Numerical Analysis of a Locking-Free Mixed Finite Element Method for a Bending Moment Formulation of Reissner-Mindlin Plate Model

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Received 26 April 2011; accepted 12 October 2011
Published online 9 February 2012 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/num.21698

This article deals with the approximation of the bending of a clamped plate, modeled by Reissner-Mindlin equations. It is known that standard finite element methods applied to this model lead to wrong results when the thickness t is small. Here, we propose a mixed formulation based on the Hellinger-Reissner principle which is written in terms of the bending moments, the shear stress, the rotations and the transverse displacement. To prove that the resulting variational formulation is well posed, we use the Babuška-Brezzi theory with appropriate t-dependent norms. The problem is discretized by standard mixed finite elements without the need of any reduction operator. Error estimates are proved. These estimates have an optimal dependence on the mesh size h and a mild dependence on the plate thickness t. This allows us to conclude that the method is locking-free. The proposed method yields direct approximation of the bending moments and the shear stress. A local postprocessing leading to H^1-type approximations of transverse displacement and rotations is introduced. Moreover, we propose a hybridization procedure, which leads to solving a significantly smaller positive definite system. Finally, we report numerical experiments which allow us to assess the performance of the method. Numer Methods Partial Differential Eq 29: 40–63, 2013

Keywords: bending moment formulation; error analysis; locking-free finite elements; Reissner-Mindlin

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Contract grant sponsor: CONICYT-Chile (FONDECYT); contract grant number: 11100180
Contract grant sponsors: BASAL project CMM, Universidad de Chile, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción

I. INTRODUCTION

The Reissner-Mindlin theory is the most used model to approximate the deformation of a thin or moderately thick elastic plate. Nowadays, it is very well understood that, due to the so called locking phenomenon, standard finite element methods applied to the classical transverse displacement-rotations formulation of this model lead to wrong results when the thickness t is small with respect to the other dimensions of the plate. Nevertheless, adopting for instance a reduced integration or a mixed interpolation technique, this phenomenon can be avoided. Indeed, several families of methods have been rigorously shown to be free from locking and optimally convergent. We mention the recent monograph by Falk [1] for a thorough description of the state of the art and further references.

Among the existing techniques, a large success has been shared by the mixed interpolation of tensorial components (MITC) methods introduced by Bathe and Dvorkin in [2] or variants of them (for instance, [3]). Other methods are based on using a Helmholtz decomposition of the shear stress, as in [4], to write an equivalent formulation of the plate equations in terms of an uncoupled system of two Poisson equations and a rotated Stokes system and using adequate finite element methods for each of these problems. An alternative approach is proposed and analyzed in [5] by Amara et al., where a conforming finite element method for the Reissner-Mindlin model satisfying various boundary conditions is introduced. In their analysis the bending moment is written in terms of three auxiliary variables belonging to classical Sobolev spaces. A mixed formulation in terms of these new variables is discretized by standard finite elements. Under some regularity assumptions on the exact solution, optimal error estimates with constants independent of the plate thickness are proved in [5].

More recently, another approach has been presented by Behrens and Guzmán in [6]. In this case the plate bending problem is written as a system of first order equations and all the resulting variables are approximated. A discretization in terms of discontinuous polynomials and enriched Raviart-Thomas elements is proposed. A hybrid form of the method allows reducing the total number of variables. Error estimates with t-independent constants are proved. These estimates are quasi optimal in regularity, since they involve a norm of the shear stress which can not be a priori bounded independently of t.

