperature variable. The coupling is by means of a buoyancy term (in the momentum equation of the Navier–Stokes system) and convective heat transfer (in the energy equation). More precisely, given suitable initial data  $(\mathbf{u}_0, \theta_0)$ , the aforementioned system of equations are given by (see [1]):

$$\begin{aligned} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mathbf{g} \theta &= \mathbf{f}_\psi & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \times (0, T), \\ \mathbf{u} &= \mathbf{0} & \text{on } \Gamma \times (0, T), \\ \mathbf{u}(0) &= \mathbf{u}_0 & \text{in } \Omega \text{ at } t = 0, \\ (p, 1)_{0,\Omega} &= 0 & \\ \partial_t \theta - \kappa \Delta \theta + \mathbf{u} \cdot \nabla \theta &= f_\theta & \text{in } \Omega \times (0, T), \\ \theta &= 0 & \text{on } \Gamma \times (0, T), \\ \theta(0) &= \theta_0 & \text{in } \Omega \text{ at } t = 0, \end{aligned} \tag{2.1}$$

where  $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^2$ ,  $p : \Omega \times (0, T) \rightarrow \mathbb{R}$  and  $\theta : \Omega \times (0, T) \rightarrow \mathbb{R}$  denote the velocity, pressure and temperature fields. The parameters  $\nu > 0$  and  $\kappa > 0$  are the viscosity fluid and the thermal conductivity, respectively.

The functions  $\mathbf{f}_\psi : \Omega \times (0, T) \rightarrow \mathbb{R}^2$ ,  $f_\theta : \Omega \times (0, T) \rightarrow \mathbb{R}$  is a set of external forces and  $\mathbf{g} : \Omega \times (0, T) \rightarrow \mathbb{R}^2$  is a force per unit mass.

In next subsection, by using the incompressibility property of the velocity field, we will write an equivalent weak formulation of the system (2.1) in terms of the stream-function and temperature variables.

### 2.2. The time dependent stream-function–temperature formulation

Let us introduce the following space of functions belonging to  $\mathbf{H}_0^1(\Omega)$  with vanishing divergence:

$$\mathbf{Z} := \{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : \operatorname{div} \mathbf{v} = 0 \}.$$

Since  $\Omega \subset \mathbb{R}^2$  is simply connected, a well known result states that a vector function  $\mathbf{v} \in \mathbf{Z}$  if and only if there exists a scalar function  $\varphi \in H^2(\Omega)$  (called *stream-function*), such that

$$\mathbf{v} = \operatorname{curl} \varphi \in \mathbf{H}_0^1(\Omega).$$

The function  $\varphi$  is defined up to a constant (see [40]). Thus, we consider the following space

$$H_0^2(\Omega) = \{ \varphi \in H^2(\Omega) : \varphi = \partial_{\mathbf{n}} \varphi = 0 \text{ on } \Gamma \}.$$

Then, choosing  $\psi(t) \in H_0^2(\Omega)$  the stream-function of the velocity field  $\mathbf{u}(t) \in \mathbf{Z}$  (i.e.  $\mathbf{u}(t) = \operatorname{curl} \psi(t)$ ) in the momentum equation of system (2.1), testing against a function  $\mathbf{v} = \operatorname{curl} \phi$  with  $\phi \in H_0^2(\Omega)$  and applying twice an integration by parts, we have

$$\int_{\Omega} \operatorname{curl}(\partial_t \psi) \cdot \operatorname{curl} \phi + \nu \int_{\Omega} \mathbf{D}^2 \psi : \mathbf{D}^2 \phi + \int_{\Omega} \Delta \psi \operatorname{curl} \psi \cdot \nabla \phi - \int_{\Omega} \mathbf{g} \theta \cdot \operatorname{curl} \phi = \int_{\Omega} \mathbf{f}_\psi \cdot \operatorname{curl} \phi \quad \forall \phi \in H_0^2(\Omega).$$

On other hand, multiplying by  $v \in H_0^1(\Omega)$  and integrating by parts in the energy equation of system (2.1), we obtain

$$\int_{\Omega} \partial_t \theta v + \kappa \int_{\Omega} \nabla \theta \cdot \nabla v + \int_{\Omega} (\operatorname{curl} \psi \cdot \nabla \theta) v = \int_{\Omega} f_\theta v \quad \forall v \in H_0^1(\Omega).$$

From the above identities, we obtain the following weak formulation for system (2.1): given  $\psi_0 \in H_0^1(\Omega)$ ,  $\theta_0 \in L^2(\Omega)$ ,  $\mathbf{g} \in L^\infty(0, T; L^\infty(\Omega))$ , and the external forces  $\mathbf{f}_\psi \in L^2(0, T; L^2(\Omega))$ ,  $f_\theta \in L^2(0, T; L^2(\Omega))$ , find  $(\psi, \theta) \in L^2(0, T; H_0^2(\Omega)) \times L^2(0, T; H_0^1(\Omega))$  such that

$$\begin{aligned} M_F(\partial_t \psi, \phi) + \nu A_F(\psi, \phi) + B_F(\psi; \psi, \phi) - C(\theta, \phi) &= F_\psi(\phi) \quad \forall \phi \in H_0^2(\Omega), \text{ for a.e. } t \in (0, T), \\ M_T(\partial_t \theta, v) + \kappa A_T(\theta, v) + B_T(\psi; \theta, v) &= F_\theta(v) \quad \forall v \in H_0^1(\Omega), \text{ for a.e. } t \in (0, T), \\ \psi(0) &= \psi_0, \quad \theta(0) = \theta_0, \end{aligned} \tag{2.2}$$

where the bilinear forms  $M_F(\cdot, \cdot)$ ,  $M_T(\cdot, \cdot)$ ,  $A_F(\cdot, \cdot)$  and  $A_T(\cdot, \cdot)$  are given by

$$M_F(\cdot, \cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}, \quad M_F(\varphi, \phi) := \int_{\Omega} \operatorname{curl} \varphi \cdot \operatorname{curl} \phi, \tag{2.3}$$

$$M_T(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}, \quad M_T(v, w) := \int_{\Omega} v w, \tag{2.4}$$

$$A_F : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}, \quad A_F(\varphi, \phi) := \int_{\Omega} \mathbf{D}^2 \varphi : \mathbf{D}^2 \phi, \tag{2.5}$$

$$A_T : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}, \quad A_T(v, w) := \int_{\Omega} \nabla v \cdot \nabla w, \tag{2.6}$$

whereas the convective trilinear forms  $B_F(\cdot; \cdot, \cdot)$  and  $B_T(\cdot; \cdot, \cdot)$  are defined by

$$B_F : H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}, \quad B_F(\zeta; \varphi, \phi) := \int_{\Omega} \Delta \zeta \operatorname{curl} \varphi \cdot \nabla \phi, \tag{2.7}$$

$$B_T : H_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}, \quad B_T(\varphi; v, w) := \int_{\Omega} (\operatorname{curl} \varphi \cdot \nabla v) w. \tag{2.8}$$

The bilinear form  $C(\cdot, \cdot)$  associated to the buoyancy term is given by

$$C : H_0^1(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}, \quad C(v, \phi) := \int_{\Omega} \mathbf{g} v \cdot \operatorname{curl} \phi \tag{2.9}$$

and the functionals  $F_\psi(\cdot)$  and  $F_\theta(\cdot)$  are given by

$$F_\psi : H_0^2(\Omega) \rightarrow \mathbb{R}, \quad F_\psi(\phi) := \int_\Omega \mathbf{f}_\psi \cdot \mathbf{curl} \phi, \tag{2.10}$$

$$F_\theta : H_0^1(\Omega) \rightarrow \mathbb{R}, \quad F_\theta(v) := \int_\Omega f_\theta v. \tag{2.11}$$

We can observe by a direct computation that the trilinear form  $B_T(\cdot; \cdot, \cdot)$  defined in (2.8) is skew-symmetric, i.e.,

$$B_T(\varphi; v, w) = -B_T(\varphi; w, v) \quad \forall \varphi \in H_0^2(\Omega) \quad \text{and} \quad \forall v, w \in H_0^1(\Omega).$$

Therefore, the bilinear form  $B_T(\cdot; \cdot, \cdot)$  is equal to its skew-symmetric part, defined by

$$B_{\text{skew}}(\varphi; v, w) := \frac{1}{2}(B_T(\varphi; v, w) - B_T(\varphi; w, v)) \quad \forall \varphi \in H_0^2(\Omega) \quad \text{and} \quad \forall v, w \in H_0^1(\Omega). \tag{2.12}$$

According with the above discussion, we rewrite system (2.2) in the following equivalent formulation: given the initial conditions  $(\psi_0, \theta_0) \in H_0^1(\Omega) \times L^2(\Omega)$  and the forces  $\mathbf{f}_\psi \in L^2(0, T; \mathbf{L}^2(\Omega))$ ,  $f_\theta \in L^2(0, T; L^2(\Omega))$  and  $\mathbf{g} \in L^\infty(0, T; \mathbf{L}^\infty(\Omega))$ , find  $(\psi, \theta) \in L^2(0, T; H_0^2(\Omega)) \times L^2(0, T; H_0^1(\Omega))$  such that

$$\begin{aligned} M_F(\partial_t \psi, \phi) + \nu A_F(\psi, \phi) + B_F(\psi; \psi, \phi) - C(\theta, \phi) &= F_\psi(\phi) & \forall \phi \in H_0^2(\Omega), \quad \text{for a.e. } t \in (0, T), \\ M_T(\partial_t \theta, v) + \kappa A_T(\theta, v) + B_{\text{skew}}(\psi; \theta, v) &= F_\theta(v) & \forall v \in H_0^1(\Omega), \quad \text{for a.e. } t \in (0, T), \\ \psi(0) &= \psi_0, & \theta(0) &= \theta_0. \end{aligned} \tag{2.13}$$

### 2.3. Well-posedness of the weak formulation

In this subsection we recall some basic properties of the continuous forms and the existence and uniqueness properties of the solution to problem (2.13).

**Lemma 2.1.** *For all  $\zeta, \varphi, \phi \in H_0^2(\Omega)$  and for each  $v, w \in H_0^1(\Omega)$ , the forms defined in (2.3)–(2.12) satisfy the following properties:*

$$\begin{aligned} |M_F(\varphi, \phi)| &\leq C_{M_F} \|\varphi\|_{1,\Omega} \|\phi\|_{1,\Omega} & \text{and} & & M_F(\phi, \phi) &\geq |\phi|_{1,\Omega}^2, \\ |M_T(v, w)| &\leq C_{M_T} \|v\|_{0,\Omega} \|w\|_{0,\Omega} & \text{and} & & M_T(v, v) &\geq \|v\|_{0,\Omega}^2, \\ |A_F(\varphi, \phi)| &\leq C_{A_F} \|\varphi\|_{2,\Omega} \|\phi\|_{2,\Omega} & \text{and} & & A_F(\phi, \phi) &\geq \alpha_{A_F} \|\phi\|_{2,\Omega}^2, \\ |A_T(v, w)| &\leq C_{A_T} \|v\|_{1,\Omega} \|w\|_{1,\Omega} & \text{and} & & A_T(v, v) &\geq \alpha_{A_T} \|v\|_{1,\Omega}^2, \\ |B_F(\zeta; \varphi, \phi)| &\leq C_{B_F} \|\zeta\|_{2,\Omega} \|\varphi\|_{2,\Omega} \|\phi\|_{2,\Omega} & \text{and} & & B_F(\zeta; \phi, \phi) &= 0, \\ |B_{\text{skew}}(\zeta; v, w)| &\leq C_{B_T} \|\zeta\|_{2,\Omega} \|v\|_{1,\Omega} \|w\|_{1,\Omega} & \text{and} & & B_{\text{skew}}(\zeta; v, v) &= 0, \\ |C(v, \phi)| &\leq \|\mathbf{g}\|_{\infty,\Omega} \|v\|_{0,\Omega} \|\phi\|_{1,\Omega}, & |F_\psi(\phi)| &\leq C_{F_\psi} \|\mathbf{f}_\psi\|_{0,\Omega} \|\phi\|_{1,\Omega}, & |F_\theta(v)| &\leq C_{F_\theta} \|f_\theta\|_{0,\Omega} \|v\|_{0,\Omega}. \end{aligned}$$

The equivalence between the (weak form of) problem (2.1) and its stream-function formulation (2.13) is well known and easy to check. The couple  $(\psi, \theta)$  satisfies (2.13) if and only if there exists a unique  $p$  such that the triple  $(\mathbf{u}, \theta, p)$  in  $L^2(0, T; \mathbf{H}_0^1(\Omega)) \times L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; L_0^2(\Omega))$  solves (the variational formulation of) (2.1), where  $\mathbf{u} = \mathbf{curl} \psi$ . Therefore the existence result for problem (2.13) follow immediately from known results for (2.1) (see for instance [5]) and the uniqueness follow by combining the arguments used in [1].

**Theorem 2.1.** *Problem (2.13) admits a unique solution  $(\psi, \theta)$ , satisfying  $\psi \in L^2(0, T; H_0^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$  and  $\theta \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ . Furthermore there exists a positive constant  $C$ , such that*

$$\begin{aligned} \|\psi\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\psi\|_{L^2(0,T;H_0^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^2(0,T;H_0^1(\Omega))} \\ \leq C (\|\mathbf{f}_\psi\|_{L^2(0,T;L^2(\Omega))} + \|f_\theta\|_{L^2(0,T;L^2(\Omega))} + \|\theta_0\|_{0,\Omega} + |\psi_0|_{1,\Omega}). \end{aligned}$$

Now, we recall the Ladyzhenskaya inequality (see for instance [12, Lemma 2.2]), needed in the sequel:

$$\|v\|_{L^4(\Omega)} \leq 2^{\frac{1}{4}} \|v\|_{1,\Omega}^{\frac{1}{2}} \|v\|_{0,\Omega}^{\frac{1}{2}} \quad \forall v \in \mathbf{H}_0^1(\Omega). \tag{2.14}$$

We close this section with the following remark.

**Remark 2.1.** For the bilinear form  $A_F(\cdot, \cdot)$  defined in (2.5), we have the following classical identity:

$$A_F(\varphi, \phi) = \int_{\Omega} \Delta\varphi \Delta\phi \quad \forall \varphi, \phi \in H_0^2(\Omega). \tag{2.15}$$

We recall that at discrete level the representations (2.5) and (2.15) will lead to different approximations, in general. In next section we will consider the representation (2.5), i.e.,  $A_F(\varphi, \phi) = \int_{\Omega} \mathbf{D}^2\varphi : \mathbf{D}^2\phi$ , in order to construct the projection  $\Pi_E^{\mathbf{D},k}$  (see (3.2)). However, we also propose an alternative discretization inspired by (2.15) in Remark 3.2 below.

### 3. Virtual elements discretization

In this section we will introduce  $C^1$ - and  $C^0$ -conforming schemes of arbitrary order  $k \geq 2$  and  $\ell \geq 1$ , for the numerical approximation of the stream-function and temperature unknowns of problem (2.13), respectively. First, we start by introducing some mesh notations together with the respective local and global virtual spaces and their degrees of freedom. Moreover, we introduce the classical VEM polynomial projections and we present the discrete multilinear forms.

#### 3.1. Polygonal decompositions and notations

Henceforth, we will denote by  $E$  a general polygon,  $e$  a general edge of  $\partial E$ ,  $h_E$  the diameter of the element  $E$  and by  $h_e$  the length of edge  $e$ . Let  $\{\Omega_h\}_{h>0}$  be a sequence of decompositions of  $\Omega$  into non-overlapping polygons  $E$ , where  $h := \max_{E \in \Omega_h} h_E$ . Moreover,  $N_E$  denotes the number of vertices of  $E$  and we define the unit normal vector  $\mathbf{n}_E$ , that points outside of  $E$  and the unit tangent vector  $\mathbf{t}_E$  to  $E$  obtained by a counterclockwise rotation of  $\mathbf{n}_E$ .

For each integer  $n \geq 0$ , we introduce the following spaces:

- For every open bounded subdomain  $\mathcal{D} \subset \mathbb{R}^2$  we define  $\mathbb{P}_n(\mathcal{D})$  as the space of polynomials on  $\mathcal{D}$  of degree up to  $n$  and we denote by  $\mathbf{P}_n(\mathcal{D})$  its vectorial version, i.e.,  $\mathbf{P}_n(\mathcal{D}) := [\mathbb{P}_n(\mathcal{D})]^2$ ;
- We define the discontinuous piecewise  $n$ -order polynomial by

$$\mathbb{P}_n(\Omega_h) := \{q \in L^2(\Omega) : q|_E \in \mathbb{P}_n(E) \quad \forall E \in \Omega_h\}.$$

Besides, for  $s > 0$ , we consider the broken spaces

$$H^s(\Omega_h) := \{\phi \in L^2(\Omega) : \phi|_E \in H^s(E) \quad \forall E \in \Omega_h\}$$

endowed with the following broken seminorm:  $|\phi|_{s,h} := \left(\sum_{E \in \Omega_h} |\phi|_{s,E}^2\right)^{1/2}$ .

For the theoretical convergence analysis, we suppose that for all  $h$ , each element  $E$  in the mesh family  $\{\Omega_h\}_{h>0}$  satisfies the following assumptions [20,21] for a uniform constant  $\rho > 0$ :

- A1** :  $E$  is star-shaped with respect to every point of a ball of radius greater or equal to  $\rho h_E$ ;
- A2** : every edge  $e \in \partial E$  has the length greater or equal to  $\rho h_E$ .

#### 3.2. Virtual element space for the stream-function

In the present section we introduce a virtual space of order  $k \geq 2$  used to approximate the stream-function unknown.

For each polygon  $E \in \Omega_h$  and every integer  $k \geq 2$ , let  $\widehat{k} := \max\{k, 3\}$  and  $\widetilde{\mathbf{W}}_k^h(E)$  be the finite dimensional space introduced in [20]:

$$\begin{aligned} \widetilde{\mathbf{W}}_k^h(E) := \{ \phi_h \in H^2(E) : \Delta^2 \phi_h \in \mathbb{P}_{k-2}(E), \phi_h|_{\partial E} \in C^0(\partial E), \phi_h|_e \in \mathbb{P}_{\widehat{k}}(e) \quad \forall e \in \partial E, \\ \nabla \phi_h|_{\partial E} \in \mathbf{C}^0(\partial E), \partial_{\mathbf{n}_E} \phi_h \in \mathbb{P}_{k-1}(e) \quad \forall e \in \partial E \}. \end{aligned}$$

Next, for  $\phi_h \in \tilde{W}_k^h(E)$ , we introduce the following set of linear operators:

- **D<sub>W</sub>1** : the values of  $\phi_h(\mathbf{v}_i)$ , for all vertex  $\mathbf{v}_i$  of the polygon  $E$ ;
- **D<sub>W</sub>2** : the values of  $h_{\mathbf{v}_i} \nabla \phi_h(\mathbf{v}_i)$ , for all vertex  $\mathbf{v}_i$  of the polygon  $E$ ;
- **D<sub>W</sub>3** : for  $k \geq 3$ , the moments on edges up to degree  $k - 3$ :

$$(q, \partial_{\mathbf{n}_E} \phi_h)_{0,e} \quad \forall q \in \mathbb{M}_{k-3}(e), \quad \forall \text{edge } e;$$

- **D<sub>W</sub>4** : for  $k \geq 4$ , the moments on edges up to degree  $k - 4$ :

$$h_e^{-1}(q, \phi_h)_{0,e} \quad \forall q \in \mathbb{M}_{k-4}(e), \quad \forall \text{edge } e;$$

- **D<sub>W</sub>5** : for  $k \geq 4$ , the moments on polygons up to degree  $k - 4$ :

$$h_E^{-2}(q, \phi_h)_{0,E} \quad \forall q \in \mathbb{M}_{k-4}(E), \quad \forall \text{polygon } E,$$

where for each vertex  $\mathbf{v}_i$ , we chose  $h_{\mathbf{v}_i}$  as the average of the diameters of the elements having  $\mathbf{v}_i$  as a vertex and  $\mathbb{M}_n(E)$  denote the scaled monomials of degree  $n$ , for each  $n \geq 0$  (for further details see [19]).

In order to construct an approximation for the bilinear form  $A_F(\cdot, \cdot)$ , we consider the operator  $P_0 : C^0(\partial E) \rightarrow \mathbb{P}_0(E)$  defined by the following average:

$$P_0 \phi_h = \frac{1}{N_E} \sum_{i=1}^{N_E} \phi_h(\mathbf{v}_i), \tag{3.1}$$

where  $\mathbf{v}_i, 1 \leq i \leq N_E$ , are the vertices of  $E$ . Then, for each polygon  $E$ , we define the projector:

$$\Pi_E^{\mathbf{D},k} : \tilde{W}_k^h(E) \rightarrow \mathbb{P}_k(E) \subset \tilde{W}_k^h(E),$$

as the solution of the local problems

$$\begin{aligned} A_F^E(\phi_h - \Pi_E^{\mathbf{D},k} \phi_h, q_k) &= 0 \quad \forall q_k \in \mathbb{P}_k(E), \\ P_0(\phi_h - \Pi_E^{\mathbf{D},k} \phi_h) &= 0, \quad P_0(\nabla(\phi_h - \Pi_E^{\mathbf{D},k} \phi_h)) = 0, \end{aligned} \tag{3.2}$$

where  $A_F^E(\cdot, \cdot)$  is the restriction of the global bilinear form  $A_F(\cdot, \cdot)$  (cf. (2.5)) on each polygon  $E$ .