In this article, we consider a bending moment formulation for the plate problem based on the Hellinger-Reissner principle. We introduce these moments (which in practice usually represent the quantities of interest in applications) as new unknowns, together with the shear stress, the rotations and the transverse displacement. We obtain a mixed variational formulation involving an elasticity-like system with weakly imposed symmetry. An advantage of this approach is that there are several well studied mixed finite element methods for the elasticity problem with weakly imposed symmetry (see for instance [7–11]). Using the Babuška-Brezzi theory, we show that the proposed variational formulation is well posed and stable in appropriate t-dependent norms. For the numerical approximation, classical Raviart-Thomas elements are used for the shear stress and piecewise constants for the transverse displacement, while for the elasticity-like problem with weakly imposed symmetry, we use PEERS finite elements [7, 8] for the bending moment and the rotations. We prove a uniform inf-sup condition with respect to the discretization parameter h and the thickness t, without the need of introducing any reduction operator. The convergence rate is proved to be optimal in terms of the mesh size h. These estimates are not fully independent of the plate thickness t. However, this dependence is very mild since it only involves a term $(\frac{h}{t})^{\epsilon}$ for arbitrarily small $\epsilon > 0$. Therefore, in practice, the method is locking-free and this is confirmed by our numerical experiments. We note that our method approximates directly the bending moments and the shear stress in classical L^2 norms, which is distinctive of this approach. In fact, standard methods based on a transverse displacement and rotation discretization only lead to approximations of the shear stress in weaker norms. In addition, we propose a local postprocessing procedure which gives piecewise linear rotations and transverse displacement that converge to the exact solution in a stronger H^1-type discrete norm. Moreover, a hybridization procedure is introduced to reduce the computer cost to that of solving an equivalent linear system that is smaller and positive definite. This process makes our approach computationally competitive with other methods.

The outline of this article is as follows: In Section II, we first recall the Reissner-Mindlin equations and some regularity results. Then, we prove the unique solvability and stability properties of the proposed formulation. In Section III, we present the finite element scheme, prove a stability result and obtain error estimates for the method. In addition, we introduce and analyze a local postprocessing procedure for transverse displacements and rotations, and the hybridization process to eliminate some variables leading to a linear system smaller and positive definite. In Section IV, we report a numerical test which allows us to assess the performance of the proposed method. We end the article with some concluding remarks.

Throughout the article we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote with c and C, with or without subscripts, tildes or hats, generic constants independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences. Moreover, we use the following notation for any tensor field $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,2}$, any vector field $\eta = (\eta_i)_{i=1,2}$ and any scalar field v:

$$\operatorname{div}\ \eta := \partial_1\eta_1 + \partial_2\eta_2, \quad \operatorname{rot}\eta := \partial_1\eta_2 - \partial_2\eta_1, \quad \nabla v := \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix}, \quad \operatorname{curl}\ v := \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix},$$

$$\mathbf{div}\ \boldsymbol{\tau} := \begin{pmatrix} \partial_1\tau_{11} + \partial_2\tau_{12} \\ \partial_1\tau_{21} + \partial_2\tau_{22} \end{pmatrix}, \quad \mathsf{Curl}\ \eta := \begin{pmatrix} \partial_2\eta_1 & -\partial_1\eta_1 \\ \partial_2\eta_2 & -\partial_1\eta_2 \end{pmatrix}, \quad \nabla\eta := \begin{pmatrix} \partial_1\eta_1 & \partial_2\eta_1 \\ \partial_1\eta_2 & \partial_2\eta_2 \end{pmatrix},$$

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji}), \qquad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2}\tau_{ii}, \qquad \boldsymbol{\tau}^a : \boldsymbol{\tau}^b := \sum_{i,j=1}^{2}\tau_{ij}^a\tau_{ij}^b.$$

Finally, we denote

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

II. THE PLATE MODEL

Consider an elastic plate of thickness t, $0 < t \leq 1$, with reference configuration $\Omega \times (-\frac{t}{2}, \frac{t}{2})$, where Ω is a convex polygonal domain of $\mathbb{R}^2$ occupied by the mid-section of the plate. The deformation of the plate is described by means of the Reissner-Mindlin model in terms of the rotations $\beta = (\beta_1, \beta_2)$ of the fibers initially normal to the plate mid-surface, the scaled shear stress $\gamma = (\gamma_1, \gamma_2)$, and the transverse displacement w. Assuming that the plate is clamped on its whole boundary $\partial\Omega$, the following equations describe the plate response to a conveniently scaled transverse load $g \in L^2(\Omega)$:

$$-\mathbf{div}\,(\mathcal{C}(\boldsymbol{\varepsilon}(\beta))) - \gamma = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$-\operatorname{div}\ \gamma = g \quad \text{in } \Omega, \tag{2.2}$$

$$\gamma = \frac{\kappa}{t^2}(\nabla w - \beta) \quad \text{in } \Omega, \tag{2.3}$$