**Remark 3.1.** The operator  $\Pi_E^{\mathbf{D},k} : \tilde{W}_k^h(E) \rightarrow \mathbb{P}_k(E)$  is explicitly computable for every  $\phi_h \in \tilde{W}_k^h(E)$ , using only the information of the linear operators **D<sub>W</sub>1** – **D<sub>W</sub>5**; see for instance [20,29].

Now, we will present the local stream-function virtual space. For any  $E \in \Omega_h$  and each integer  $k \geq 2$ , we consider the following local enhanced virtual space

$$W_k^h(E) := \left\{ \phi_h \in \tilde{W}_k^h(E) : (q^*, \phi_h - \Pi_E^{\mathbf{D},k} \phi_h)_{0,E} = 0 \quad \forall q^* \in \mathbb{M}_{k-3}^*(E) \cup \mathbb{M}_{k-2}^*(E) \right\}, \tag{3.3}$$

where  $\mathbb{M}_{k-3}^*(E)$  and  $\mathbb{M}_{k-2}^*(E)$  are scaled monomials of degree  $k - 3$  and  $k - 2$ , respectively (see [41]), with the convention that  $\mathbb{M}_{-1}^*(E) := \emptyset$ . For further details, see for instance [20] (see also [19,29,42]).

For  $k \geq 2$ , we introduce an additional projector, which will be used to build an approximation of the bilinear form  $M_F(\cdot, \cdot)$ . Such projector  $\Pi_E^{\mathbf{c},k} : \tilde{W}_k^h(E) \rightarrow \mathbb{P}_k(E) \subset \tilde{W}_k^h(E)$  is defined as the solution of the local problems:

$$\begin{aligned} M_F^E(\phi_h - \Pi_E^{\mathbf{c},k} \phi_h, q_k) &= 0 \quad \forall q_k \in \mathbb{P}_k(E), \\ P_0(\nabla(\phi_h - \Pi_E^{\mathbf{c},k} \phi_h)) &= 0, \end{aligned}$$

where  $M_F^E(\cdot, \cdot)$  is the restriction of the global bilinear form  $M_F(\cdot, \cdot)$  (cf. (2.3)) on each polygon  $E$ .

We summarize the main properties of the local virtual space  $W_k^h(E)$  defined in (3.3) (for the proof, we refer to [19,20,29,41]).

- $\mathbb{P}_k(E) \subset W_k^h(E) \subset \tilde{W}_k^h(E)$ ;
- The sets of linear operators **D<sub>W</sub>1** – **D<sub>W</sub>5** constitutes a set of degrees of freedom for  $W_k^h(E)$ ;
- The operators  $\Pi_E^{\mathbf{D},k} : \tilde{W}_k^h(E) \rightarrow \mathbb{P}_k(E)$  and  $\Pi_E^{\mathbf{c},k} : \tilde{W}_k^h(E) \rightarrow \mathbb{P}_k(E)$  are computable using only the degrees of freedom **D<sub>W</sub>1** – **D<sub>W</sub>5**.

Now, we present our global virtual space to approximate the stream-function of the Boussinesq system (2.13). For each decomposition  $\Omega_h$  of  $\Omega$  into simple polygons  $E$ , we define

$$\mathbf{W}_k^h := \{ \phi_h \in \mathbf{H}_0^2(\Omega) : \phi_h|_E \in \mathbf{W}_k^h(E) \quad \forall E \in \Omega_h \}.$$

### 3.3. Virtual element space for the temperature

In this subsection we will introduce a  $C^0$ -virtual element space of high order  $\ell \geq 1$  to approximate the temperature field of problem (2.13). To this end, for each polygon  $E \in \Omega_h$ , we consider the following finite dimensional space (see [41,43,44]):

$$\tilde{\mathbf{H}}_\ell^h(E) := \{ w_h \in \mathbf{H}^1(E) \cap C^0(\partial E) : \Delta w_h \in \mathbb{P}_\ell(E), \quad w_h|_e \in \mathbb{P}_\ell(e) \quad \forall e \in \partial E \}.$$

For each  $w_h \in \tilde{\mathbf{H}}_\ell^h(E)$  we consider the following set of linear operators:

- **D<sub>H1</sub>** : the values of  $w_h(\mathbf{v}_i)$ , for all vertex  $\mathbf{v}_i$  of the polygon  $E$ .
- **D<sub>H2</sub>** : for  $\ell \geq 2$ , the moments on edges up to degree  $\ell - 2$ :

$$h_e^{-1}(q, w_h)_{0,e} \quad \forall q \in \mathbb{M}_{\ell-2}(e), \quad \forall \text{edge } e;$$

- **D<sub>H3</sub>** : for  $\ell \geq 2$ , the moments on element  $E$  up to degree  $\ell - 2$ :

$$h_E^{-2}(q, w_h)_{0,E} \quad \forall q \in \mathbb{M}_{\ell-2}(E), \quad \forall \text{polygon } E,$$

where  $\mathbb{M}_n(E)$  denote the scaled monomials of degree  $n$ , for each  $n \geq 0$  (for further details see [41,44]). Now, we define the projector  $\Pi_E^{\nabla,\ell} : \tilde{\mathbf{H}}_\ell^h(E) \rightarrow \mathbb{P}_\ell(E) \subset \tilde{\mathbf{H}}_\ell^h(E)$ , as the solution of the local problems:

$$\begin{aligned} A_T^E(w_h - \Pi_E^{\nabla,\ell} w_h, r_\ell) &= 0 \quad \forall r_\ell \in \mathbb{P}_\ell(E), \\ P_0(w_h - \Pi_E^{\nabla,\ell} w_h) &= 0, \end{aligned} \tag{3.4}$$

where  $A_T^E(\cdot, \cdot)$  is the restriction of the global bilinear form  $A_T(\cdot, \cdot)$  (cf. (2.6)) on each polygon  $E$  and the operator  $P_0(\cdot)$  is defined in (3.1). We have that the operator  $\Pi_E^{\nabla,\ell} : \tilde{\mathbf{H}}_\ell^h(E) \rightarrow \mathbb{P}_\ell(E)$  is computable using the set **D<sub>H1</sub>** – **D<sub>H3</sub>** (see for instance, [41,43,44]). In addition, by using this projection and the definition of space  $\tilde{\mathbf{H}}_\ell^h(E)$ , we introduce our local virtual space to approximate the temperature field:

$$\mathbf{H}_\ell^h(E) := \left\{ w_h \in \tilde{\mathbf{H}}_\ell^h(E) : (r^*, w_h - \Pi_E^{\nabla,\ell} w_h)_{0,E} = 0 \quad \forall r^* \in \mathbb{M}_\ell^*(E) \cup \mathbb{M}_{\ell-1}^*(E) \right\},$$

where  $\mathbb{M}_\ell^*(E)$  and  $\mathbb{M}_{\ell-1}^*(E)$  are scaled monomials of degree  $\ell$  and  $\ell - 1$ , respectively, with the convention that  $\mathbb{M}_{-1}^*(E) := \emptyset$  (see [41,44]).

Now, we summarize the main properties of the local virtual spaces  $\mathbf{H}_\ell^h(E)$  (for a proof we refer to [41,43,44]):

- $\mathbb{P}_\ell(E) \subset \mathbf{H}_\ell^h(E) \subset \tilde{\mathbf{H}}_\ell^h(E)$ ;
- The sets of linear operators **D<sub>H1</sub>** – **D<sub>H3</sub>** constitutes a set of degrees of freedom for  $\mathbf{H}_\ell^h(E)$ ;
- The operator  $\Pi_E^{\nabla,\ell} : \mathbf{H}_\ell^h(E) \rightarrow \mathbb{P}_\ell(E)$  is also computable using the degrees of freedom **D<sub>H1</sub>** – **D<sub>H3</sub>**.

Next, we present our global virtual space to approximate the fluid temperature of the Boussinesq system (2.13). For each decomposition  $\Omega_h$  of  $\Omega$  into simple polygons  $E$ , we define

$$\mathbf{H}_\ell^h := \{ w_h \in \mathbf{H}_0^1(\Omega) : w_h|_E \in \mathbf{H}_\ell^h(E) \quad \forall E \in \Omega_h \}.$$

### 3.4. L<sup>2</sup>-Projections and the discrete forms

In this subsection we introduce some functions built from the classical L<sup>2</sup>-polynomial projections, which will be useful to construct an approximation for the continuous multilinear forms defined in Section 2.2. We start recalling the usual L<sup>2</sup>(E)-projection onto the scalar polynomial space  $\mathbb{P}_n(E)$ , with  $n \in \mathbb{N} \cup \{0\}$ : for each  $\phi \in L^2(E)$ , the function  $\Pi_E^n \phi \in \mathbb{P}_n(E)$  is defined as the unique function, such that

$$(q_n, \phi - \Pi_E^n \phi)_{0,E} = 0 \quad \forall q_n \in \mathbb{P}_n(E). \tag{3.5}$$



An analogous definition holds for the  $\mathbf{L}^2(E)$ -projection onto the vectorial polynomial space  $\mathbf{P}_n(E)$ , which we will denote by  $\mathbf{\Pi}_E^n$ .

We recall that for all sufficiently regular  $\phi$  (for the right hand side to make sense) there exists  $C > 0$ , independent of  $E$  and  $h_E$ , such that (see [30, Page 10]):

$$\|\mathbf{\Pi}_E^n \phi\|_{\mathbf{L}^4(E)} \leq C \|\phi\|_{\mathbf{L}^4(E)} \quad \text{and} \quad \|\mathbf{\Pi}_E^n \phi\|_{0,E} \leq \|\phi\|_{0,E}. \quad (3.6)$$

The same properties hold for the vectorial version.

The following lemma establishes that certain polynomial functions are computable on  $\mathbf{W}_k^h(E)$ , using only the information of the degrees of freedom  $\mathbf{D}_W\mathbf{1} - \mathbf{D}_W\mathbf{5}$  (see for instance [20,29]).

**Lemma 3.1.** For  $k \geq 2$ , let  $\mathbf{\Pi}_E^{k-2} : \mathbf{L}^2(E) \rightarrow \mathbb{P}_{k-2}(E)$  and  $\mathbf{\Pi}_E^{k-1} : \mathbf{L}^2(E) \rightarrow \mathbf{P}_{k-1}(E)$  be the operators defined by the relation (3.5) and by its vectorial version. Then, for each  $\phi_h \in \mathbf{W}_k^h(E)$  the polynomial functions

$$\mathbf{\Pi}_E^{k-2} \phi_h, \quad \mathbf{\Pi}_E^{k-2} \Delta \phi_h, \quad \mathbf{\Pi}_E^{k-1} \nabla \phi_h \quad \text{and} \quad \mathbf{\Pi}_E^{k-1} \mathbf{curl} \phi_h$$

are computable using only the information of the degrees of freedom  $\mathbf{D}_W\mathbf{1} - \mathbf{D}_W\mathbf{5}$ .

For the space  $\mathbf{H}_\ell^h(E)$  and its degrees of freedom  $\mathbf{D}_H\mathbf{1} - \mathbf{D}_H\mathbf{3}$ , we have the following result (see for instance [43, 44]).

**Lemma 3.2.** For  $\ell \geq 1$ , let  $\mathbf{\Pi}_E^{\ell-1} : \mathbf{L}^2(E) \rightarrow \mathbb{P}_{\ell-1}(E)$ ,  $\mathbf{\Pi}_E^\ell : \mathbf{L}^2(E) \rightarrow \mathbb{P}_\ell(E)$  and  $\mathbf{\Pi}_E^{\ell-1} : \mathbf{L}^2(E) \rightarrow \mathbf{P}_{\ell-1}(E)$  be the operators defined by the relation (3.5) and by its vectorial version, respectively. Then, for each  $w_h \in \mathbf{H}_\ell^h(E)$  the polynomial functions

$$\mathbf{\Pi}_E^{\ell-1} w_h, \quad \mathbf{\Pi}_E^\ell w_h \quad \text{and} \quad \mathbf{\Pi}_E^{\ell-1} \nabla w_h$$

are computable using only the information of the degrees of freedom  $\mathbf{D}_H\mathbf{1} - \mathbf{D}_H\mathbf{3}$ .

Now, using the functions introduced above, we will construct the discrete version of the forms defined in Section 2.2. First, let  $s_E^c : \mathbf{W}_k^h(E) \times \mathbf{W}_k^h(E) \rightarrow \mathbb{R}$  and  $s_E^D : \mathbf{W}_k^h(E) \times \mathbf{W}_k^h(E) \rightarrow \mathbb{R}$  be any symmetric positive definite bilinear forms to be chosen to satisfy:

$$\begin{aligned} c_0 M_F^E(\phi_h, \phi_h) &\leq s_E^c(\phi_h, \phi_h) \leq c_1 M_F^E(\phi_h, \phi_h) & \forall \phi_h \in \text{Ker}(\mathbf{\Pi}_E^{c,k}), \\ c_2 A_F^E(\phi_h, \phi_h) &\leq s_E^D(\phi_h, \phi_h) \leq c_3 A_F^E(\phi_h, \phi_h) & \forall \phi_h \in \text{Ker}(\mathbf{\Pi}_E^{D,k}), \end{aligned} \quad (3.7)$$

with  $c_0, c_1, c_2$  and  $c_3$  are positive constants independent of  $h$  and  $E$ . We will choose the following representation satisfying (3.7) (see [29, Proposition 3.5]):

$$s_E^D(\varphi_h, \phi_h) := h_E^{-2} \sum_{i=1}^{N_E^{\text{dof}}} \text{dof}_i^{\mathbf{W}_k^h(E)}(\varphi_h) \text{dof}_i^{\mathbf{W}_k^h(E)}(\phi_h) \quad \text{and} \quad s_E^c(\varphi_h, \phi_h) := \sum_{i=1}^{N_E^{\text{dof}}} \text{dof}_i^{\mathbf{W}_k^h(E)}(\varphi_h) \text{dof}_i^{\mathbf{W}_k^h(E)}(\phi_h),$$

where  $N_E^{\text{dof}} := \dim(\mathbf{W}_k^h(E))$  and the operator  $\text{dof}_j^{\mathbf{W}_k^h(E)}(\phi)$  associates to each smooth enough function  $\phi$  the  $j$ th local degree of freedom  $\text{dof}_j^{\mathbf{W}_k^h(E)}(\phi)$ , with  $1 \leq j \leq N_E^{\text{dof}}$ .

On each polygon  $E$ , we define the local discrete bilinear forms  $M_F^{h,E}(\cdot, \cdot)$  and  $A_F^{h,E}(\cdot, \cdot)$  as follows

$$M_F^{h,E}(\varphi_h, \phi_h) := M_F^E(\mathbf{\Pi}_E^{c,k} \varphi_h, \mathbf{\Pi}_E^{c,k} \phi_h) + s_E^c((\mathbf{I} - \mathbf{\Pi}_E^{c,k})\varphi_h, (\mathbf{I} - \mathbf{\Pi}_E^{c,k})\phi_h) \quad \forall \varphi_h, \phi_h \in \mathbf{W}_k^h(E), \quad (3.8)$$

$$A_F^{h,E}(\varphi_h, \phi_h) := A_F^E(\mathbf{\Pi}_E^{D,k} \varphi_h, \mathbf{\Pi}_E^{D,k} \phi_h) + s_E^D((\mathbf{I} - \mathbf{\Pi}_E^{D,k})\varphi_h, (\mathbf{I} - \mathbf{\Pi}_E^{D,k})\phi_h) \quad \forall \varphi_h, \phi_h \in \mathbf{W}_k^h(E). \quad (3.9)$$

For the approximation of the local trilinear form  $B_F^E(\cdot; \cdot, \cdot)$ , we consider

$$B_F^{h,E}(\zeta_h; \varphi_h, \phi_h) := \int_E [(\mathbf{\Pi}_E^{k-2} \Delta \zeta_h)(\mathbf{\Pi}_E^{k-1} \mathbf{curl} \varphi_h)] \cdot \mathbf{\Pi}_E^{k-1} \nabla \phi_h \quad \forall \zeta_h, \varphi_h, \phi_h \in \mathbf{W}_k^h(E). \quad (3.10)$$

For the treatment of the right-hand side associate to the fluid equation, we set the following local load term:

$$F_\psi^{h,E}(\phi_h) = \int_E \mathbf{\Pi}_E^{k-1} \mathbf{f}_\psi(t) \cdot \mathbf{curl} \phi_h \equiv \int_E \mathbf{f}_\psi(t) \cdot \mathbf{\Pi}_E^{k-1} \mathbf{curl} \phi_h \quad \forall \phi_h \in \mathbf{W}_k^h(E), \quad \text{for a.e. } t \in (0, T).$$

The following result establishes the usual  $k$ -consistency and stability properties for the discrete local forms  $M_F^{h,E}(\cdot, \cdot)$  and  $A_F^{h,E}(\cdot, \cdot)$ .

**Proposition 3.1.** *The local bilinear forms defined in (2.3), (2.5), (3.8) and (3.9), satisfy the following properties:*

- *k-consistency: for all  $E \in \Omega_h$ , we have that*

$$M_F^{h,E}(q, \phi_h) = M_F^E(q, \phi_h), \quad A_F^{h,E}(q, \phi_h) = A_F^E(q, \phi_h) \quad \forall q \in \mathbb{P}_k(E), \quad \forall \phi_h \in \mathbf{W}_k^h(E).$$

- *stability and boundedness: there exist positive constants  $\alpha_i, i = 1, \dots, 4$ , independent of  $E$ , such that:*

$$\begin{aligned} \alpha_1 M_F^E(\phi_h, \phi_h) &\leq M_F^{h,E}(\phi_h, \phi_h) \leq \alpha_2 M_F^E(\phi_h, \phi_h) & \forall \phi_h \in \mathbf{W}_k^h(E), \\ \alpha_3 A_F^E(\phi_h, \phi_h) &\leq A_F^{h,E}(\phi_h, \phi_h) \leq \alpha_4 A_F^E(\phi_h, \phi_h) & \forall \phi_h \in \mathbf{W}_k^h(E). \end{aligned}$$

**Proof.** The proof follows standard arguments in the VEM literature (see [21,42,43]).  $\square$

Now, we continue with the construction of the forms associated to the energy equation. First, let  $s_E^0(\cdot, \cdot)$  and  $s_E^\nabla(\cdot, \cdot)$  be any symmetric positive definite bilinear forms such that

$$\begin{aligned} c_4 M_T^E(v_h, v_h) &\leq s_E^0(v_h, v_h) \leq c_5 M_T^E(v_h, v_h) & \forall v_h \in \text{Ker}(\Pi_E^\ell), \\ c_6 A_T^E(v_h, v_h) &\leq s_E^\nabla(v_h, v_h) \leq c_7 A_T^E(v_h, v_h) & \forall v_h \in \text{Ker}(\Pi_E^{\nabla,\ell}), \end{aligned} \quad (3.11)$$

for some positive constants  $c_4, c_5, c_6$  and  $c_7$ , independent of  $h$  and  $E$ . We will choose the classical representation for these stabilizing forms satisfying property (3.11) (see [44–46]):

$$s_E^0(v_h, w_h) := h_E^2 \sum_{j=1}^{\dim(\mathbf{H}_\ell^h(E))} \text{dof}_j^{\mathbf{H}_\ell^h(E)}(v_h) \text{dof}_j^{\mathbf{H}_\ell^h(E)}(w_h), \quad s_E^\nabla(v_h, w_h) := \sum_{j=1}^{\dim(\mathbf{H}_\ell^h(E))} \text{dof}_j^{\mathbf{H}_\ell^h(E)}(v_h) \text{dof}_j^{\mathbf{H}_\ell^h(E)}(w_h),$$

where the operator  $\text{dof}_j^{\mathbf{H}_\ell^h(E)}(v)$  associates to each smooth enough function  $v$  the  $j$ th local degree of freedom  $\text{dof}_j^{\mathbf{H}_\ell^h(E)}(v)$ , with  $1 \leq j \leq \dim(\mathbf{H}_\ell^h(E))$ . Then, we set the following approximation for the forms  $M_T^E(\cdot, \cdot)$  and  $A_T^E(\cdot, \cdot)$  (cf. (2.4) and (2.6))

$$\begin{aligned} M_T^{h,E}(v_h, w_h) &:= M_T^E(\Pi_E^\ell v_h, \Pi_E^\ell w_h) + s_E^0((\mathbf{I} - \Pi_E^\ell)v_h, (\mathbf{I} - \Pi_E^\ell)w_h) & \forall v_h, w_h \in \mathbf{H}_\ell^h(E), \\ A_T^{h,E}(v_h, w_h) &:= \int_E \Pi_E^{\ell-1} \nabla v_h \cdot \Pi_E^{\ell-1} \nabla w_h + s_E^\nabla((\mathbf{I} - \Pi_E^{\nabla,\ell})v_h, (\mathbf{I} - \Pi_E^{\nabla,\ell})w_h) & \forall v_h, w_h \in \mathbf{H}_\ell^h(E). \end{aligned}$$

We have that the bilinear forms  $M_T^{h,E}(\cdot, \cdot)$  and  $A_T^{h,E}(\cdot, \cdot)$  satisfy the classical  $\ell$ -consistency and stability properties (analogous to Proposition 3.1). For further details, see [21,43,44].

To approximate of bilinear form  $C^E(\cdot, \cdot)$ , we set

$$C^{h,E}(w_h, \phi_h) := \int_E \mathbf{g} \Pi_E^{\ell-1} w_h \cdot \Pi_E^{k-1} \mathbf{curl} \phi_h \quad \forall w_h \in \mathbf{H}_\ell^h(E), \quad \forall \phi_h \in \mathbf{W}_k^h(E).$$

Now, we consider the following discrete trilinear form

$$B_T^{h,E}(\varphi_h; v_h, w_h) := \int_E (\Pi_E^{k-1} \mathbf{curl} \varphi_h \cdot \Pi_E^{\ell-1} \nabla v_h) \Pi_E^{\ell-1} w_h \quad \forall \varphi_h \in \mathbf{W}_k^h(E), \quad \forall w_h, v_h \in \mathbf{H}_\ell^h(E).$$

Then, for the skew-symmetric trilinear form  $B_{\text{skew}}^E(\cdot; \cdot, \cdot)$  (cf. (2.12)), we set the following approximation:

$$B_{\text{skew}}^{h,E}(\varphi_h; v_h, w_h) := \frac{1}{2} (B_T^{h,E}(\varphi_h; v_h, w_h) - B_T^{h,E}(\varphi_h; w_h, v_h)) \quad \forall \varphi_h \in \mathbf{W}_k^h(E), \quad \forall w_h, v_h \in \mathbf{H}_\ell^h(E).$$

For the treatment of the right-hand side associated to the temperature discretization, we set following local load term

$$F_\theta^{h,E}(v_h) := \int_E \Pi_E^{\ell-1} f_\theta(t) v_h \equiv \int_E f_\theta(t) \Pi_E^{\ell-1} v_h \quad \forall v_h \in \mathbf{H}_\ell^h(E) \quad \text{for a.e. } t \in (0, T).$$

Thus, for all  $\zeta_h, \varphi_h, \phi_h \in \mathbf{W}_k^h$ , we define the associated global forms  $M_F^h, A_F^h, B_F^h, F_\psi^h$  in the usual way, by summing the local forms on all mesh elements. Analogously, we define the associated global forms  $M_T^h, C^h, B_{\text{skew}}^h, F_\theta^h$  for all  $v_h, w_h \in \mathbf{H}_\ell^h$ . For instance

$$M_F^h : \mathbf{W}_k^h \times \mathbf{W}_k^h \rightarrow \mathbb{R}, \quad M_F^h(\varphi_h, \phi_h) := \sum_{E \in \Omega_h} M_F^{h,E}(\varphi_h, \phi_h).$$

We recall that the forms defined above are computable using the degrees of freedom. In addition, we have that the trilinear forms are immediately extendable to the whole continuous spaces.

In next result we summarize some properties of the discrete global forms defined above.

**Lemma 3.3.** *For each  $\zeta_h, \varphi_h, \phi_h \in \mathbf{W}_k^h$  and each  $v_h, w_h \in \mathbf{H}_\ell^h$ , the global forms defined above satisfy the following properties:*

$$\begin{aligned} |M_F^h(\varphi_h, \phi_h)| &\leq \widehat{C}_{M_F} \|\varphi_h\|_{1,\Omega} \|\phi_h\|_{1,\Omega} & \text{and} & & M_F^h(\phi_h, \phi_h) &\geq \widehat{\alpha}_{M_F} \|\phi_h\|_{1,\Omega}^2, \\ |M_T^h(v_h, w_h)| &\leq \widehat{C}_{M_T} \|v_h\|_{0,\Omega} \|w_h\|_{0,\Omega} & \text{and} & & M_T^h(v_h, v_h) &\geq \widehat{\alpha}_{M_T} \|v_h\|_{0,\Omega}^2, \\ |A_F^h(\varphi_h, \phi_h)| &\leq \widehat{C}_{A_F} \|\varphi_h\|_{2,\Omega} \|\phi_h\|_{2,\Omega} & \text{and} & & A_F^h(\phi_h, \phi_h) &\geq \widehat{\alpha}_{A_F} \|\phi_h\|_{2,\Omega}^2, \\ |A_T^h(v_h, w_h)| &\leq \widehat{C}_{A_T} \|v_h\|_{1,\Omega} \|w_h\|_{1,\Omega} & \text{and} & & A_T^h(v_h, v_h) &\geq \widehat{\alpha}_{A_T} \|v_h\|_{1,\Omega}^2, \\ |B_F^h(\zeta_h; \varphi_h, \phi_h)| &\leq \widehat{C}_{B_F} \|\zeta_h\|_{2,\Omega} \|\varphi_h\|_{2,\Omega} \|\phi_h\|_{2,\Omega} & \text{and} & & B_F^h(\zeta_h; \phi_h, \phi_h) &= 0, \\ |B_{\text{skew}}^h(\zeta_h; v_h, w_h)| &\leq \widehat{C}_{B_T} \|\zeta_h\|_{2,\Omega} \|v_h\|_{1,\Omega} \|w_h\|_{1,\Omega} & \text{and} & & B_{\text{skew}}^h(\zeta_h; v_h, v_h) &= 0, \\ |C^h(v_h, \phi_h)| &\leq \|\mathbf{g}\|_{\infty,\Omega} \|v_h\|_{0,\Omega} \|\phi_h\|_{1,\Omega}, & |F_\psi^h(\phi_h)| &\leq \widehat{C}_{F_\psi} \|\mathbf{f}_\psi\|_{0,\Omega} \|\phi_h\|_{1,\Omega} & \text{and} \\ |F_\theta^h(v_h)| &\leq \widehat{C}_{F_\theta} \|f_\theta\|_{0,\Omega} \|v_h\|_{0,\Omega}, \end{aligned}$$

where all the constants involved are positive and independent of mesh size  $h$ .

We close this section with the following remarks.

**Remark 3.2.** We can propose an alternative discretization inspired by (2.15), which is given by:

$$A_F^h(\varphi_h, \phi_h) := \sum_{E \in \Omega_h} \int_E \Delta \Pi_E^{\mathbf{D},k} \varphi_h \Delta \Pi_E^{\mathbf{D},k} \phi_h + s_E^{\mathbf{D}} ((I - \Pi_E^{\mathbf{D},k}) \varphi_h, (I - \Pi_E^{\mathbf{D},k}) \phi_h) \quad \forall \varphi_h, \phi_h \in \mathbf{W}_k^h,$$

which is also fully computable by using the degrees of freedom  $\mathbf{D}_{\mathbf{W}1} - \mathbf{D}_{\mathbf{W}5}$ . Nevertheless, in the present work we will stick to the choice (3.9).

**Remark 3.3.** If  $\mathbf{f}_\psi$  is given as an explicit function, then we can consider the following alternative discrete load term

$$F_\psi^h(\phi_h) := \sum_{E \in \Omega_h} \int_E \text{rot } \mathbf{f}_\psi(t) \Pi_E^{k-2} \phi_h \quad \forall \phi_h \in \mathbf{W}_k^h,$$

which is also computable using the degrees of freedom  $\mathbf{D}_{\mathbf{W}1} - \mathbf{D}_{\mathbf{W}5}$ .

#### 4. Fully-discrete formulation and its well posedness

In order to present a full discretization of problem (2.13) we introduce a sequence of time steps  $t_n = n\Delta t$ ,  $n = 0, 1, 2, \dots, N$ , where  $\Delta t = T/N$  is the time step. Moreover, we consider the following approximations at each time  $t_n$ :  $\psi_h^n \approx \psi_h(t_n)$  and  $\theta_h^n \approx \theta_h(t_n)$ . For the external forces, we introduce the following notation:  $\mathbf{f}_\psi^n := \mathbf{f}_\psi(t_n)$ ,  $f_\theta^n := f_\theta(t_n)$  and  $\mathbf{g}^n := \mathbf{g}(t_n)$ .

We consider the backward Euler method coupled with the VE discretization presented in Section 3, which read as follows: given  $(\psi_h^0, \theta_h^0)$ , find  $\{(\psi_h^n, \theta_h^n)\}_{n=1}^N \in \mathbf{W}_k^h \times \mathbf{H}_\ell^h$ , such that

$$\begin{aligned} M_F^h \left( \frac{\psi_h^n - \psi_h^{n-1}}{\Delta t}, \phi_h \right) + \nu A_F^h(\psi_h^n, \phi_h) + B_F^h(\psi_h^n; \psi_h^n, \phi_h) - C^h(\theta_h^n, \phi_h) &= F_\psi^h(\phi_h) \quad \forall \phi_h \in \mathbf{W}_k^h, \\ M_T^h \left( \frac{\theta_h^n - \theta_h^{n-1}}{\Delta t}, v_h \right) + \kappa A_T^h(\theta_h^n, v_h) + B_{\text{skew}}^h(\psi_h^n; \theta_h^n, v_h) &= F_\theta^h(v_h) \quad \forall v_h \in \mathbf{H}_\ell^h. \end{aligned} \quad (4.1)$$

The functions  $(\psi_h^0, \theta_h^0)$  are initial approximations of  $(\psi_h, \theta_h)$  at  $t = 0$ . For instance, we will consider  $\psi_h^0 := \mathcal{S}_h \psi_0$  (see (5.1) below) and  $\theta_h^0 := \mathcal{P}_h \theta_0$ , with  $\mathcal{P}_h(\cdot)$  being the energy operator associated to the  $\mathbf{H}^1$ -inner product (for further details, see for instance [47, Equation (9)]).

We now recall local inverse inequalities for the virtual spaces  $W_k^h(E)$  and  $H_\ell^h(E)$  (see [48,49]):

$$|\phi_h|_{2,E} \leq C_{\text{inv}} h_E^{-1} |\phi_h|_{1,E} \quad \forall \phi_h \in W_k^h(E) \quad \text{and} \quad |v_h|_{1,E} \leq C_{\text{inv}} h_E^{-1} \|v_h\|_{0,E} \quad \forall v_h \in H_\ell^h(E). \quad (4.2)$$

In what follows, we will provide the well-posedness of the fully-discrete formulation (4.1).

**Theorem 4.1.** *Let  $\widehat{\alpha} := \min \{\widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T}\}$  and  $\gamma := \min \{\widehat{\alpha}_{A_F \nu}, \widehat{\alpha}_{A_T \kappa}\}$ , where  $\widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T}, \widehat{\alpha}_{A_F}$  and  $\widehat{\alpha}_{A_T}$  are the constants in Lemma 3.3. Assume that*

$$\widehat{\alpha} + \Delta t (\gamma - C_g) > 0, \quad (4.3)$$

where  $C_g := \|\mathbf{g}\|_{L^\infty(0,T;L^\infty(\Omega))}$ . Then the fully-discrete scheme (4.1) admits at least one solution  $(\psi_h^n, \theta_h^n) \in W_k^h \times H_\ell^h$  at every time step  $t_n$ , with  $n = 1, \dots, N$ .

**Proof.** For simplicity we set  $X_{k,\ell}^h := W_k^h \times H_\ell^h$  and we endow this space with the following equivalent norm:

$$\|(\phi_h, w_h)\| := (\|\phi_h\|_{1,\Omega}^2 + \|w_h\|_{0,\Omega}^2)^{\frac{1}{2}} \quad \forall (\phi_h, w_h) \in X_{k,\ell}^h.$$

Next, for  $1 \leq n \leq N$ , let  $(\psi_h^{n-1}, \theta_h^{n-1}) \in X_{k,\ell}^h$ . Thus, for any  $(\psi_h, \theta_h) \in X_{k,\ell}^h$ , we consider the operator  $\Phi : X_{k,\ell}^h \rightarrow (X_{k,\ell}^h)^*$  defined by

$$\begin{aligned} \langle \Phi(\psi_h, \theta_h), (\phi_h, w_h) \rangle &:= M_F^h(\psi_h, \phi_h) - M_F^h(\psi_h^{n-1}, \phi_h) + \nu \Delta t A_F^h(\psi_h, \phi_h) + \Delta t B_F^h(\psi_h; \psi_h, \phi_h) \\ &\quad - \Delta t F_\psi^h(\phi_h) + M_T^h(\theta_h, w_h) - M_T^h(\theta_h^{n-1}, w_h) + \kappa \Delta t A_T^h(\theta_h, w_h) \\ &\quad + \Delta t B_{\text{skew}}^h(\psi_h; \theta_h, w_h) - \Delta t F_\theta^h(w_h) - \Delta t C^h(\theta_h, \phi_h) \quad \forall (\phi_h, w_h) \in X_{k,\ell}^h. \end{aligned} \quad (4.4)$$

From the definition of operator  $\Phi$ , we observe that for each  $1 \leq n \leq N$  a solution  $(\psi_h^n, \theta_h^n) \in X_{k,\ell}^h$  of problem (4.1) is characterized by  $\Phi(\psi_h^n, \theta_h^n) = \mathbf{0}$ . Thus, we will prove that this operator satisfies the hypothesis of the fixed point result [40, Chap. IV, Corollary 1.1].

First we will prove its continuity. Indeed, by using of operator  $\Phi$  and Lemma 3.3, for all  $(\phi_h, w_h) \in X_{k,\ell}^h$  have

$$\begin{aligned} \langle \Phi(\psi_h, \theta_h) - \Phi(\psi_h^*, \theta_h^*), (\phi_h, w_h) \rangle &:= M_F^h(\psi_h - \psi_h^*, \phi_h) + \nu \Delta t A_F^h(\psi_h - \psi_h^*, \phi_h) \\ &\quad + \Delta t (B_F^h(\psi_h; \psi_h, \phi_h) - B_F^h(\psi_h^*; \psi_h^*, \phi_h)) \\ &\quad + M_T^h(\theta_h - \theta_h^*, w_h) + \kappa \Delta t A_T^h(\theta_h - \theta_h^*, w_h) \\ &\quad + \Delta t (B_{\text{skew}}^h(\psi_h; \theta_h, w_h) - B_{\text{skew}}^h(\psi_h^*; \theta_h^*, w_h)) + \Delta t C^h(\theta_h^* - \theta_h, \phi_h) \\ &\leq \widehat{C}_{M_F} \|\psi_h - \psi_h^*\|_{1,\Omega} \|\phi_h\|_{1,\Omega} + \nu \Delta t \widehat{C}_{A_F} \|\psi_h - \psi_h^*\|_{2,\Omega} \|\phi_h\|_{2,\Omega} \\ &\quad + \Delta t (B_F^h(\psi_h; \psi_h, \phi_h) - B_F^h(\psi_h^*; \psi_h^*, \phi_h)) \\ &\quad + \widehat{C}_{M_T} \|\theta_h - \theta_h^*\|_{0,\Omega} \|w_h\|_{0,\Omega} + \kappa \Delta t \widehat{C}_{A_T} \|\theta_h - \theta_h^*\|_{1,\Omega} \|w_h\|_{1,\Omega} \\ &\quad + \Delta t (B_{\text{skew}}^h(\psi_h; \theta_h, w_h) - B_{\text{skew}}^h(\psi_h^*; \theta_h^*, w_h)) + \Delta t \|\mathbf{g}\|_{\infty,\Omega} \|\theta_h - \theta_h^*\|_{0,\Omega} \|\phi_h\|_{1,\Omega}. \end{aligned} \quad (4.5)$$

Now, we add and subtract the term  $B_F^h(\psi_h^*; \psi_h, \phi_h)$ , then by using the linearity in each entry and the continuity of the trilinear form  $B_F^h(\cdot; \cdot, \cdot)$  (cf. Lemma 3.3), we obtain

$$\begin{aligned} B_F^h(\psi_h; \psi_h, \phi_h) - B_F^h(\psi_h^*; \psi_h^*, \phi_h) &= B_F^h(\psi_h - \psi_h^*; \psi_h, \phi_h) + B_F^h(\psi_h^*; \psi_h - \psi_h^*, \phi_h) \\ &\leq \widehat{C}_{B_F} (\|\psi_h - \psi_h^*\|_{2,\Omega} \|\psi_h\|_{2,\Omega} + \|\psi_h^*\|_{2,\Omega} \|\psi_h - \psi_h^*\|_{2,\Omega}) \|\phi_h\|_{2,\Omega}. \end{aligned}$$

Following analogous steps, we get

$$\begin{aligned} B_{\text{skew}}^h(\psi_h; \theta_h, w_h) - B_{\text{skew}}^h(\psi_h^*; \theta_h^*, w_h) &= B_{\text{skew}}^h(\psi_h - \psi_h^*; \theta_h, w_h) + B_{\text{skew}}^h(\psi_h^*; \theta_h - \theta_h^*, w_h) \\ &\leq \widehat{C}_{B_T} (\|\psi_h - \psi_h^*\|_{2,\Omega} \|\theta_h\|_{1,\Omega} + \|\psi_h^*\|_{2,\Omega} \|\theta_h - \theta_h^*\|_{1,\Omega}) \|w_h\|_{1,\Omega}. \end{aligned}$$

By combining (4.5), the above estimates, the inverse inequalities (4.2) and the Cauchy–Schwarz inequality, for all  $(\phi_h, w_h) \in X_{k,\ell}^h$ , it holds

$$|\langle \Phi(\psi_h, \theta_h) - \Phi(\psi_h^*, \theta_h^*), (\phi_h, w_h) \rangle| \leq C(1 + \Delta t h_{\min}^{-2} + \Delta t h_{\min}^{-3}) \|(\psi_h - \psi_h^*, \theta_h - \theta_h^*)\| \|(\phi_h, w_h)\|.$$

Therefore, we deduce that for  $h$  and  $\Delta t$  fixed

$$\|\Phi(\psi_h, \theta_h) - \Phi(\psi_h^*, \theta_h^*)\|_{(X_{k,\ell}^h)^*} \longrightarrow 0, \quad \text{when } (\psi_h, \theta_h) \xrightarrow{\|\cdot\|} (\psi_h^*, \theta_h^*),$$

i.e.,  $\Phi$  is continuous.

On the other hand, by employing again Lemma 3.3 and the Young inequality, for all  $(\psi_h, \theta_h) \in X_{k,\ell}^h$ , we obtain

$$\begin{aligned} \langle \Phi(\psi_h, \theta_h), (\psi_h, \theta_h) \rangle &\geq \widehat{\alpha}_{M_F} \|\psi_h\|_{1,\Omega}^2 - \frac{\widehat{C}_{M_F}^2}{2\widehat{\alpha}_{M_F}} \|\psi_h^{n-1}\|_{1,\Omega}^2 - \frac{\widehat{\alpha}_{M_F}}{2} \|\psi_h\|_{1,\Omega}^2 \\ &\quad + \widehat{\alpha}_{A_F\nu} \Delta t \|\psi_h\|_{2,\Omega}^2 - \frac{\widehat{C}_{F_\psi}^2 \Delta t}{2\widehat{\alpha}_{A_F\nu}} \|\mathbf{f}_\psi^n\|_{0,\Omega}^2 \\ &\quad - \frac{\widehat{\alpha}_{A_F\nu} \Delta t}{2} \|\psi_h\|_{2,\Omega}^2 + \widehat{\alpha}_{M_T} \|\theta_h\|_{0,\Omega}^2 - \frac{\widehat{C}_{M_T}^2}{2\widehat{\alpha}_{M_T}} \|\theta_h^{n-1}\|_{0,\Omega}^2 - \frac{\widehat{\alpha}_{M_T}}{2} \|\theta_h\|_{0,\Omega}^2 + \widehat{\alpha}_{A_T\kappa} \Delta t \|\theta_h\|_{1,\Omega}^2 \\ &\quad - \frac{\widehat{C}_{F_\theta}^2 \Delta t}{2\widehat{\alpha}_{A_T\kappa}} \|f_\theta^n\|_{0,\Omega}^2 - \frac{\widehat{\alpha}_{A_T\kappa} \Delta t}{2} \|\theta_h\|_{1,\Omega}^2 - \frac{\Delta t C_g}{2} (\|\psi_h\|_{1,\Omega}^2 + \|\theta_h\|_{0,\Omega}^2) \\ &\geq \frac{1}{2} \min \{ \widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T} \} (\|\psi_h\|_{1,\Omega}^2 + \|\theta_h\|_{0,\Omega}^2) + \frac{\Delta t}{2} \min \{ \widehat{\alpha}_{A_F\nu}, \widehat{\alpha}_{A_T\kappa} \} (\|\psi_h\|_{2,\Omega}^2 + \|\theta_h\|_{1,\Omega}^2) \\ &\quad - \frac{\Delta t C_g}{2} (\|\psi_h\|_{1,\Omega}^2 + \|\theta_h\|_{0,\Omega}^2) - \frac{\widehat{C}_{M_F}^2}{2\widehat{\alpha}_{M_F}} \|\psi_h^{n-1}\|_{1,\Omega}^2 - \frac{\widehat{C}_{M_T}^2}{2\widehat{\alpha}_{M_T}} \|\theta_h^{n-1}\|_{0,\Omega}^2 \\ &\quad - \frac{\widehat{C}_{F_\psi}^2 \Delta t}{2\widehat{\alpha}_{A_F\nu}} \|\mathbf{f}_\psi^n\|_{0,\Omega}^2 - \frac{\widehat{C}_{F_\theta}^2 \Delta t}{2\widehat{\alpha}_{A_T\kappa}} \|f_\theta^n\|_{0,\Omega}^2 \\ &\geq \frac{1}{2} (\widehat{\alpha} + \Delta t (\gamma - C_g)) (\|\psi_h\|_{1,\Omega}^2 + \|\theta_h\|_{0,\Omega}^2) - \frac{\widehat{C}_{M_F}^2}{2\widehat{\alpha}_{M_F}} \|\psi_h^{n-1}\|_{1,\Omega}^2 - \frac{\widehat{C}_{M_T}^2}{2\widehat{\alpha}_{M_T}} \|\theta_h^{n-1}\|_{0,\Omega}^2 \\ &\quad - \frac{\widehat{C}_{F_\psi}^2 \Delta t}{2\widehat{\alpha}_{A_F\nu}} \|\mathbf{f}_\psi^n\|_{0,\Omega}^2 - \frac{\widehat{C}_{F_\theta}^2 \Delta t}{2\widehat{\alpha}_{A_T\kappa}} \|f_\theta^n\|_{0,\Omega}^2, \end{aligned}$$

where we have used the facts that  $\|\psi_h\|_{1,\Omega} \leq \|\psi_h\|_{2,\Omega}$ ,  $\|\theta_h\|_{0,\Omega} \leq \|\theta_h\|_{1,\Omega}$  and  $\frac{\Delta t}{2} \min \{ \widehat{\alpha}_{A_F\nu}, \widehat{\alpha}_{A_T\kappa} \} (\|\psi_h\|_{2,\Omega}^2 + \|\theta_h\|_{1,\Omega}^2) \geq 0$ .