$$w = 0, \ \beta = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where $\kappa := Ek/2(1+\nu)$ is the shear modulus, with E being the Young modulus, ν the Poisson ratio, and k a correction factor usually taken as 5/6 for clamped plates, $\boldsymbol{\varepsilon}(\beta) := \frac{1}{2}(\nabla\beta + (\nabla\beta)^{\mathrm{t}})$ is the standard strain tensor, and $\mathcal{C}$ is the tensor of bending moduli, given by (for isotropic materials)

$$\mathcal{C}\boldsymbol{\tau} := \frac{E}{12(1-\nu^2)}[(1-\nu)\boldsymbol{\tau} + \nu\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}], \qquad \boldsymbol{\tau} \in L^2(\Omega)^{2\times 2}.$$

The tensor $\mathcal{C}$ is invertible with its inverse given by

$$\mathcal{C}^{-1}\boldsymbol{\tau} := \frac{12}{E}[(1+\nu)\boldsymbol{\tau} - \nu\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}], \qquad \boldsymbol{\tau} \in L^2(\Omega)^{2\times 2}.$$

To write a variational formulation of the Reissner-Mindlin plate problem, we introduce as a new unknown the bending moment $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})_{i,j=1,2}$ defined by

$$\boldsymbol{\sigma} := \mathcal{C}(\boldsymbol{\varepsilon}(\beta)).$$

We rewrite the equation above as follows:

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \nabla\beta + \left(\frac{1}{2}\operatorname{rot}\beta\right)\mathbf{J}.$$

Then, introducing the auxiliary unknown $r := -\frac{1}{2}\operatorname{rot}\beta$, multiplying by a test function $\boldsymbol{\tau}$ and integrating by parts, we obtain

$$\int_\Omega (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\tau} + \int_\Omega \beta \cdot \mathbf{div}\,\boldsymbol{\tau} + \int_\Omega r(\tau_{12} - \tau_{21}) = 0. \tag{2.5}$$

Now, by testing (2.1)–(2.3) with adequate functions, integrating by parts, using (2.5) and (2.4), and imposing weakly the symmetry of $\boldsymbol{\sigma}$, we obtain the following mixed variational formulation:

Find $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ *such that*

$$\int_\Omega (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\tau} + \frac{t^2}{\kappa}\int_\Omega \gamma\cdot\xi + \int_\Omega \beta\cdot(\mathbf{div}\,\boldsymbol{\tau} + \xi) + \int_\Omega r(\tau_{12}-\tau_{21}) + \int_\Omega w\operatorname{div}\xi = 0,$$

$$\int_\Omega \eta\cdot(\mathbf{div}\,\boldsymbol{\sigma} + \gamma) + \int_\Omega s(\sigma_{12}-\sigma_{21}) + \int_\Omega v\operatorname{div}\gamma = -\int_\Omega gv,$$

for all $((\boldsymbol{\tau},\xi),(\eta,s,v)) \in \mathbf{H}\times\mathbf{Q}$.

The spaces above are defined as follows:

$$\mathbf{H} := H(\mathbf{div};\Omega)\times H(\operatorname{div};\Omega),$$

$$\mathbf{Q} := L^2(\Omega)^2 \times L^2(\Omega)\times L^2(\Omega),$$

with

$$H(\mathbf{div};\Omega) := \{\boldsymbol{\tau}\in L^2(\Omega)^{2\times 2} : \ \mathbf{div}\,\boldsymbol{\tau}\in L^2(\Omega)^2\},$$

and

$$H(\mathrm{div}\,;\Omega) := \{\xi \in L^2(\Omega)^2 : \ \mathrm{div}\,\xi \in L^2(\Omega)\}.$$

We endow $\mathbf{H}$ with the following t-dependent norm:

$$\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau} + \xi\|_{0,\Omega} + t\|\xi\|_{0,\Omega} + \|\mathrm{div}\,\xi\|_{0,\Omega},$$

while for the space $\mathbf{Q}$ we use

$$\|(\eta, s, v)\|_{\mathbf{Q}} := \|\eta\|_{0,\Omega} + \|s\|_{0,\Omega} + \|v\|_{0,\Omega}.$$

Finally, we endow $\mathbf{H} \times \mathbf{Q}$ with the corresponding product norm.