Thus, from assumption (4.3), we can set

$$\rho := (\widehat{\alpha} + \Delta t (\gamma - C_g))^{-\frac{1}{2}} \left( \frac{\widehat{C}_{M_F}^2}{\widehat{\alpha}_{M_F}} \|\psi_h^{n-1}\|_{1,\Omega}^2 + \frac{\widehat{C}_{M_T}^2}{\widehat{\alpha}_{M_T}} \|\theta_h^{n-1}\|_{0,\Omega}^2 + \frac{\widehat{C}_{F_\psi}^2 \Delta t}{\widehat{\alpha}_{A_F\nu}} \|\mathbf{f}_\psi^n\|_{0,\Omega}^2 + \frac{\widehat{C}_{F_\theta}^2 \Delta t}{\widehat{\alpha}_{A_T\kappa}} \|f_\theta^n\|_{0,\Omega}^2 \right)^{\frac{1}{2}},$$

and  $\mathcal{S} := \{(\varphi_h, w_h) \in X_{k,\ell}^h : \|\!(\varphi_h, w_h)\|\| \leq \rho\}$ . Then, we have that

$$\langle \Phi(\psi_h, \theta_h), (\psi_h, \theta_h) \rangle \geq 0 \quad \text{for any } (\psi_h, \theta_h) \in \partial\mathcal{S}.$$

Then, by employing the fixed point Theorem [40, Chap. IV, Corollary 1.1], there exists  $(\psi_h^n, \theta_h^n) \in \mathcal{S}$ , such that  $\Phi(\psi_h^n, \theta_h^n) = \mathbf{0}$ , i.e., the fully-discrete problem (4.1) admits at least one solution  $(\psi_h^n, \theta_h^n) \in \mathcal{S}$  at every time step  $t_n$ .  $\square$

**Remark 4.1.** From assumption (4.3) it follows that if  $C_g \leq \gamma$  then the condition (4.3) is always satisfied. Instead, if  $C_g > \gamma$ , that is when the buoyancy term is strong when compared to the diffusion terms, a “small time step condition”  $\Delta t < \widehat{\alpha}/(C_g - \gamma)$  is needed in order to guarantee the existence of a discrete solution.

The following result establishes that the fully-discrete scheme (4.1) is unconditionally stable.

**Theorem 4.2.** Assume that  $\mathbf{f}_\psi \in L^2(0, T; L^2(\Omega))$ ,  $f_\theta \in L^2(0, T; L^2(\Omega))$ ,  $\mathbf{g} \in L^\infty(0, T; L^\infty(\Omega))$ . Moreover, suppose that the initial data satisfy  $\psi_0 \in H_0^2(\Omega)$  and  $\theta_0 \in H_0^1(\Omega)$ . Then, the fully-discrete scheme (4.1) is unconditionally

stable and satisfy the following estimate for any  $0 < m \leq N$

$$\begin{aligned} & \|(\phi_h^m, \theta_h^m)\|_{H^1(\Omega) \times L^2(\Omega)} + \left( \Delta t \sum_{n=1}^m \|(\psi_h^n, \theta_h^n)\|_{H^2(\Omega) \times H^1(\Omega)}^2 \right)^{\frac{1}{2}} \\ & \leq C \left( \left( \Delta t \sum_{n=1}^m \|(\mathbf{f}_\psi^n, f_\theta^n)\|_{L^2(\Omega) \times L^2(\Omega)}^2 \right)^{\frac{1}{2}} + \|(\psi_0, \theta_0)\|_{H^2(\Omega) \times H^1(\Omega)} \right) =: \delta, \end{aligned}$$

where  $C > 0$  is independent of  $h$  and  $\Delta t$ .

**Proof.** Let  $(\psi_h^n, \theta_h^n) \in \mathbf{W}_k^h \times \mathbf{H}_\ell^h$  be a solution of fully-discrete problem (4.1). We consider the following equivalent norms:

$$\|\phi_h\|_{F,h} := (M_F^h(\phi_h, \phi_h))^{1/2}, \quad \|v_h\|_{T,h} := (M_T^h(v_h, v_h))^{1/2} \quad \phi_h \in \mathbf{W}_k^h, \quad \forall v_h \in \mathbf{H}_\ell^h. \quad (4.6)$$

Taking  $v_h = \theta_h^n \in \mathbf{H}_\ell^h$  in the second equation of (4.1), using Lemma 3.3, the Young inequality and some identities of real numbers, we obtain

$$\frac{1}{2\Delta t} (\|\theta_h^n\|_{T,h}^2 - \|\theta_h^{n-1}\|_{T,h}^2) + \widehat{\alpha}_{A_T} \kappa \|\theta_h^n\|_{1,\Omega}^2 \leq C_{F\theta} \|f_\theta^n\|_{0,\Omega} \|\theta_h^n\|_{1,\Omega} \leq C \|f_\theta^n\|_{0,\Omega}^2 + \frac{1}{2} \widehat{\alpha}_{A_T} \kappa \|\theta_h^n\|_{1,\Omega}^2.$$

Then, multiplying by  $2\Delta t$ , using the equivalence of norms and summing for  $n = 1, \dots, m$ , we have that

$$\|\theta_h^m\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^m \|\theta_h^n\|_{1,\Omega}^2 \leq C \left( \Delta t \sum_{n=1}^m \|f_\theta^n\|_{0,\Omega}^2 + \|\theta_h^0\|_{0,\Omega}^2 \right). \quad (4.7)$$

Analogously, taking  $\phi_h = \psi_h^n \in \mathbf{W}_k^h$  in the first equation of (4.1) and repeating the same arguments, we obtain

$$\|\psi_h^n\|_{F,h}^2 - \|\psi_h^{n-1}\|_{F,h}^2 + \widehat{\alpha}_{A_F} \nu \Delta t \|\psi_h^n\|_{2,\Omega}^2 \leq C \Delta t C_g \|\theta_h^n\|_{0,\Omega}^2 + C \Delta t \|\mathbf{f}_\psi^n\|_{0,\Omega}^2, \quad (4.8)$$

where the constant  $C_g$  is defined in Theorem 4.1.

Now, summing for  $n = 1, \dots, m$ , inserting (4.7) in (4.8) and using the equivalence of norms and, we get

$$\|\psi_h^m\|_{1,\Omega}^2 + \Delta t \sum_{n=1}^m \|\psi_h^n\|_{2,\Omega}^2 \leq C \left( \Delta t \sum_{n=1}^m (\|\mathbf{f}_\psi^n\|_{0,\Omega}^2 + \|f_\theta^n\|_{0,\Omega}^2) + \|\psi_h^0\|_{1,\Omega}^2 + \|\theta_h^0\|_{0,\Omega}^2 \right), \quad (4.9)$$

where the constant  $C_g$  was included in the constant  $C$  to shorten the bound.

Finally, the desired result follows adding (4.7) and (4.9).  $\square$

The following result establishes that the solution of scheme (4.1) is unique for small values of  $\Delta t$ .

**Theorem 4.3.** Let  $\widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T}, \widehat{C}_{B_F}$  and  $\widehat{C}_{B_T}$  be the constants in Lemma 3.3. Moreover, let  $\delta$  be the upper bound in Theorem 4.2,  $C_g$  be the constant defined in Theorem 4.1 and  $C_{\text{inv}}$  be the constant in (4.2). Assume that

$$\Delta t < \min\{\widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T}\} \min \left\{ \frac{h_{\min}^2}{2C_{\text{inv}}^2(\widehat{C}_{B_F} + \widehat{C}_{B_T})\delta}, \frac{1}{2C_g} \right\}. \quad (4.10)$$

Then, for each  $n = 1, \dots, N$  the solution of the fully-discrete scheme (4.1) is unique.

**Proof.** Let  $1 \leq n \leq N$  and  $(\psi_{h1}^n, \theta_{h1}^n), (\psi_{h2}^n, \theta_{h2}^n) \in \mathbf{W}_k^h \times \mathbf{H}_\ell^h$  be two solutions of problem (4.1). Then, setting  $\widetilde{\psi}_h^n := \psi_{h1}^n - \psi_{h2}^n, \widetilde{\theta}_h^n := \theta_{h1}^n - \theta_{h2}^n$  and using the definition of operator (4.4), for all  $(\phi_h, v_h) \in \mathbf{W}_k^h \times \mathbf{H}_\ell^h$ , we have that

$$\begin{aligned} & M_F^h(\widetilde{\psi}_h^n, \phi_h) + M_T^h(\widetilde{\theta}_h^n, v_h) + \nu \Delta t A_F^h(\widetilde{\psi}_h^n, \phi_h) + \kappa \Delta t A_T^h(\widetilde{\theta}_h^n, v_h) - \Delta t C^h(\widetilde{\theta}_h^n, \phi_h) \\ & + \Delta t (B_F^h(\psi_{h1}^n; \psi_{h1}^n, \phi_h) - B_F^h(\psi_{h2}^n; \psi_{h2}^n, \phi_h)) + \Delta t (B_{\text{skew}}^h(\psi_{h1}^n; \theta_{h1}^n, v_h) - B_{\text{skew}}^h(\psi_{h2}^n; \theta_{h2}^n, v_h)) = 0. \end{aligned} \quad (4.11)$$

Adding and subtracting  $B_F^h(\psi_{h2}^n; \psi_{h1}^n, \phi_h)$  and  $B_{\text{skew}}^h(\psi_{h2}^n; \theta_{h1}^n, v_h)$  we obtain

$$\begin{aligned} & B_F^h(\psi_{h1}^n; \psi_{h1}^n, \phi_h) - B_F^h(\psi_{h2}^n; \psi_{h2}^n, \phi_h) = B_F^h(\widetilde{\psi}_h^n; \psi_{h1}^n, \phi_h) + B_F^h(\psi_{h2}^n; \widetilde{\psi}_h^n, \phi_h) \\ & B_{\text{skew}}^h(\psi_{h1}^n; \theta_{h1}^n, v_h) - B_{\text{skew}}^h(\psi_{h2}^n; \theta_{h2}^n, v_h) = B_{\text{skew}}^h(\widetilde{\psi}_h^n; \theta_{h1}^n, v_h) + B_{\text{skew}}^h(\psi_{h2}^n; \widetilde{\theta}_h^n, v_h). \end{aligned}$$

Next, taking  $\phi_h = \widetilde{\psi}_h^n$  and  $v_h = \widetilde{\theta}_h^n$  in (4.11), from the above identities, the skew-symmetry of trilinear forms, the continuity and coercivity properties of the multilinear forms involved (cf. Lemma 3.3), it follows

$$\begin{aligned} & \widehat{\alpha}_{M_F} \|\widetilde{\psi}_h^n\|_{1,\Omega}^2 + \widehat{\alpha}_{M_T} \|\widetilde{\theta}_h^n\|_{0,\Omega}^2 + \widehat{\alpha}_{A_F} \nu \Delta t \|\widetilde{\psi}_h^n\|_{2,\Omega}^2 + \widehat{\alpha}_{A_T} \kappa \Delta t \|\widetilde{\theta}_h^n\|_{1,\Omega}^2 \\ & \leq \Delta t \widehat{C}_{B_F} \|\widetilde{\psi}_h^n\|_{2,\Omega} \|\psi_{h_1}^n\|_{2,\Omega} \|\widetilde{\psi}_h^n\|_{2,\Omega} + \Delta t \widehat{C}_{B_T} \|\widetilde{\psi}_h^n\|_{2,\Omega} \|\theta_{h_1}^n\|_{1,\Omega} \|\widetilde{\theta}_h^n\|_{1,\Omega} + \Delta t C_g \|\theta_h^n\|_{0,\Omega} \|\widetilde{\psi}_h^n\|_{1,\Omega} \\ & \leq \Delta t \widehat{C}_{B_F} \|\widetilde{\psi}_h^n\|_{2,\Omega}^2 + \frac{1}{2} \Delta t \widehat{C}_{B_T} \|\theta_{h_1}^n\|_{1,\Omega} (\|\widetilde{\psi}_h^n\|_{2,\Omega}^2 + \|\widetilde{\theta}_h^n\|_{1,\Omega}^2) + \frac{1}{2} \Delta t C_g (\|\widetilde{\psi}_h^n\|_{1,\Omega}^2 + \|\widetilde{\theta}_h^n\|_{0,\Omega}^2) \\ & \leq \Delta t (\widehat{C}_{B_F} \|\psi_{h_1}^n\|_{2,\Omega} + \widehat{C}_{B_T} \|\theta_{h_1}^n\|_{1,\Omega}) \|\widetilde{\psi}_h^n, \widetilde{\theta}_h^n\|_{H^2(\Omega) \times H^1(\Omega)}^2 + \Delta t C_g \|\widetilde{\psi}_h^n, \widetilde{\theta}_h^n\|_{H^1(\Omega) \times L^2(\Omega)}^2. \end{aligned}$$

Now, employing local inverse inequalities (4.2) in the above estimate and Theorem 4.2, we get

$$\begin{aligned} \min\{\widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T}\} \|\widetilde{\psi}_h^n, \widetilde{\theta}_h^n\|_{H^1(\Omega) \times L^2(\Omega)}^2 & \leq [C_{\text{inv}} h_{\min}^{-1} \Delta t ((\widehat{C}_{B_F} + \widehat{C}_{B_T}) C_{\text{inv}} h_{\min}^{-1} \delta) + \Delta t C_g] \\ & \quad \times \|\widetilde{\psi}_h^n, \widetilde{\theta}_h^n\|_{H^1(\Omega) \times L^2(\Omega)}^2. \end{aligned}$$

From the assumption (4.10), we have that

$$\frac{1}{\min\{\widehat{\alpha}_{M_F}, \widehat{\alpha}_{M_T}\}} (C_{\text{inv}} h_{\min}^{-1} (\widehat{C}_{B_F} + \widehat{C}_{B_T}) C_{\text{inv}} h_{\min}^{-1} \delta + C_g) < 1. \tag{4.12}$$

Thus,  $\widetilde{\psi}_h^n = 0$  and  $\widetilde{\theta}_h^n = 0$ , which implies  $\psi_{h_1}^n = \psi_{h_2}^n$  and  $\theta_{h_1}^n = \theta_{h_2}^n$ . The proof is complete.  $\square$

**Remark 4.2.** Exploiting the fact that we are in the two dimensional case and using sharper Sobolev bounds for the convective terms (i.e., employing the Hölder inequality, Sobolev bounds with adequate exponents and an inverse inequality), we could get a power  $h_{\min}^{-\epsilon}$ , for all  $\epsilon > 0$ , instead of  $h_{\min}^{-1}$  in the term  $h_{\min}^{-1} \delta$  (see Eq. (4.12)).

### 5. Convergence analysis

This section is devoted to the convergence analysis of the fully-discrete formulation (4.1) introduced in the previous section. We start recalling some preliminary results of approximation in the polynomial and virtual spaces. Moreover, we introduce an energy operator associated to the  $H^2$ -inner product with its corresponding approximation properties. Later on, we state technical results, which will be useful to provide the convergence result of our fully-discrete virtual scheme.

#### 5.1. Preliminary results

First, we recall the following polynomial approximation result (see for instance [50]). Here below  $E$  represents as usual a generic element of  $\{\Omega_h\}_{h>0}$ , which we recall satisfies assumptions **A1**, **A2** in Section 3.1.

**Proposition 5.1.** *Let  $m \in \mathbb{R}$  and  $n \in \mathbb{N} \cup \{0\}$ . Then, for each  $\phi \in H^m(E)$ , there exist  $\phi_\pi \in \mathbb{P}_n(E)$ , and  $C > 0$  independent of  $h_E$ , such that*

$$\|\phi - \phi_\pi\|_{t,E} \leq C h_E^{m-t} |\phi|_{m,E}, \quad 0 \leq m \leq n + 1, \quad t = 0, \dots, [m],$$

with  $[m]$  denoting the largest integer equal or smaller than  $m$ .

Standard arguments and (3.6) lead easily to following approximation properties for the projectors  $\Pi_E^n$  (an analogous result can be obtained the vectorial version).

**Proposition 5.2.** *Let  $m \in \mathbb{R}$ ,  $n \in \mathbb{N} \cup \{0\}$  and let  $\Pi_E^n$  be the projection defined in (3.5). Then, for each  $\phi \in H^m(E)$ , there exists a constant  $C$ , independent of  $E$  and  $h_E$ , such that*

$$\|\phi - \Pi_E^n \phi\|_{t,E} \leq C h_E^{m-t} |\phi|_{m,E}, \quad 0 \leq m \leq n + 1, \quad t = 0, \dots, [m],$$

with  $[m]$  denoting the largest integer equal or smaller than  $m$ .

Now, we continue with the following approximation for the stream-function and temperature virtual element spaces, which can be found in [19,43,44,51–53], respectively.

**Proposition 5.3.** *Let  $m \in \mathbb{R}$ . Then, for each  $\phi \in H^m(\Omega)$ , there exist  $\phi_I \in W_k^h$  and  $C_I > 0$ , independent of  $h$ , such that*

$$\|\phi - \phi_I\|_{t,\Omega} \leq C_I h^{m-t} |\phi|_{m,\Omega}, \quad t = 0, 1, 2, \quad 2 < m \leq k + 1, \quad k \geq 2.$$

For the temperature variable, we present local and global approximation properties.

**Proposition 5.4.** *Let  $m \in \mathbb{R}$ . Then, for each  $v \in H^m(\Omega)$ , there exist  $v_I \in H_\ell^h$  and  $C_I > 0$ , independent of  $h$ , such that*

$$\|v - v_I\|_{t,E} \leq C_I h_E^{m-t} |v|_{m,E} \quad \forall E \in \Omega_h; \quad \|v - v_I\|_{t,\Omega} \leq C_I h^{m-t} |v|_{m,\Omega}, \quad t = 0, 1, \quad 1 < m \leq \ell + 1, \quad \ell \geq 1.$$

Now, we will introduce the following discrete biharmonic projection associated with the stream-function discretization. For each  $\varphi \in H_0^2(\Omega)$ , we consider the operator  $\mathcal{S}_h : H_0^2(\Omega) \rightarrow W_k^h$ , defined as the solution of problem:

$$A_F^h(\mathcal{S}_h \varphi, \phi_h) = A_F(\varphi, \phi_h) \quad \forall \phi_h \in W_k^h, \tag{5.1}$$

where  $A_F(\cdot, \cdot)$  was defined in (2.5) and we recall that  $A_F^h(\cdot, \cdot)$  is the global version of the form defined in (3.9). By using the ellipticity and continuity of the bilinear form  $A_F^h(\cdot, \cdot)$  (cf. Lemma 3.3) and the Lax–Milgram Lemma, we have that the above problem (5.1) is well-posed.

By using Propositions 3.1, 5.1 and 5.3, the following approximation result for the energy projection  $\mathcal{S}_h(\cdot)$  holds true (see [54, Lemma 5.3]).

**Proposition 5.5.** *For each  $\varphi \in H_0^2(\Omega)$ , there exists a unique function  $\mathcal{S}_h \varphi \in W_k^h$  satisfying (5.1). Moreover, if  $\varphi \in H^{2+s}(\Omega)$ , with  $\frac{1}{2} < s \leq k - 1$ , then the following approximation property holds:*

$$\|\varphi - \mathcal{S}_h \varphi\|_{1,\Omega} + h^{\tilde{s}} \|\varphi - \mathcal{S}_h \varphi\|_{2,\Omega} \leq C h^{\tilde{s}+s} |\varphi|_{2+s,\Omega},$$

where  $C$  is a positive constant, independent of  $h$  and  $\tilde{s} \in (\frac{1}{2}, 1]$  depends on the largest reentrant angle of the domain  $\Omega$ . In particular, when  $\Omega$  is a convex domain it holds  $\tilde{s} = 1$ .

In what follows, we will establish four technical lemmas involving the trilinear forms associated to transport/convection and the bilinear form associated to the buoyancy term; these results will be useful in Section 5.2.

**Lemma 5.1.** *For all  $\zeta_h; \varphi_h, \phi_h \in W_k^h$ , there exists  $\widehat{C}_{BF} > 0$ , independent of  $h$ , such that*

$$|B_F^h(\zeta_h; \varphi_h, \phi_h)| \leq \widehat{C}_{BF} \|\zeta_h\|_{2,\Omega} \|\varphi_h\|_{2,\Omega} \|\phi_h\|_{2,\Omega}^{\frac{1}{2}} \|\phi_h\|_{1,\Omega}^{\frac{1}{2}}.$$

**Proof.** We use the definition of the trilinear form  $B_F^h(\cdot; \cdot, \cdot)$  (cf. (3.10)), the Hölder inequality, the continuity of the operators  $\Pi_E^{k-2}$  and  $\Pi_E^{k-1}$  with respect to the  $L^2$ - and  $L^4$ -norms, respectively (cf. properties (3.6)), and the Hölder inequality for sequences, to obtain

$$\begin{aligned} B_F^h(\zeta_h; \varphi_h, \phi_h) &\leq \sum_{E \in \Omega_h} \|\Pi_E^{k-2} \Delta \zeta_h\|_{0,E} \|\Pi_E^{k-1} \mathbf{curl} \varphi_h\|_{L^4(E)} \|\Pi_E^{k-1} \nabla \phi_h\|_{L^4(E)} \\ &\leq C \|\Delta \zeta_h\|_{0,\Omega} \|\mathbf{curl} \varphi_h\|_{L^4(\Omega)} \|\nabla \phi_h\|_{L^4(\Omega)} \\ &\leq C \|\Delta \zeta_h\|_{0,\Omega} \|\varphi_h\|_{2,\Omega} \|\nabla \phi_h\|_{L^4(\Omega)}, \end{aligned}$$

where we have used the Sobolev inclusion  $H^1(\Omega) \hookrightarrow L^4(\Omega)$ . Now, applying the Ladyzhenskaya inequality (2.14) with  $\mathbf{v} = \nabla \phi_h$  we obtain the desired result.  $\square$

**Lemma 5.2.** *For all  $\zeta, \varphi, \phi \in H_0^2(\Omega)$ , we have that*

$$B_F^h(\varphi; \varphi, \phi) - B_F^h(\zeta; \zeta, \phi) = B_F^h(\varphi; \varphi - \zeta + \phi, \phi) + B_F^h(\varphi - \zeta + \phi; \zeta, \phi) - B_F^h(\phi; \zeta, \phi).$$

**Proof.** The proof follows by adding and subtracting suitable terms, and using the trilinearity and skew-symmetry properties of the form  $B_F^h(\cdot; \cdot, \cdot)$ .  $\square$



Next lemmas give us the measure of the variational crime in the discretization of the trilinear forms  $B_F(\cdot; \cdot, \cdot)$  and  $B_{\text{skew}}(\cdot; \cdot, \cdot)$  and the bilinear form  $C(\cdot, \cdot)$ .

**Lemma 5.3.** *Let  $\varphi(t) \in H_0^2(\Omega) \cap H^{2+s}(\Omega)$ , with  $\frac{1}{2} < s \leq k - 1$ , for almost all  $t \in (0, T)$ . Then, there exists  $C > 0$ , independent of mesh size  $h$ , such that*

$$|B_F(\varphi; \varphi, \phi_h) - B_F^h(\varphi; \varphi, \phi_h)| \leq Ch^s (\|\varphi\|_{1+s, \Omega} + \|\varphi\|_{2, \Omega}) \|\varphi\|_{2+s, \Omega} \|\phi_h\|_{2, \Omega} \quad \forall \phi_h \in \mathbf{W}_k^h.$$

**Proof.** The proof has been established in [54, Lemma 5.4].  $\square$

**Lemma 5.4.** *Let  $\frac{1}{2} < \gamma \leq \min\{k - 1, \ell\}$ . Assume that  $\varphi(t) \in H_0^2(\Omega) \cap H^{2+\gamma}(\Omega)$  and  $v(t) \in H_0^1(\Omega) \cap H^{1+\gamma}(\Omega)$ , for almost all  $t \in (0, T)$ . Then, there exists  $C > 0$ , independent of mesh size  $h$ , such that, a.e.  $t \in (0, T)$ ,*

$$|B_{\text{skew}}(\varphi; v, w_h) - B_{\text{skew}}^h(\varphi; v, w_h)| \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega} \quad \forall w_h \in \mathbf{H}_\ell^h. \tag{5.2}$$

Moreover, assume that  $\mathbf{g}(t) \in \mathbf{H}^\gamma(\Omega) \cap \mathbf{L}^\infty(\Omega)$ , for almost all  $t \in (0, T)$ . Then, a.e.  $t \in (0, T)$ ,

$$|C(v, \phi_h) - C^h(v, \phi_h)| \leq Ch^\gamma \max\{\|\mathbf{g}\|_{\gamma, \Omega}, \|\mathbf{g}\|_{\infty, \Omega}\} \|v\|_{1+\gamma, \Omega} \|\phi_h\|_{1, \Omega} \quad \forall \phi_h \in \mathbf{W}_k^h. \tag{5.3}$$

**Proof.** To prove estimate (5.2), we split the consistency error as

$$B_{\text{skew}}(\varphi; v, w_h) - B_{\text{skew}}^h(\varphi; v, w_h) = \frac{1}{2} (\beta_1(w_h) + \beta_2(w_h)), \tag{5.4}$$

where

$$\beta_1(w_h) := \sum_{E \in \Omega_h} \left( B_T^E(\varphi; v, w_h) - B_T^{h,E}(\varphi; v, w_h) \right) \quad \text{and} \quad \beta_2(w_h) := \sum_{E \in \Omega_h} \left( B_T^E(\varphi; w_h, v) - B_T^{h,E}(\varphi; w_h, v) \right).$$

In what follows, we will establish bounds for the terms  $\beta_1(w_h)$  and  $\beta_2(w_h)$ . Indeed, for the term  $\beta_1(w_h)$  we have

$$\begin{aligned} \beta_1(w_h) &= \sum_{E \in \Omega_h} \int_E (\mathbf{curl} \varphi \cdot \nabla v) w_h - \int_E (\mathbf{\Pi}_E^{k-1} \mathbf{curl} \varphi \cdot \mathbf{\Pi}_E^{\ell-1} \nabla v) \mathbf{\Pi}_E^{\ell-1} w_h \\ &= \sum_{E \in \Omega_h} \int_E (\mathbf{curl} \varphi \cdot \nabla v) (w_h - \mathbf{\Pi}_E^{\ell-1} w_h) + \sum_{E \in \Omega_h} \int_E (\mathbf{curl} \varphi \cdot (\nabla v - \mathbf{\Pi}_E^{\ell-1} \nabla v)) \mathbf{\Pi}_E^{\ell-1} w_h \\ &\quad + \sum_{E \in \Omega_h} \int_E ((\mathbf{curl} \varphi - \mathbf{\Pi}_E^{k-1} \mathbf{curl} \varphi) \cdot \mathbf{\Pi}_E^{\ell-1} \nabla v) \mathbf{\Pi}_E^{\ell-1} w_h \\ &=: T_1 + T_2 + T_3. \end{aligned} \tag{5.5}$$

In order to bound the terms  $T_1$ , first we consider the case  $1/2 < \gamma \leq 1$ . Then, by using approximation property of  $\mathbf{\Pi}_E^{\ell-1}$  and the Hölder inequality, it follows

$$\begin{aligned} T_1 &\leq \sum_{E \in \Omega_h} \|\mathbf{curl} \varphi\|_{L^4(E)} \|\nabla v\|_{L^4(E)} \|w_h - \mathbf{\Pi}_E^{\ell-1} w_h\|_{0,E} \\ &\leq C \sum_{E \in \Omega_h} \|\mathbf{curl} \varphi\|_{L^4(E)} \|\nabla v\|_{L^4(E)} h_E |w_h|_{1,E} \\ &\leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}. \end{aligned}$$

On the other hand, for the case  $1 < \gamma \leq \ell$ , we use orthogonality property of  $\mathbf{\Pi}_E^{\ell-1}$ , the Hölder inequality (for sequences), to obtain

$$T_1 = \sum_{E \in \Omega_h} \int_E (\mathbf{curl} \varphi \cdot \nabla v - \mathbf{\Pi}_E^{\ell-1} (\mathbf{curl} \varphi \cdot \nabla v)) (w_h - \mathbf{\Pi}_E^{\ell-1} w_h).$$

Now, we apply [55, Theorem 7.4], with  $s = \gamma - 1$ ,  $s_1 = s_2 = \gamma$  and  $p = p_1 = p_2 = 2$  to obtain  $\mathbf{curl} \varphi \cdot \nabla v \in H^{\gamma-1}(\Omega)$  and  $|\mathbf{curl} \varphi \cdot \nabla v|_{\gamma-1, \Omega} \leq C \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega}$ .

Thus, by using Proposition 5.2 and the above facts, we arrive

$$T_1 \leq Ch^{\gamma-1} |\mathbf{curl} \varphi \cdot \nabla v|_{\gamma-1, \Omega} h \|w_h\|_{1, \Omega} \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}.$$

Collecting the above inequalities, for  $\frac{1}{2} < \gamma \leq \ell$ , we have

$$T_1 \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}. \tag{5.6}$$

Now, for the term  $T_2$  we proceed as follows. First, we apply the Hölder inequality, then by using stability and approximation properties of the  $L^2$ -projectors (cf. properties (3.6) and Proposition 5.2), Sobolev embedding and the Hölder inequality for sequences, we get

$$T_2 \leq \sum_{E \in \Omega_h} \|\mathbf{curl} \varphi\|_{L^4(E)} \|\nabla v - \Pi_E^{\ell-1} \nabla v\|_{0, E} \|\Pi_E^{\ell-1} w_h\|_{L^4(E)} \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}. \tag{5.7}$$

For the term  $T_3$ , we follow similar arguments, to obtain

$$T_3 \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}. \tag{5.8}$$

From the bounds (5.5)–(5.8), we conclude that

$$\beta_1(w_h) \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}. \tag{5.9}$$

Now, we will focus on the term  $\beta_2(w_h)$ . To estimate this term, first we add and subtract suitable expressions to obtain

$$\begin{aligned} \beta_2(w_h) &= \sum_{E \in \Omega_h} \int_E (\mathbf{curl} \varphi \cdot \nabla w_h) v - \int_E (\Pi_E^{k-1} \mathbf{curl} \varphi \cdot \Pi_E^{\ell-1} \nabla w_h) \Pi_E^{\ell-1} v \\ &= \sum_{E \in \Omega_h} \int_E v (\mathbf{curl} \varphi) \cdot (\nabla w_h - \Pi_E^{\ell-1} \nabla w_h) + \sum_{E \in \Omega_h} \int_E (\mathbf{curl} \varphi - \Pi_E^{k-1} \mathbf{curl} \varphi) \cdot v \Pi_E^{\ell-1} \nabla w_h \\ &\quad + \sum_{E \in \Omega_h} \int_E (\Pi_E^{k-1} \mathbf{curl} \varphi \cdot \Pi_E^{\ell-1} \nabla w_h) (v - \Pi_E^{\ell-1} v) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Applying orthogonality and approximation properties of  $\Pi_E^{\ell-1}$ , we have

$$\begin{aligned} I_1 &= \sum_{E \in \Omega_h} \int_E (v (\mathbf{curl} \varphi) - \Pi_E^{\ell-1} (v (\mathbf{curl} \varphi))) \cdot (\nabla w_h - \Pi_E^{\ell-1} \nabla w_h) \\ &\leq C \sum_{E \in \Omega_h} h_E^\gamma |v (\mathbf{curl} \varphi)|_{\gamma, E} |w_h|_{1, E} \leq Ch^\gamma |v (\mathbf{curl} \varphi)|_{\gamma, \Omega} \|w_h\|_{1, \Omega}. \end{aligned}$$

Then, applying again [55, Theorem 7.4], now with  $s = \gamma$ ,  $s_1 = \gamma + 1$ ,  $s_2 = \gamma$  and  $p = p_1 = p_2 = 2$ , we get

$$|v (\mathbf{curl} \varphi)|_{\gamma, \Omega} \leq C \|v\|_{1+\gamma, \Omega} \|\varphi\|_{1+\gamma, \Omega}.$$

From the two bounds above, we obtain

$$I_1 \leq Ch^\gamma \|\varphi\|_{1+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}.$$

The terms  $I_2$  and  $I_3$  can be estimated using similar arguments. We conclude that

$$\beta_2(w_h) \leq Ch^\gamma \|\varphi\|_{2+\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|w_h\|_{1, \Omega}. \tag{5.10}$$

The proof of (5.2) follows from (5.4), (5.9) and (5.10).

Next, we will prove property (5.3). Let  $\phi_h \in W_k^h$ , then adding and subtracting the term  $\mathbf{g}v \cdot \Pi_E^{k-1} \mathbf{curl} \phi_h$  and by using orthogonality, stability and approximations properties of the  $L^2$ -projections, we have

$$\begin{aligned} C(v, \phi_h) - C^h(v, \phi_h) &= \sum_{E \in \Omega_h} \int_E (\mathbf{g}v - \Pi_E^{k-1}(\mathbf{g}v)) \cdot (\mathbf{curl} \phi_h - \Pi_E^{k-1} \mathbf{curl} \phi_h) \\ &\quad + \int_E \mathbf{g}(v - \Pi_E^{\ell-1}v) \cdot \Pi_E^{k-1} \mathbf{curl} \phi_h \\ &\leq C \sum_{E \in \Omega_h} (h_E^\gamma |\mathbf{g}v|_{\gamma, E} \|\mathbf{curl} \phi_h\|_{0, E} + h_E^\gamma \|\mathbf{g}\|_{L^\infty(E)} \|v\|_{\gamma, E} \|\mathbf{curl} \phi_h\|_{0, E}) \\ &\leq Ch^\gamma (\|\mathbf{g}\|_{\gamma, \Omega} \|v\|_{1+\gamma, \Omega} \|\phi_h\|_{1, \Omega} + h_E^\gamma \|\mathbf{g}\|_{\infty, \Omega} \|v\|_{\gamma, \Omega} \|\phi_h\|_{1, \Omega}), \end{aligned}$$

where we have used analogous step to those used to bound  $I_1$ . The proof is complete.  $\square$

We finish this subsection recalling a discrete Gronwall inequality, which will be useful to derive the error estimate of the fully-discrete virtual scheme (4.1).

**Lemma 5.5.** *Let  $D \geq 0$ ,  $a_j, b_j, c_j$  and  $\lambda_j$  be non negative numbers for any integer  $j \geq 0$ , such that*

$$a_n + \Delta t \sum_{j=0}^n b_j \leq \Delta t \sum_{j=0}^n \lambda_j a_j + \Delta t \sum_{j=0}^n c_j + D, \quad n \geq 0.$$

Suppose that  $\Delta t \lambda_j < 1$  for all  $j$ , and set  $\sigma_j := (1 - \Delta t \lambda_j)^{-1}$ . Then, the following bound holds

$$a_n + \Delta t \sum_{j=0}^n b_j \leq \exp\left(\Delta t \sum_{j=0}^n \sigma_j \lambda_j\right) \left(\Delta t \sum_{j=0}^n c_j + D\right).$$

**Proof.** See [56, Lemma 5.1].  $\square$

### 5.2. Error estimates for the fully-discrete scheme

In this subsection we will provide a convergence result for the fully-discrete problem (4.1) under suitable regularity conditions for the exact solution.

We start denoting  $(\psi(t_n), \theta(t_n))$  as  $(\psi^n, \theta^n)$  at each time level  $t_n$ , and splitting the stream-function error as follows:

$$\psi^n - \psi_h^n = (\psi^n - S_h \psi^n) - (\psi_h^n - S_h \psi^n) =: \eta_\psi^n - \varphi_\psi^n.$$

For the temperature variable we will exploit the virtual interpolant presented in Proposition 5.4, to split the error as:

$$\theta^n - \theta_h^n = (\theta^n - \theta_I^n) - (\theta_h^n - \theta_I^n) =: \eta_\theta^n - \varphi_\theta^n,$$

where  $\theta_I^n$  is the interpolant of  $\theta^n$  in the virtual space  $H_\ell^h$ .

Error estimates for the terms  $\eta_\theta^n$  and  $\eta_\psi^n$  are given by Propositions 5.4 and 5.5, respectively. Therefore, we will focus on the terms  $\varphi_\psi^n$  and  $\varphi_\theta^n$ .

We start establishing error equations of the momentum and energy identities. Indeed, by using the fully-discrete scheme (4.1), the continuous weak formulation (2.13) and the biharmonic energy projection  $S_h$  defined in (5.1), we have the following error equation for the momentum identity (where we have taken  $\phi_h = \varphi_\psi^n \in W_k^h$ )

$$\begin{aligned} M_F^h \left( \frac{\varphi_\psi^n - \varphi_\psi^{n-1}}{\Delta t}, \varphi_\psi^n \right) + \nu A_F^h(\varphi_\psi^n, \varphi_\psi^n) &= \left( F_\psi^h(\varphi_\psi^n) - F_\psi(\varphi_\psi^n) \right) + \left( B_F(\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\psi_h^n; \psi_h^n, \varphi_\psi^n) \right) \\ &\quad + \left( M_F(\partial_t \psi^n, \varphi_\psi^n) - M_F^h \left( \frac{S_h \psi^n - S_h \psi^{n-1}}{\Delta t}, \varphi_\psi^n \right) \right) + \left( C^h(\theta_h^n, \varphi_\psi^n) - C(\theta^n, \varphi_\psi^n) \right) \\ &=: T_F + T_B + T_M + T_C. \end{aligned} \tag{5.11}$$

Analogously, recalling that  $\varphi_\theta^n = \theta_h^n - \theta_I^n$ , and using the definition of the continuous and discrete problems (cf. (2.13) and (4.1), respectively) for the energy equation, we have that

$$\begin{aligned} M_T^h \left( \frac{\varphi_\theta^n - \varphi_\theta^{n-1}}{\Delta t}, \varphi_\theta^n \right) + \kappa A_T^h(\varphi_\theta^n, \varphi_\theta^n) &= \left( F_\theta^h(\varphi_\theta^n) - F_\theta(\varphi_\theta^n) \right) + \left( B_{\text{skew}}(\psi^n; \theta^n, \varphi_\theta^n) - B_{\text{skew}}^h(\psi_h^n; \theta_h^n, \varphi_\theta^n) \right) \\ &+ \left( M_T(\partial_t \theta^n, \varphi_\theta^n) - M_T^h \left( \frac{\theta_I^n - \theta_I^{n-1}}{\Delta t}, \varphi_\theta^n \right) \right) + \kappa \left( A_T(\theta^n, \varphi_\theta^n) - A_T^h(\theta_I^n, \varphi_\theta^n) \right) \\ &=: I_F + I_B + I_M + I_A. \end{aligned} \tag{5.12}$$

The next step is to establish error estimates for the momentum and energy Eqs. (5.11) and (5.12). The following two lemmas provide such bounds and will be useful to obtain the convergence result for the fully-discrete problem (4.1).

**Lemma 5.6** (Error Estimate for the Momentum Equation). *Suppose that the external forces satisfy  $\mathbf{f}_\psi \in L^\infty(0, T; \mathbf{H}^s(\Omega))$  and  $\mathbf{g} \in L^\infty(0, T; \mathbf{H}^{\min\{s,r\}}(\Omega) \cap \mathbf{L}^\infty(\Omega))$ , with  $\frac{1}{2} < s \leq k-1$  and  $1 \leq r \leq \ell$ . Let  $(\psi^n, \theta^n) \in \mathbf{H}_0^2(\Omega) \times \mathbf{H}_0^1(\Omega)$  be the solution of problem (2.13) at time  $t = t_n$ . Moreover, assume that*

$$\begin{aligned} \psi &\in L^\infty(0, T; \mathbf{H}^{2+s}(\Omega)), \quad \partial_t \psi \in L^1(0, T; \mathbf{H}^{1+s}(\Omega)), \quad \partial_{tt} \psi \in L^1(0, T; \mathbf{H}^1(\Omega)), \\ &\text{and } \theta \in L^\infty(0, T; \mathbf{H}^r(\Omega)). \end{aligned}$$

Let  $(\psi_h^n, \theta_h^n) \in \mathbf{W}_k^h \times \mathbf{H}_\ell^h$  be the virtual element solution generated by scheme (4.1). Then, the following error estimate holds

$$\begin{aligned} \frac{1}{2\Delta t} (\|\varphi_\psi^n\|_{F,h}^2 - \|\varphi_\psi^{n-1}\|_{F,h}^2) + \frac{\widehat{\alpha}_{AF} \nu}{2} \|\varphi_\psi^n\|_{2,\Omega}^2 &\leq C [1 + \nu^{-3} (\|\eta_\psi^n\|_{2,\Omega}^4 + \|\psi^n\|_{2,\Omega}^4)] \|\varphi_\psi^n\|_{F,h}^2 \\ &+ C [\nu^{-1} (\|\psi^n\|_{2,\Omega}^2 + \|\eta_\psi^n\|_{2,\Omega}^2)] \|\eta_\psi^n\|_{2,\Omega}^2 + \|\mathbf{g}^n\|_{\infty,\Omega}^2 (\|\eta_\theta^n\|_{0,\Omega}^2 + \|\varphi_\theta^n\|_{0,\Omega}^2) \\ &+ Ch^{2s} (\|\mathbf{f}_\psi\|_{L^\infty(t_{n-1}, t_n; \mathbf{H}^s(\Omega_h))}^2 + \nu^{-1} \|\psi^n\|_{2+s,\Omega}^2) + Ch^{2\min\{s,r\}} \max\{\|\mathbf{g}^n\|_{\min\{s,r\},\Omega}^2, \|\mathbf{g}^n\|_{\infty,\Omega}^2\} \|\theta^n\|_{r,\Omega}^2 \\ &+ C \|\partial_{tt} \psi\|_{L^1(t_{n-1}, t_n; \mathbf{H}^1(\Omega))} \|\varphi_\psi^n\|_{F,h} + \frac{C}{\Delta t} h^s \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; \mathbf{H}^{1+s}(\Omega))} \|\varphi_\psi^n\|_{F,h}. \end{aligned} \tag{5.13}$$

**Proof.** We will estimate each term in (5.11). Indeed, by using the definition of the functionals  $F_\psi(\cdot)$  and  $F_\psi^h(\cdot)$ , the Cauchy–Schwarz and Young inequalities for the term  $T_F$  holds

$$T_F \leq \frac{C}{2c} h^{2s} \|\mathbf{f}_\psi\|_{L^\infty(t_{n-1}, t_n; \mathbf{H}^s(\Omega_h))}^2 + \frac{c}{2} \|\varphi_\psi^n\|_{1,\Omega}^2. \tag{5.14}$$

For the term  $T_M$ , we proceed similarly as in [54, Theorem 5.6] to obtain

$$\begin{aligned} T_M &:= M_F(\partial_t \psi^n, \varphi_\psi^n) - M_F^h \left( \frac{\mathcal{S}_h \psi^n - \mathcal{S}_h \psi^{n-1}}{\Delta t}, \varphi_\psi^n \right) = M_F \left( \partial_t \psi^n - \frac{\psi^n - \psi^{n-1}}{\Delta t}, \varphi_\psi^n \right) \\ &+ \sum_{E \in \Omega_h} M_F^E \left( \frac{\psi^n - \psi^{n-1}}{\Delta t} - \left( \frac{\Pi_E^{\mathbf{D},k}(\psi^n - \psi^{n-1})}{\Delta t} \right), \varphi_\psi^n \right) \\ &+ \sum_{E \in \Omega_h} M_F^{E,h} \left( \left( \frac{\Pi_E^{\mathbf{D},k}(\psi^n - \psi^{n-1})}{\Delta t} \right) - \frac{\mathcal{S}_h \psi^n - \mathcal{S}_h \psi^{n-1}}{\Delta t}, \varphi_\psi^n \right) \\ &\leq C \|\partial_{tt} \psi\|_{L^1(t_{n-1}, t_n; \mathbf{H}^1(\Omega))} \|\varphi_\psi^n\|_{1,\Omega} + \frac{C}{\Delta t} h^s \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; \mathbf{H}^{1+s}(\Omega))} \|\varphi_\psi^n\|_{1,\Omega}. \end{aligned} \tag{5.15}$$

Next, to estimate  $T_C$ , we add and subtract the term  $C^h(\theta^n, \varphi_\psi^n)$  to get

$$\begin{aligned} T_C &:= C^h(\theta_h^n, \varphi_\psi^n) - C(\theta^n, \varphi_\psi^n) = C^h(\theta_h^n - \theta^n, \varphi_\psi^n) + (C^h(\theta^n, \varphi_\psi^n) - C(\theta^n, \varphi_\psi^n)) \\ &= (C^h(\varphi_\theta^n, \varphi_\psi^n) - C^h(\eta_\theta^n, \varphi_\psi^n)) + (C^h(\theta^n, \varphi_\psi^n) - C(\theta^n, \varphi_\psi^n)) \\ &\leq \|\mathbf{g}^n\|_{\infty, \Omega} (\|\varphi_\theta^n\|_{0, \Omega} + \|\eta_\theta^n\|_{0, \Omega}) \|\varphi_\psi^n\|_{1, \Omega} + Ch^{\min\{s, r\}} \max\{\|\mathbf{g}^n\|_{\min\{s, r\}, \Omega}, \|\mathbf{g}^n\|_{\infty, \Omega}\} \|\theta^n\|_{r, \Omega} \|\varphi_\psi^n\|_{1, \Omega} \\ &\leq C\|\mathbf{g}^n\|_{\infty, \Omega}^2 (\|\varphi_\theta^n\|_{0, \Omega}^2 + \|\varphi_\psi^n\|_{1, \Omega}^2) + Ch^{2\min\{s, r\}} \max\{\|\mathbf{g}^n\|_{\min\{s, r\}, \Omega}^2, \|\mathbf{g}^n\|_{\infty, \Omega}^2\} \|\theta^n\|_{r, \Omega}^2 + c\|\varphi_\psi^n\|_{1, \Omega}^2, \end{aligned} \tag{5.16}$$

where we have used the Hölder inequality, bound (5.3) (with  $\gamma = \min\{s, r\}$ ) and the Young inequality.

For the term  $T_B$ , we have

$$\begin{aligned} T_B &:= B_F(\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\psi_h^n; \psi_h^n, \varphi_\psi^n) = (B_F(\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\psi^n; \psi^n, \varphi_\psi^n)) \\ &\quad + (B_F^h(\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\psi_h^n; \psi_h^n, \varphi_\psi^n)) =: T_{B1} + T_{B2}. \end{aligned} \tag{5.17}$$

Now, we will bound the terms  $T_{B1}$  and  $T_{B2}$ . Indeed, from Lemma 5.3 and the Young inequality we have that

$$\begin{aligned} T_{B1} &:= B_F(\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\psi^n; \psi^n, \varphi_\psi^n) \leq C h^s (\|\psi^n\|_{2+s, \Omega} + \|\psi^n\|_{2, \Omega}) \|\psi^n\|_{2+s, \Omega} \|\varphi_\psi^n\|_{2, \Omega} \\ &\leq \frac{4C_\psi}{\widehat{\alpha}_{AF} \nu} h^{2s} \|\psi^n\|_{2+s, \Omega}^2 + \frac{\widehat{\alpha}_{AF} \nu}{8} \|\varphi_\psi^n\|_{2, \Omega}^2 \\ &\leq C_\psi \nu^{-1} h^{2s} \|\psi^n\|_{2+s, \Omega}^2 + \frac{\widehat{\alpha}_{AF} \nu}{8} \|\varphi_\psi^n\|_{2, \Omega}^2, \end{aligned} \tag{5.18}$$

where we have included the term  $(\|\psi^n\|_{2+s, \Omega} + \|\psi^n\|_{2, \Omega})$  in the constant  $C_\psi$  in order to shorten the inequality.

On the other hand, to bound the expression  $T_{B2}$ , we apply Lemma 5.2, recall that  $\varphi_\psi^n = \psi_h^n - \mathcal{S}_h \psi^n$  and  $\eta_\psi^n = \psi^n - \mathcal{S}_h \psi^n$ , to arrive

$$\begin{aligned} T_{B2} &:= B_F^h(\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\psi_h^n; \psi_h^n, \varphi_\psi^n) \\ &= B_F^h(\psi^n; \psi^n - \psi_h^n + \varphi_\psi^n, \varphi_\psi^n) + B_F^h(\psi^n - \psi_h^n + \varphi_\psi^n; \psi_h^n, \varphi_\psi^n) - B_F^h(\varphi_\psi^n; \psi_h^n, \varphi_\psi^n) \\ &= B_F^h(\psi^n; \eta_\psi^n, \varphi_\psi^n) + B_F^h(\eta_\psi^n; \psi_h^n, \varphi_\psi^n) - B_F^h(\varphi_\psi^n; \psi_h^n, \varphi_\psi^n). \end{aligned} \tag{5.19}$$

By using Lemma 3.3, together with the Young inequality, we have

$$B_F^h(\psi^n; \eta_\psi^n, \varphi_\psi^n) \leq \frac{\widehat{\alpha}_{AF} \nu}{8} \|\varphi_\psi^n\|_{2, \Omega}^2 + C\nu^{-1} \|\psi^n\|_{2, \Omega}^2 \|\eta_\psi^n\|_{2, \Omega}^2.$$

Now, adding and subtracting suitable terms, and employing Lemma 3.3 along with the Young inequality, we obtain

$$\begin{aligned} B_F^h(\eta_\psi^n; \psi_h^n, \varphi_\psi^n) &= B_F^h(\eta_\psi^n; \psi^n + (\psi_h^n - \psi^n), \varphi_\psi^n) \\ &= B_F^h(\eta_\psi^n; \psi^n, \varphi_\psi^n) + B_F^h(\eta_\psi^n; \varphi_\psi^n - \eta_\psi^n, \varphi_\psi^n) \\ &= B_F^h(\eta_\psi^n; \psi^n, \varphi_\psi^n) - B_F^h(\eta_\psi^n; \eta_\psi^n, \varphi_\psi^n) \\ &\leq \widehat{C}_{BF} (\|\psi^n\|_{2, \Omega} + \|\eta_\psi^n\|_{2, \Omega}) \|\eta_\psi^n\|_{2, \Omega} \|\varphi_\psi^n\|_{2, \Omega} \\ &\leq \frac{\widehat{\alpha}_{AF} \nu}{8} \|\varphi_\psi^n\|_{2, \Omega}^2 + C\nu^{-1} (\|\psi^n\|_{2, \Omega}^2 + \|\eta_\psi^n\|_{2, \Omega}^2) \|\eta_\psi^n\|_{2, \Omega}^2. \end{aligned}$$

Once again adding and subtracting adequate terms, using Lemma 5.1 and the Young inequality, we get

$$\begin{aligned} -B_F^h(\varphi_\psi^n; \psi_h^n, \varphi_\psi^n) &= B_F^h(\varphi_\psi^n; (\psi^n - \psi_h^n) - \psi^n, \varphi_\psi^n) = B_F^h(\varphi_\psi^n; \eta_\psi^n, \varphi_\psi^n) - B_F^h(\varphi_\psi^n; \psi^n, \varphi_\psi^n) \\ &\leq \widehat{C}_{BF} \|\varphi_\psi^n\|_{2, \Omega} (\|\eta_\psi^n\|_{2, \Omega} + \|\psi^n\|_{2, \Omega}) \|\varphi_\psi^n\|_{2, \Omega}^{\frac{1}{2}} \|\varphi_\psi^n\|_{1, \Omega}^{\frac{1}{2}} \\ &\leq \frac{\widehat{\alpha}_{AF} \nu}{16} \|\varphi_\psi^n\|_{2, \Omega}^2 + 2C\nu^{-1} (\|\eta_\psi^n\|_{2, \Omega}^2 + \|\psi^n\|_{2, \Omega}^2) \|\varphi_\psi^n\|_{2, \Omega} \|\varphi_\psi^n\|_{1, \Omega} \\ &\leq \frac{\widehat{\alpha}_{AF} \nu}{16} \|\varphi_\psi^n\|_{2, \Omega}^2 + 2\nu^{-2} C_4 \nu^{-1} (\|\eta_\psi^n\|_{2, \Omega}^2 + \|\psi^n\|_{2, \Omega}^2)^2 \|\varphi_\psi^n\|_{1, \Omega}^2 + \frac{\widehat{\alpha}_{AF} \nu}{16} \|\varphi_\psi^n\|_{2, \Omega}^2 \\ &\leq \frac{\widehat{\alpha}_{AF} \nu}{8} \|\varphi_\psi^n\|_{2, \Omega}^2 + 4C_4 \nu^{-3} (\|\eta_\psi^n\|_{2, \Omega}^4 + \|\psi^n\|_{2, \Omega}^4) \|\varphi_\psi^n\|_{1, \Omega}^2. \end{aligned}$$

Combining the estimates (5.17)–(5.19) and the three previous inequalities, we have

$$T_B \leq C_1 v^{-1} h^{2s} \|\psi^n\|_{2+s,\Omega}^2 \|\psi^n\|_{2,\Omega}^2 + \frac{\widehat{\alpha}_{AF} v}{2} \|\varphi_\psi^n\|_{2,\Omega}^2 + C v^{-1} \|\psi^n\|_{2,\Omega}^2 \|\eta_\psi^n\|_{2,\Omega}^2 + C v^{-1} (\|\psi^n\|_{2,\Omega}^2 + \|\eta_\psi^n\|_{2,\Omega}^2) \|\eta_\psi^n\|_{2,\Omega}^2 + C_4 v^{-3} (\|\eta_\psi^n\|_{2,\Omega}^4 + \|\psi^n\|_{2,\Omega}^4) \|\varphi_\psi^n\|_{1,\Omega}^2. \tag{5.20}$$

Now, from estimates (5.11), (5.14)–(5.16) and (5.20), the definition and equivalence of the norm  $\|\cdot\|_{F,h}$  (cf. (4.6)), together with the coercivity of bilinear form  $A_F^h(\cdot, \cdot)$  we obtain the desired estimate.  $\square$

**Lemma 5.7** (Error Estimate for the Energy Equation). *Let  $\frac{1}{2} < s \leq k - 1$  and  $1 \leq r \leq \ell$ . Suppose that  $f_\theta \in L^\infty(0, T; H^r(\Omega))$ . Moreover, let  $(\psi^n, \theta^n) \in H_0^2(\Omega) \times H_0^1(\Omega)$  be the solution of problem (2.13) at time  $t = t_n$  and assume that*

$$\theta \in L^\infty(0, T; H^{1+r}(\Omega) \cap W_\infty^1(\Omega)), \quad \partial_t \theta \in L^1(0, T; H^r(\Omega)), \quad \partial_{tt} \theta \in L^1(0, T; L^2(\Omega))$$

and  $\psi \in L^\infty(0, T; H^{2+s}(\Omega))$ .

Let  $(\psi_h^n, \theta_h^n) \in W_k^h \times H_\ell^h$  be the virtual element solution generated by scheme (4.1). Then, the following error estimate holds

$$\begin{aligned} \frac{1}{2\Delta t} (\|\varphi_\theta^n\|_{T,h}^2 - \|\varphi_\theta^{n-1}\|_{T,h}^2) + \frac{\widehat{\alpha}_{AT} \kappa}{2} \|\varphi_\theta^n\|_{1,\Omega}^2 &\leq C \|\varphi_\theta^n\|_{0,\Omega}^2 + \kappa^{-1} \|\theta^n\|_{1,\Omega}^2 \|\eta_\psi^n\|_{2,\Omega}^2 \\ &+ C [\kappa^{-1} (\|\psi^n\|_{2,\Omega}^2 + \|\eta_\psi^n\|_{2,\Omega}^2)] \|\eta_\theta^n\|_{1,\Omega}^2 + C \|\varphi_\theta^n\|_{T,h}^2 \\ &+ C h^{2r} \|f_\theta\|_{L^\infty(t_{n-1}, t_n; H^r(\Omega_h))}^2 + C \kappa^{-1} h^{2\min\{s,r\}} \|\psi^n\|_{2+s,\Omega}^2 \|\theta^n\|_{2+r,\Omega}^2 \\ &+ C \|\partial_{tt} \theta\|_{L^1(t_{n-1}, t_n; L^2(\Omega))} \|\varphi_\theta^n\|_{T,h} + \frac{C}{\Delta t} h^r \|\partial_t \theta\|_{L^1(t_{n-1}, t_n; H^r(\Omega))} \|\varphi_\theta^n\|_{T,h}. \end{aligned} \tag{5.21}$$

**Proof.** We will establish estimates for each term in the error equation (5.12). We start with the term  $I_F$ , which is bounded by using the Cauchy–Schwarz inequality and approximation properties of projection  $\Pi_E^\ell$ , as follows:

$$I_F := F_\theta^h(\varphi_\theta^n) - F_\theta(\varphi_\theta^n) \leq \frac{C}{2c} h^{2r} \|f_\theta\|_{L^\infty(t_{n-1}, t_n; H^r(\Omega_h))}^2 + \frac{c}{2} \|\varphi_\theta^n\|_{0,\Omega}^2. \tag{5.22}$$

For the term  $I_M$ , we proceed similarly as in [47, Theorem 3.3] to obtain

$$\begin{aligned} I_M &:= M_T(\partial_t \theta^n, \varphi_\theta^n) - M_F^h\left(\frac{\theta_I^n - \theta_I^{n-1}}{\Delta t}, \varphi_\theta^n\right) \\ &\leq C \|\partial_{tt} \theta\|_{L^1(t_{n-1}, t_n; L^2(\Omega))} \|\varphi_\theta^n\|_{0,\Omega} + \frac{C}{\Delta t} h^r \|\partial_t \theta\|_{L^1(t_{n-1}, t_n; H^r(\Omega))} \|\varphi_\theta^n\|_{0,\Omega}. \end{aligned} \tag{5.23}$$

Analogously, as in (5.17) we split the term  $I_B$  as follows:

$$\begin{aligned} I_B &:= B_{\text{skew}}(\psi^n; \theta^n, \varphi_\theta^n) - B_{\text{skew}}^h(\psi_h^n; \theta_h^n, \varphi_\theta^n) = (B_{\text{skew}}(\psi^n; \theta^n, \varphi_\theta^n) - B_{\text{skew}}^h(\psi^n; \theta^n, \varphi_\theta^n)) \\ &+ (B_{\text{skew}}^h(\psi^n; \theta^n, \varphi_\theta^n) - B_{\text{skew}}^h(\psi_h^n; \theta_h^n, \varphi_\theta^n)) =: I_{B1} + I_{B2}. \end{aligned} \tag{5.24}$$

Now, applying the bound (5.2), with  $\gamma = \min\{s, r\}$  and using the Young inequality, we obtain

$$\begin{aligned} I_{B1} &:= B_{\text{skew}}(\psi^n; \theta^n, \varphi_\theta^n) - B_{\text{skew}}^h(\psi^n; \theta^n, \varphi_\theta^n) \leq C h^{\min\{s,r\}} \|\psi^n\|_{2+s,\Omega} \|\theta^n\|_{1+r,\Omega} \|\varphi_\theta^n\|_{1,\Omega} \\ &\leq C \kappa^{-1} h^{2\min\{s,r\}} \|\psi^n\|_{2+s,\Omega}^2 \|\theta^n\|_{1+r,\Omega}^2 + \frac{\widehat{\alpha}_{AT} \kappa}{10} \|\varphi_\theta^n\|_{1,\Omega}^2. \end{aligned} \tag{5.25}$$

On the other hand, similarly as in (5.19) and (5.20), we can derive

$$\begin{aligned} I_{B2} &= B_{\text{skew}}^h(\psi^n; \eta_\psi^n, \varphi_\theta^n) + B_{\text{skew}}^h(\eta_\psi^n; \theta_h^n, \varphi_\theta^n) - B_{\text{skew}}^h(\varphi_\psi^n; \theta_h^n, \varphi_\theta^n) \\ &\leq \frac{\widehat{\alpha}_{AT} \kappa}{10} \|\varphi_\theta^n\|_{1,\Omega}^2 + C \kappa^{-1} \|\psi^n\|_{2,\Omega}^2 \|\eta_\theta^n\|_{1,\Omega}^2 + \frac{\widehat{\alpha}_{AT} \kappa}{10} \|\varphi_\theta^n\|_{1,\Omega}^2 + C \kappa^{-1} (\|\theta^n\|_{1,\Omega}^2 + \|\eta_\theta^n\|_{1,\Omega}^2) \|\eta_\psi^n\|_{2,\Omega}^2 \\ &- B_{\text{skew}}^h(\varphi_\psi^n; \theta_h^n, \varphi_\theta^n). \end{aligned} \tag{5.26}$$

However, since the discrete trilinear form  $B_{\text{skew}}^h(\cdot; \cdot, \cdot)$  does not satisfy an analogous property to Lemma 5.1, we will bound the last term in (5.26) by a different way. Indeed, adding and subtracting adequate terms, using

the definition of trilinear form, the Hölder inequality and employing the continuity of the  $L^2$ -projections involved (cf. (3.6)), we obtain

$$\begin{aligned}
 -B_{\text{skew}}^h(\varphi_\psi^n; \theta_h^n, \varphi_\theta^n) &= B_{\text{skew}}^h(\varphi_\psi^n; \eta_\theta^n, \varphi_\theta^n) + B_{\text{skew}}^h(\varphi_\psi^n; -\theta^n, \varphi_\theta^n) \\
 &= \frac{1}{2} (B_T^h(\varphi_\psi^n; \eta_\theta^n, \varphi_\theta^n) - B_T^h(\varphi_\psi^n; \varphi_\theta^n, \eta_\theta^n)) + B_{\text{skew}}^h(\varphi_\psi^n; -\theta^n, \varphi_\theta^n) \\
 &\leq C \sum_{E \in \Omega_h} \|\mathbf{\Pi}_E^{\ell-1} \nabla \eta_\theta^n\|_{L^\infty(E)} \|\mathbf{curl} \varphi_\psi^n\|_{0,E} \|\varphi_\theta^n\|_{0,E} \\
 &\quad + C \sum_{E \in \Omega_h} \|\mathbf{\Pi}_E^{\ell-1} \eta_\theta^n\|_{L^\infty(E)} \|\mathbf{curl} \varphi_\psi^n\|_{0,E} \|\nabla \varphi_\theta^n\|_{0,E} + B_{\text{skew}}^h(\varphi_\psi^n; -\theta^n, \varphi_\theta^n).
 \end{aligned} \tag{5.27}$$

Now, applying an inverse inequality for polynomials, the continuity of  $\mathbf{\Pi}_E^{\ell-1}$ , and Proposition 5.4, for  $r \geq 1$  we get

$$\|\mathbf{\Pi}_E^{\ell-1} \nabla \eta_\theta^n\|_{L^\infty(E)} \leq Ch_E^{-1} \|\mathbf{\Pi}_E^{\ell-1} \nabla \eta_\theta^n\|_{0,E} \leq Ch_E^{-1} \|\eta_\theta^n\|_{1,E} \leq C \|\theta^n\|_{1+r,E} \leq C_{\text{reg}}.$$

Analogously, we have that

$$\|\mathbf{\Pi}_E^{\ell-1} \eta_\theta^n\|_{L^\infty(E)} \leq C \|\theta^n\|_{1+r,E} \leq C_{\text{reg}}.$$

Next, under assumption  $\theta^n \in \mathbf{W}_\infty^1(\Omega)$ , the definition of the form  $B_{\text{skew}}^h(\cdot; \cdot, \cdot)$  and the Cauchy–Schwarz inequality, we get

$$B_{\text{skew}}^h(\varphi_\psi^n; -\theta^n, \varphi_\theta^n) \leq C_1 \|\theta^n\|_{\mathbf{W}_\infty^1(\Omega)} \|\varphi_\psi^n\|_{1,\Omega} \|\varphi_\theta^n\|_{0,\Omega} \leq C_{\text{reg}} \|\varphi_\psi^n\|_{1,\Omega} \|\varphi_\theta^n\|_{0,\Omega}.$$

Inserting the above estimates in (5.27), and applying the Cauchy–Schwarz and Young inequalities, it follows

$$-B_{\text{skew}}^h(\varphi_\psi^n; \theta_h^n, \varphi_\theta^n) \leq 3C_{\text{reg}} \|\varphi_\psi^n\|_{1,\Omega} \|\varphi_\theta^n\|_{1,\Omega} \leq C\kappa^{-1} \|\varphi_\psi^n\|_{1,\Omega}^2 + \frac{\widehat{\alpha}_{A_T \mathcal{K}}}{10} \|\varphi_\theta^n\|_{1,\Omega}^2. \tag{5.28}$$

Then, combining the estimates (5.24), (5.25), (5.26) and (5.28), we obtain

$$\begin{aligned}
 I_B &\leq C\kappa^{-1} h^{2\min\{s,r\}} \|\psi^n\|_{2+s,\Omega}^2 \|\theta^n\|_{1+r,\Omega}^2 + C\kappa^{-1} \|\psi^n\|_{2,\Omega}^2 \|\eta_\theta^n\|_{1,\Omega}^2 \\
 &\quad + C\kappa^{-1} (\|\theta^n\|_{1,\Omega}^2 + \|\eta_\theta^n\|_{1,\Omega}^2) \|\eta_\psi^n\|_{2,\Omega}^2 + \frac{4\widehat{\alpha}_{A_T \mathcal{K}}}{10} \|\varphi_\theta^n\|_{1,\Omega}^2 + C(\kappa^{-1} + 1) \|\varphi_\psi^n\|_{1,\Omega}^2.
 \end{aligned} \tag{5.29}$$

Now, for the term  $I_A$ , we add and subtract  $\theta_\pi^n \in \mathbb{P}_\ell(E)$  such that Proposition 5.1 holds true, then applying the consistency property of  $A_T^{h,E}(\cdot, \cdot)$ , the triangle inequality and Proposition 5.4, we have that

$$\begin{aligned}
 I_A &= \kappa \sum_{E \in \Omega_h} \left( A_T^E(\theta^n, \varphi_\theta^n) - A_T^{h,E}(\theta_\pi^n, \varphi_\theta^n) \right) = \kappa \sum_{E \in \Omega_h} \left( A_T^E(\theta^n - \theta_\pi^n, \varphi_\theta^n) + A_T^{h,E}(\theta_\pi^n - \theta_\pi^n, \varphi_\theta^n) \right) \\
 &\leq C\kappa h^r \|\theta^n\|_{1+r,\Omega} \|\varphi_\theta^n\|_{1,\Omega} \\
 &\leq Ch^{2r} \|\theta^n\|_{1+r,\Omega}^2 + \frac{\widehat{\alpha}_{A_T \mathcal{K}}}{10} \|\varphi_\theta^n\|_{1,\Omega}^2.
 \end{aligned} \tag{5.30}$$

Now, from bounds (5.12), (5.22), (5.23), (5.29) and (5.30), the definition and equivalence of the norms  $\|\cdot\|_{T,h}$  (cf. (4.6)) and  $\|\cdot\|_{0,\Omega}$ , together with the coercivity of bilinear form  $A_T^h(\cdot, \cdot)$ , we obtain the estimate (5.21).  $\square$

The following result establishes an error estimate for the fully-discrete virtual scheme (4.1).

**Theorem 5.1.** *Suppose that the external forces satisfy  $\mathbf{f}_\psi \in L^\infty(0, T; \mathbf{H}^s(\Omega))$ ,  $\mathbf{f}_\theta \in L^\infty(0, T; \mathbf{H}^r(\Omega))$  and  $\mathbf{g} \in L^\infty(0, T; \mathbf{H}^{\min\{s,r\}}(\Omega) \cap \mathbf{L}^\infty(\Omega))$ , with  $\frac{1}{2} < s \leq k - 1$  and  $1 \leq r \leq \ell$ . Let  $(\psi^n, \theta^n) \in \mathbf{H}_0^2(\Omega) \times \mathbf{H}_0^1(\Omega)$  be the solution of problem (2.13) at time  $t = t_n$ . Moreover, assume that*

$$\begin{aligned}
 \psi &\in L^\infty(0, T; \mathbf{H}^{2+s}(\Omega)), & \partial_t \psi &\in L^1(0, T; \mathbf{H}^{1+s}(\Omega)), & \partial_{tt} \psi &\in L^1(0, T; \mathbf{H}^1(\Omega)), \\
 \theta &\in L^\infty(0, T; \mathbf{H}^{1+r}(\Omega) \cap \mathbf{W}_\infty^1(\Omega)), & \partial_t \theta &\in L^1(0, T; \mathbf{H}^r(\Omega)), & \partial_{tt} \theta &\in L^1(0, T; L^2(\Omega)).
 \end{aligned}$$

Let  $(\psi_h^n, \theta_h^n) \in \mathbf{W}_k^h \times \mathbf{H}_\ell^h$  be the virtual element solution generated by scheme (4.1). Then, the following estimate holds

$$\|\psi^n - \psi_h^n, \theta^n - \theta_h^n\|_{\mathbf{H}^1(\Omega) \times L^2(\Omega)}^2 + \Delta t \sum_{j=1}^n \|(\psi^j - \psi_h^j, \theta^j - \theta_h^j)\|_{\mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega)}^2 \leq C(h^{2\min\{s,r\}} + \Delta t^2),$$

where the constant  $C$  is positive and depends on the physical parameters  $\nu, \kappa$ , final time  $T$ , mesh regularity parameter, the regularity of the Boussinesq solution fields  $(\psi, \theta)$  and the external forces  $\mathbf{f}_\psi, f_\theta, \mathbf{g}$ , but is independent of mesh size  $h$  and time steps  $\Delta t$ .

**Proof.** The desired estimate will follow combining Lemmas 5.6 and 5.7 with the discrete Gronwall inequality. Indeed, we proceed to multiply by  $2\Delta t$  the estimates (5.13) and (5.21), then by employing the Young inequality to the resulting bounds and iterating  $j = 0, \dots, n$ , we have

$$\begin{aligned} & \|\varphi_\psi^n\|_{F,h}^2 + \|\varphi_\theta^n\|_{T,h}^2 + \Delta t \sum_{j=0}^n \|\varphi_\psi^j\|_{2,\Omega}^2 + \Delta t \sum_{j=0}^n \|\varphi_\theta^j\|_{1,\Omega}^2 \\ & \leq C \Delta t \sum_{j=0}^n \left[ 1 + \nu^{-3} \left( \|\eta_\psi^j\|_{2,\Omega}^4 + \|\psi^j\|_{2,\Omega}^4 \right) \right] \|\varphi_\psi^j\|_{F,h}^2 + C \Delta t \sum_{j=0}^n \left[ 1 + \|\mathbf{g}^j\|_{\infty,\Omega} \right] \|\varphi_\theta^j\|_{T,h}^2 \\ & \quad + C \Delta t \sum_{j=0}^n \left[ \nu^{-1} (\|\psi^j\|_{2,\Omega}^2 + \|\eta_\psi^j\|_{2,\Omega}^2) + \kappa^{-1} \|\theta^j\|_{1,\Omega}^2 \right] \|\eta_\psi^j\|_{2,\Omega}^2 \\ & \quad + C \Delta t \sum_{j=0}^n \left[ \kappa^{-1} (\|\psi^j\|_{2,\Omega}^2 + \|\eta_\psi^j\|_{2,\Omega}^2) + \|\mathbf{g}^j\|_{\infty,\Omega} \right] \|\eta_\theta^j\|_{1,\Omega}^2 \\ & \quad + C \Delta t h^{2s} \left( \|\mathbf{f}_\psi\|_{L^\infty(0,t_n;H^s(\Omega_h))}^2 + \|\partial_t \psi\|_{L^1(0,t_n;H^{1+s}(\Omega))}^2 + \nu^{-1} \|\psi\|_{L^\infty(0,t_n;H^{2+s}(\Omega))}^2 \right) \\ & \quad + \Delta t h^{2\min\{s,r\}} \max\{\|\mathbf{g}\|_{L^\infty(0,t_n;H^{\min\{s,r\}}(\Omega))}^2, \|\mathbf{g}\|_{L^\infty(0,t_n;L^\infty(\Omega))}^2\} \|\theta\|_{L^\infty(0,t_n;H^r(\Omega))}^2 \\ & \quad + C \Delta t h^{2r} \left( \|f_\theta\|_{L^\infty(0,t_n;H^r(\Omega_h))}^2 + \|\partial_t \theta\|_{L^1(0,t_n;H^r(\Omega))}^2 \right) \\ & \quad + C \Delta t \kappa^{-1} h^{2\min\{s,r\}} \left( \|\psi\|_{L^\infty(0,t_n;H^{2+s}(\Omega))}^2 + \|\theta\|_{L^\infty(0,t_n;H^{1+r}(\Omega))}^2 \right) \\ & \quad + C \Delta t^2 \left( \|\partial_{tt} \theta\|_{L^1(0,t_n;L^2(\Omega))}^2 + \|\partial_{tt} \psi\|_{L^1(0,t_n;H^1(\Omega))}^2 \right) + \widehat{\alpha}_{MF} \|\varphi_\psi^0\|_{1,\Omega}^2 + \widehat{\alpha}_{MT} \|\varphi_\theta^0\|_{0,\Omega}^2. \end{aligned}$$

Thus, applying the discrete Gronwall inequality (cf. Lemma 5.5), choosing  $(\psi_h^0, \theta_h^0) = (\psi_I(0), \theta_I(0))$  and using Propositions 5.3 and 5.4 along with the equivalence of norms, we have

$$\left( \|\varphi_\psi^n\|_{1,\Omega}^2 + \|\varphi_\theta^n\|_{0,\Omega}^2 \right) + \Delta t \sum_{j=1}^n \left( \|\varphi_\psi^j\|_{2,\Omega}^2 + \|\varphi_\theta^j\|_{1,\Omega}^2 \right) \leq C(h^{2\min\{s,r\}} + \Delta t^2),$$

with  $\frac{1}{2} < s \leq k - 1$ ,  $1 \leq r \leq \ell$  and  $C > 0$  is independent of mesh size  $h$  and time step  $\Delta t$ .

Finally, the desired result follows from the above estimate, triangular inequality, together with Propositions 5.4 and 5.5.  $\square$

**Remark 5.1.** In the present framework, the main advantage of using an energy projector  $\mathcal{S}_h \psi^n$ , as we do for the stream-function space, is to obtain a shorter proof. Nevertheless, for the temperature variable we do not use an energy projector, but resort to a standard interpolant  $\theta^n$ . The reason is that we need also some local approximation properties for the temperature field that the energy projection operator, being global in nature, would not have.

### 6. Numerical results

In this section we carry out numerical experiments in order to support our analytical results and illustrate the performance of the proposed fully-discrete virtual scheme (4.1) for the Boussinesq system. In all examples, we use the lowest order virtual element spaces  $W_2^h$  and  $H_1^h$ , for the stream-function and temperature fields, respectively. At each discrete time, the nonlinear fully-discrete system (4.1) is linearized by using the Newton method. For the first time step, we take as initial guess  $(\psi_h^{1n}, \theta_h^{1n}) = (0, 0)$ , and for all  $n \geq 1$  we take  $(\psi_h^{1n}, \theta_h^{1n}) = (\psi_h^{n-1}, \theta_h^{n-1})$ . The iterations are finalized when the  $\ell^\infty$ -norm of the global incremental discrete solution drop below a fixed tolerance of  $\text{Tol} = 10^{-8}$ .

The domain  $\Omega$  is partitioned using the following sequences of polygonal meshes (an example for each family is shown in Fig. 1):



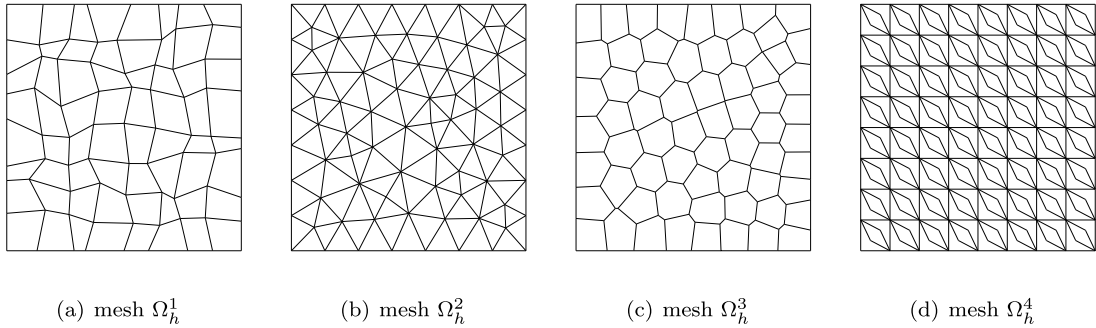


Fig. 1. Domain discretized with different meshes.

- $\Omega_h^1$ : Distorted quadrilaterals meshes;
- $\Omega_h^2$ : Triangular meshes;
- $\Omega_h^3$ : Voronoi meshes;
- $\Omega_h^4$ : Distorted concave rhombic quadrilaterals.

In order to test the convergence properties of the proposed VEM, we measure some errors as the difference between the exact solutions  $(\psi, \theta)$  and adequate projections of the numerical solution  $(\psi_h^n, \theta_h^n)$ . More precisely, we consider the following quantities:

$$E(\psi, L^2, H^2) := \left( \Delta t \sum_{n=1}^N |\psi(t_n) - \Pi^{\mathbf{D},2} \psi_h^n|_{2,h}^2 \right)^{1/2}, \quad E(\theta, L^2, H^1) := \left( \Delta t \sum_{n=1}^N |\theta(t_n) - \Pi^{\nabla,1} \theta_h^n|_{1,h}^2 \right)^{1/2}, \tag{6.1}$$

$$E(\psi, L^\infty, H^1) := |\psi(T) - \Pi^{\mathbf{D},2} \psi_h^N|_{1,h}, \quad E(\theta, L^\infty, L^2) := \|\theta(T) - \Pi^{\nabla,1} \theta_h^N\|_{0,\Omega}. \tag{6.2}$$

Accordingly to Theorem 5.1, the expected convergence rate for the sum of the above norms is  $\mathcal{O}(h + \Delta t)$ .

### 6.1. Accuracy assessment

In our first example, we illustrate the accuracy in space and time of the proposed VEM (4.1), considering a manufactured exact solution on the square domain  $\Omega := (0, 1)^2$ , the time interval  $[0, 1]$  and force per unit mass  $\mathbf{g} = (0, -1)^T$ . We solve the Boussinesq system (2.1), taking the load terms  $\mathbf{f}_\psi$  and  $f_\theta$ , boundary and initial conditions in such a way that the analytical solution is given by:

$$\mathbf{u}(x, y, t) = \begin{pmatrix} u_1(x, y, t) \\ u_2(x, y, t) \end{pmatrix} = \begin{pmatrix} (e^{10(t-1)} - e^{-10}) x^2(1-x)^2(2y - 6y^2 + 4y^3) \\ -(e^{10(t-1)} - e^{-10}) y^2(1-y)^2(2x - 6x^2 + 4x^3) \end{pmatrix},$$

$$p(x, y, t) = (e^{10(t-1)} - e^{-10})(\sin(x) \cos(y) + (\cos(1) - 1) \sin(1)),$$

$$\psi(x, y, t) = (e^{10(t-1)} - e^{-10}) x^2(1-x)^2 y^2(1-y)^2 \quad \text{and} \quad \theta(x, y, t) = u_1(x, y, t) + u_2(x, y, t).$$

In order to see the linear trend of the stream-function and temperature errors (6.1), predicted by Theorem 5.1, we refine simultaneously in space and time. More precisely, for each mesh family we consider the mesh refinements with  $h = 1/4, 1/8, 1/16, 1/32$ , and we use the same uniform refinements for the time variable. In particular, for the mesh  $\Omega_h^1$ , it can be seen along the diagonal of Table 1, the expected first order convergence for the stream-function and temperature errors (6.1).

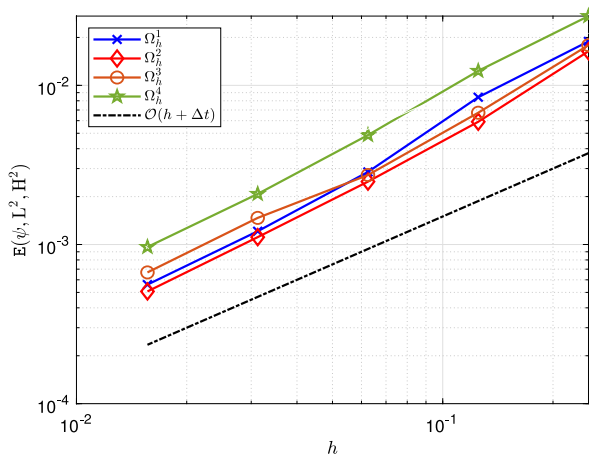
In Fig. 2, we display the errors (6.1) for the same simultaneous time and space refinements ( $h = \Delta t = 2^{-i}$ , with  $i = 2, \dots, 5$ ), using the four mesh families. We notice that the rates of convergence predicted in Theorem 5.1 are attained by both unknowns.

In order to study the trend of the stream-function and temperature errors (6.2), we show in Table 2 the results considering again the mesh  $\Omega_h^1$ , with  $h = \Delta t = 2^{-i}$ , with  $i = 2, \dots, 5$ . In particular, we can observe that the rate of convergence in the mesh size  $h$  seems higher than one; this is not fully surprising, since standard interpolation estimates (in space) for the norms in (6.2) indicate that, potentially, the discrete space could approximate the exact

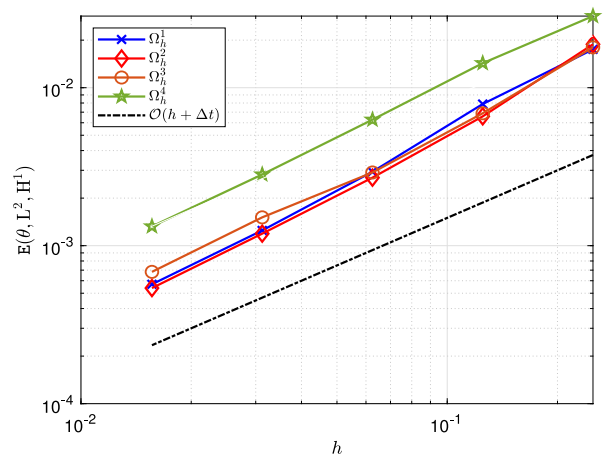
**Table 1**

Accuracy assessment. Errors (6.1) using the VEM (4.1), with polynomial degrees  $(k, \ell) = (2, 1)$ , physical parameters  $\nu = \kappa = 1$  and the mesh family  $\Omega_h^1$ .

$E(\psi, L^2; H^2)$						
dofs	$h$	$\Delta t$				
		1/4	1/8	1/16	1/32	1/64
36	1/4	1.88912e-2	1.42183e-2	1.16131e-2	1.02912e-2	9.63665e-3
196	1/8	1.11333e-2	8.42107e-3	6.91546e-3	6.15400e-3	5.77765e-3
900	1/16	4.92223e-3	3.53363e-3	2.85747e-3	2.54826e-3	2.40427e-3
3844	1/32	3.61175e-3	2.11884e-3	1.46063e-3	1.21158e-3	1.11670e-3
15 876	1/64	3.21002e-3	1.64565e-3	9.22443e-4	6.49802e-4	5.59824e-4
$E(\theta, L^2; H^1)$						
dofs	$h$	$\Delta t$				
		1/4	1/8	1/16	1/32	1/64
36	1/4	1.74892e-2	1.34200e-2	1.11391e-2	9.96756e-3	9.38232e-3
196	1/8	1.02277e-2	7.88174e-3	6.66404e-3	6.05736e-3	5.75702e-3
900	1/16	5.32067e-3	3.65373e-3	2.93777e-3	2.64415e-3	2.51594e-3
3844	1/32	3.80377e-3	2.18463e-3	1.49484e-3	1.24874e-3	1.16084e-3
15 876	1/64	3.37157e-3	1.71644e-3	9.52229e-4	6.64250e-4	5.69713e-4



(a) Stream-function errors



(b) Temperature errors

**Fig. 2.** Accuracy assessment. Errors (6.1) for simultaneous space and time refinements, using the VEM (4.1) with polynomial degrees  $(k, \ell) = (2, 1)$ , physical parameters  $\nu = \kappa = 1$  and the mesh families  $\Omega_h^i, i = 1, \dots, 4$ .

solution with order  $\mathcal{O}(h^2)$ . In order to better investigate this aspect, in Fig. 3 we display the errors (6.2) for space and time refinements given by  $h = 2^{-i}$  and  $\Delta t = 4^{-i}$ , with  $i = 2, \dots, 5$ , respectively, using the four mesh families. We notice that the rates of convergence seem indeed quadratic with respect to  $h$ .

6.2. Performance of the VEM for small viscosity

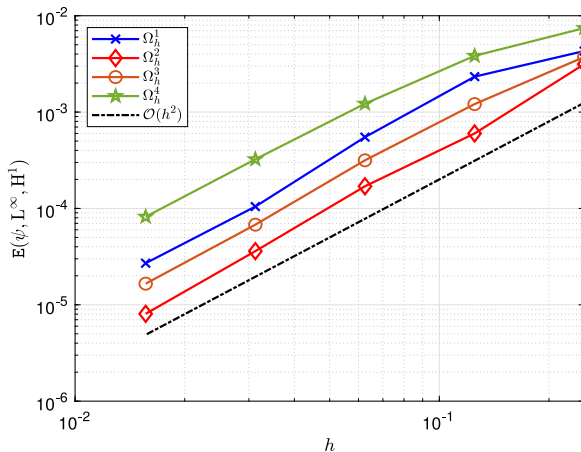
In this test we consider the square domain  $\Omega := (0, 1)^2$ , the time interval  $[0, 1]$  and force per unit mass  $\mathbf{g} = (0, -1)^T$ . We solve the Boussinesq system (2.1), taking the load terms  $\mathbf{f}_\psi$  and  $f_\theta$ , boundary and initial conditions in such a way that the analytical solution is given by:

$$\mathbf{u}(x, y, t) = \begin{pmatrix} u_1(x, y, t) \\ u_2(x, y, t) \end{pmatrix} = \begin{pmatrix} -\cos(t) \sin(\pi x) \sin(\pi y) \\ -\cos(t) \cos(\pi x) \cos(\pi y) \end{pmatrix},$$

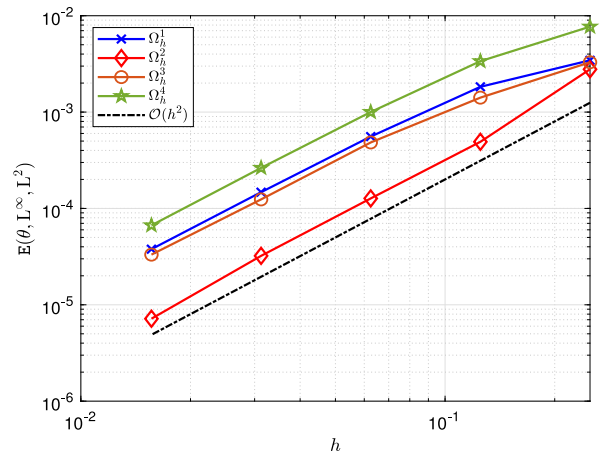
**Table 2**

Accuracy assessment. Errors (6.2) using the VEM (4.1), with polynomial degrees  $(k, \ell) = (2, 1)$ , the physical parameters  $\nu = \kappa = 1$  and the mesh family  $\Omega_h^1$ .

$E(\psi, L^\infty; H^1)$						
dofs	$h$	$\Delta t$				
		1/4	1/8	1/16	1/32	1/64
36	1/4	4.30301e-3	4.50090e-3	4.65255e-3	4.74590e-3	4.79749e-3
196	1/8	2.03865e-3	2.20110e-3	2.33234e-3	2.41662e-3	2.46443e-3
900	1/16	2.38767e-4	2.11074e-4	3.61809e-4	4.80109e-4	5.49619e-4
3844	1/32	7.26027e-4	4.35284e-4	2.05347e-4	6.71747e-5	4.99331e-5
15876	1/64	8.16241e-4	5.20174e-4	2.84604e-4	1.34953e-4	5.10645e-5
$E(\theta, L^\infty; L^2)$						
36	1/4	3.44760e-3	3.94792e-3	4.28939e-3	4.48462e-3	4.58811e-3
196	1/8	9.85211e-4	1.44875e-3	1.82900e-3	2.06308e-3	2.19159e-3
900	1/16	5.96219e-4	2.98014e-4	3.26274e-4	4.64998e-4	5.57065e-4
3844	1/32	8.26668e-4	4.90632e-4	2.31786e-4	9.52686e-5	9.44396e-5
15876	1/64	8.90387e-4	5.68492e-4	3.13988e-4	1.53393e-4	6.48063e-5



(a) Stream-function errors



(b) Temperature errors

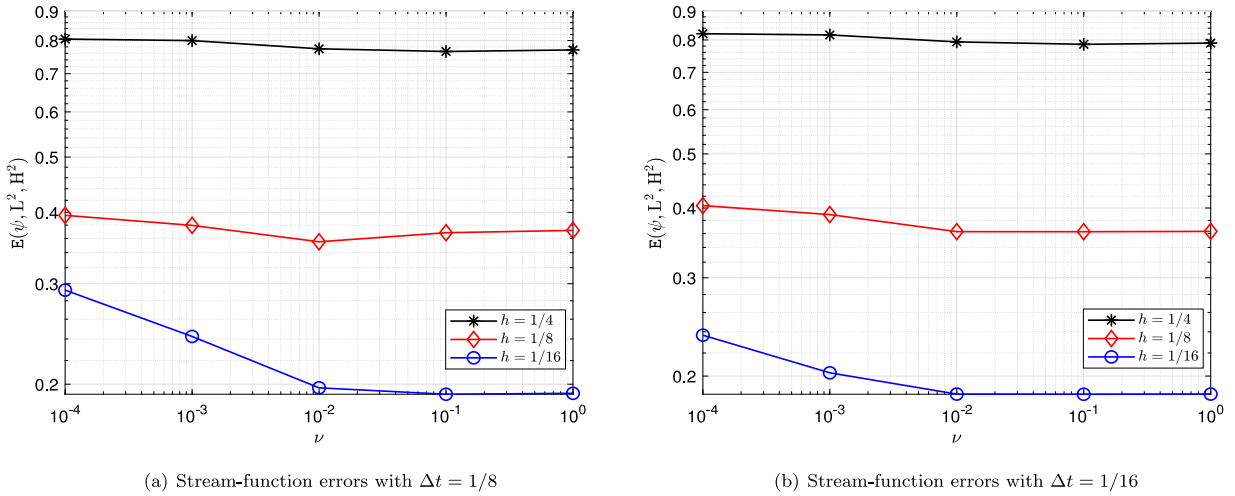
**Fig. 3.** Accuracy assessment. Errors (6.2), using the VEM (4.1) with polynomial degrees  $(k, \ell) = (2, 1)$ , the physical parameters  $\nu = \kappa = 1$  and the mesh families  $\Omega_h^i, i = 1, \dots, 4$ .

$$p(x, y, t) = \cos(t)(\sin(\pi x) + \cos(\pi y) - 2/\pi),$$

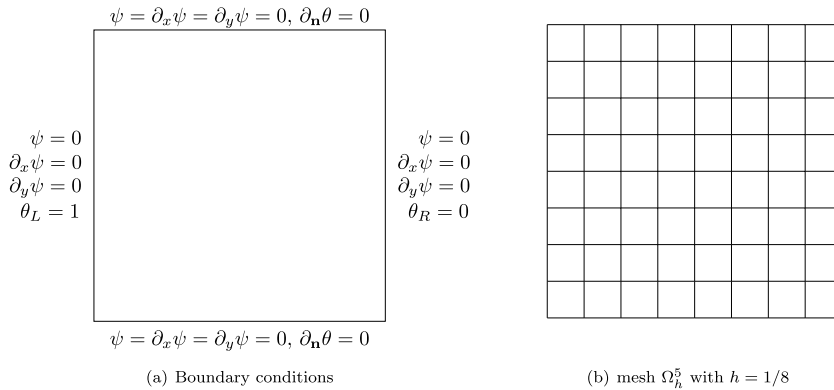
$$\psi(x, y, t) = \frac{1}{\pi} \cos(t) \sin(\pi x) \cos(\pi y) \quad \text{and} \quad \theta(x, y, t) = u_1(x, y, t) + u_2(x, y, t).$$

The purpose of this experiment is to investigate the performance of the VEM (4.1) for small viscosity parameters. In Fig. 4, we post the errors (6.1) of the stream-function variable obtained with the mesh sizes  $h = 1/4, 1/8, 1/16$  of  $\Omega_h^2$ , considering different values of  $\nu$  and fixing the time step  $\Delta t$  as  $1/8$  and  $1/16$  (see Figs. 4(a) and 4(b), respectively). It can be observed that the solutions of our VEM are accurate even for small values of  $\nu$ . Larger stream-function errors appear for very small viscosity values.

We observe that this results are in accordance with the general observation that exactly divergence-free Galerkin methods are more robust with respect to small diffusion parameters, see for instance [57] (and also [30] in the VEM context). On the other hand, note that the scheme proposed here has no explicit stabilization of the convection term



**Fig. 4.** Small viscosity test. Errors (6.1) of the VEM (4.1), for different values of  $\nu$  and  $\kappa = 1$ , using the meshes  $\Omega_h^2$ , polynomial degrees  $(k, \ell) = (2, 1)$ .



**Fig. 5.** Natural convection cavity. Boundary conditions and domain discretized with mesh  $\Omega_h^5$ .

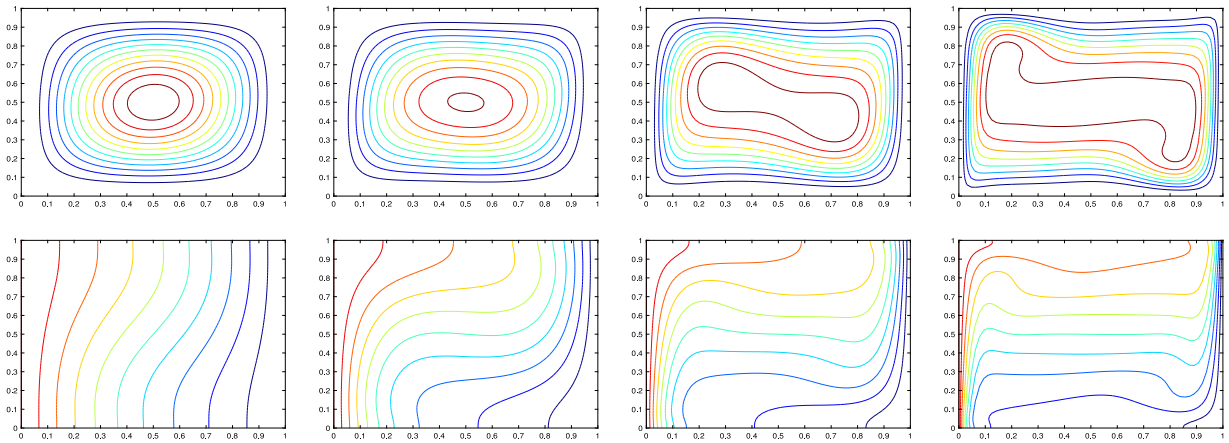
since this is not the focus of the present work (for instance, the natural norm associated to the stability of the discrete problem does not guarantee a robust control on the convection).

### 6.3. Natural convection in a cavity with the left wall heating

In this last example we consider the 2D natural convection benchmark problem, describing the behaviour of a incompressible flow in a squared cavity, which is heated at the left wall (see [6,18,58–60]). In particular, we consider the unitary square domain  $\Omega = (0, 1)^2$ . The boundary conditions are given as follows: the temperature in the left and right walls are  $\theta_L = 1$  and  $\theta_R = 0$ , respectively, while in the horizontal walls is  $\partial_n\theta = 0$  (i.e., insulated, there is no heat transfer through these walls), no-slip boundary conditions are imposed for the fluid flow at all walls. In terms of the stream-function these conditions are given by:  $\psi = \partial_x\psi = \partial_y\psi = 0$  on  $\Gamma \times (0, T)$ , as shown in Fig. 5. The initial conditions are chosen as  $\psi_0 = -x + y$  and  $\theta_0 = 1$  (so that the initial data does not satisfy the boundary conditions).

We consider the forces  $\mathbf{f}_\psi = \mathbf{0}$ ,  $f_\theta = 0$  and  $\mathbf{g} = \text{PrRa}(0, 1)^T$ , where Pr and Ra denote the Prandtl and Rayleigh numbers, respectively. For the numerical experiment, we set the physical parameters as:  $\nu = \text{Pr} = 0.71$ ,  $\text{Ra} \in [10^3, 10^6]$  and  $\kappa = 1$ .

In order to compare our results with the existing bibliography, we decompose the domain  $\Omega$  using mesh  $\Omega_h^5$  conformed by uniform squares (see Fig. 5(b)). Moreover, the time step is  $\Delta t = 10^{-3}$  and final time  $T = 1$ .



**Fig. 6.** Natural convection cavity: streamlines (top panels) and isotherms (bottom panels), for  $Ra = 10^3, 10^4, 10^5$  and  $10^6$ , respectively (from left to right), using the mesh  $\Omega_h^5$  ( $h = 1/64$ ).

**Table 3**

Natural convection cavity. Comparison of maximum vertical velocity  $u_{1h} := \Pi_h^1 \partial_y \psi$  at  $y = 0.5$  with the VEM (4.1) and mesh  $\Omega_h^5$  ( $h = 1/64$ ).

Ra	VEM	Ref. [6]	Ref. [18]	Ref. [58]	Ref. [59]	Ref. [60]
$10^4$	19.56(64)	19.63(64)	19.51(41)	19.63(71)	19.90(71)	19.79(101)
$10^5$	68.46(64)	68.48(64)	68.22(81)	68.85(71)	70.00(71)	70.63(101)
$10^6$	216.37(64)	220.46(64)	216.75(81)	221.6(71)	228.0(71)	227.11(101)

**Table 4**

Natural convection cavity. Comparison of maximum horizontal velocity  $u_{2h} := -\Pi_h^1 \partial_x \psi$  at  $x = 0.5$  with the VEM (4.1) and mesh  $\Omega_h^5$  ( $h = 1/64$ ).

Ra	VEM	Ref. [6]	Ref. [18]	Ref. [59]	Ref. [60]
$10^4$	16.15(64)	16.19(64)	16.18(41)	16.10(71)	16.10(101)
$10^5$	34.80(64)	34.74(64)	34.81(81)	34.0(71)	34.00(101)
$10^6$	65.91(64)	64.81(64)	65.33(81)	65.40(71)	65.40(101)

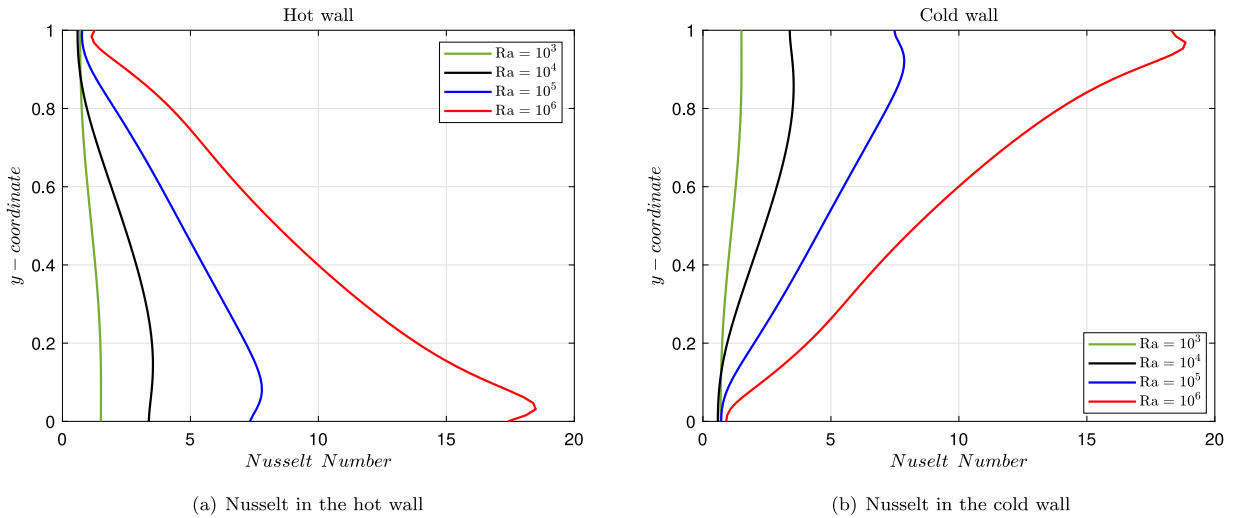
Streamlines and isotherms of the discrete solution obtained with our VEM (4.1) are posted in Fig. 6, using  $Ra = 10^3, 10^4, 10^5, 10^6$  and mesh size  $h = 1/64$ . The results show well agreement with the results presented in the benchmark solutions in [6,18,58–60].

Tables 3 and 4 present a quantitative comparison between our results and those obtained by the benchmark solutions in the above papers. Table 3 shows the maximum vertical velocity at  $y = 0.5$ , for  $Ra = 10^4, 10^5$  and  $10^6$ , while Table 4 shows the maximum horizontal velocity at  $x = 0.5$ , using the same values of the Rayleigh number. Here the numbers in the parenthesis denotes the numbers of elements along each edge of the domain, and is therefore an indication on the mesh finesse. We can observe that the results show good agreement, even for higher Rayleigh numbers.

Finally, for the natural convection problem we investigate the heat transfer coefficient along the vertical walls of the cavity in terms of the local Nusselt number ( $Nu_{local}$ ), which is defined by:  $Nu_{local}(x, y) := -\partial_n \theta(x, y)$ . Fig. 7 describes the variation of local Nusselt number at hot wall and cold wall, for different values of the Rayleigh number. It can be seen that the results show good agreement with the results presented in [6,18,58–60].

## 7. Conclusions

In this work we have designed and analyzed a high order fully-discrete virtual element for the nonstationary Boussinesq system in terms of the stream-function and temperature fields. We combined the  $C^1$ - and  $C^0$ -conforming virtual element approaches with a backward Euler scheme and proposed a fully-coupled formulation which is



**Fig. 7.** Natural convection cavity. Nusselt number along the hot wall (left) and the cold wall (right) for varying Rayleigh numbers, using the VEM (4.1) and mesh  $\Omega_h^5$ , with  $h = 1/64$ .

implicit in the nonlinear terms. By using fixed-point arguments we proved the existence of discrete solutions and, under a small time step condition, we have shown uniqueness of such solutions. The ensuing numerical method is unconditionally stable. Error estimates in  $L^2(H^2) \cap L^\infty(H^1)$  and  $L^2(H^1) \cap L^\infty(L^2)$  are provided for the stream-function and temperature, respectively. A set of benchmark numerical experiments have been reported, illustrating the good performance of the method and the theoretical rates of convergence. We observed that the present stream-function–temperature approach provides an attractive and competitive alternative to solve the two dimensional nonstationary Boussinesq problem; there are only two scalar unknowns, and the incompressibility constraint is automatically satisfied. Thus, the present approach leads to a smaller system compared with the classical velocity–pressure–temperature form. From Test 6.2, we observed numerically that our VE scheme presents certain robustness with respect to small diffusion parameters; this behaviour can be attributed to the fact that the incompressibility condition is satisfied automatically, a scenario in which the partial decoupling of the velocity and pressure errors leads a positive effect on the velocity computation. Furthermore, we observed that the resulting trilinear forms (continuous and discrete) in the momentum equation are naturally skew-symmetric, allowing more direct stability and convergence arguments (cf. Sections 4 and 5). The advantages described above come at the price of a scheme without velocity and pressure fields (which need to be recovered), possibly a larger condition number due to the higher order derivatives involved, and a more complex extension to the three dimensional case. Further developments of this work could be to derive error estimates, possibly with the addition of stabilizing terms, that are quasi-robust in the viscosity parameter. Another challenging aspect would be to tackle the three dimensional case. Simpler, but yet interesting, extensions could be to consider temperature-dependent coefficients [5] or include a pressure recovery technique [29].

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

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