We rewrite this variational problem as follows:

Find $((\boldsymbol{\sigma},\gamma),(\beta, r, w)) \in \mathbf{H} \times \mathbf{Q}$ *such that*

$$a((\boldsymbol{\sigma},\gamma),(\boldsymbol{\tau},\xi)) + b((\boldsymbol{\tau},\xi),(\beta, r, w)) = 0 \qquad \forall(\boldsymbol{\tau},\xi) \in \mathbf{H}, \tag{2.6}$$

$$b((\boldsymbol{\sigma},\gamma),(\eta, s, v)) = F(\eta, s, v) \qquad \forall(\eta, s, v) \in \mathbf{Q}, \tag{2.7}$$

where the bilinear forms $a : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and $b : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$ and the linear functional $F : \mathbf{Q} \to \mathbb{R}$ are defined by

$$\begin{aligned} a((\boldsymbol{\sigma},\gamma),(\boldsymbol{\tau},\xi)) &:= \int_\Omega (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\tau} + \frac{t^2}{\kappa}\int_\Omega \gamma \cdot \xi \\ &= \frac{12}{E}[(1+\nu)\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} - \nu \int_\Omega \mathrm{tr}(\boldsymbol{\sigma})\mathrm{tr}(\boldsymbol{\tau})] + \frac{t^2}{\kappa}\int_\Omega \gamma \cdot \xi, \end{aligned} \tag{2.8}$$

$$b((\boldsymbol{\tau},\xi),(\eta, s, v)) := \int_\Omega \eta \cdot (\mathbf{div}\,\boldsymbol{\tau} + \xi) + \int_\Omega s(\tau_{12} - \tau_{21}) + \int_\Omega v\,\mathrm{div}\,\xi, \tag{2.9}$$

and

$$F(\eta, s, v) := -\int_\Omega g v,$$

for all $(\boldsymbol{\sigma},\gamma), (\boldsymbol{\tau},\xi) \in \mathbf{H}$ and $(\eta, s, v) \in \mathbf{Q}$.

Next, we will prove that problem (2.6)–(2.7) satisfies the hypotheses of the Babuška-Brezzi theory, which yields the unique solvability and continuous dependence on the data of this variational formulation.

We first observe that the bilinear forms a and b and the linear functional F are bounded with constants independent of the plate thickness t.

Let

$$\mathbf{V} := \{(\boldsymbol{\tau},\xi) \in \mathbf{H} : \ b((\boldsymbol{\tau},\xi),(\eta, s, v)) = 0 \ \forall(\eta, s, v) \in \mathbf{Q}\}$$

be the so-called continuous kernel; hence (cf. (2.9))

$$\mathbf{V} = \{(\boldsymbol{\tau},\xi) \in \mathbf{H} : \ \xi + \mathbf{div}\,\boldsymbol{\tau} = 0, \ \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}} \text{ and } \mathrm{div}\,\xi = 0 \text{ in } \Omega\}.$$

The following lemma shows that the bilinear form a is $\mathbf{V}$-elliptic uniformly in t.

Lemma 2.1. *There exists $\alpha > 0$, independent of t, such that*

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)) \geq \alpha \|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}^2 \qquad \forall(\boldsymbol{\tau},\xi) \in \mathbf{V}.$$

Proof. Given $(\boldsymbol{\tau},\xi) \in \mathbf{V}$, using that $\operatorname{tr}(\boldsymbol{\tau})^2 \leq 2(\boldsymbol{\tau} : \boldsymbol{\tau})$, from (2.8) we obtain

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)) \geq \frac{12(1-\nu)}{E}\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{t^2}{\kappa}\|\xi\|_{0,\Omega}^2.$$

Thus, since $\|\mathbf{div}\,\boldsymbol{\tau} + \xi\|_{0,\Omega} = 0$ and $\|\operatorname{div}\xi\|_{0,\Omega} = 0$, we have that

$$a((\boldsymbol{\tau},\xi),(\boldsymbol{\tau},\xi)) \geq \alpha \|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}^2,$$

with $\alpha := \min\{6(1-\nu)/E, 1/2\kappa\}$. Therefore, we end the proof. ∎

To obtain the corresponding inf-sup condition, we first prove the following lemma.

Lemma 2.2. *There exists $c > 0$, independent of t, such that, $\forall s \in L^2(\Omega)$, there exists $\boldsymbol{\tau}^s \in H(\mathbf{div}\,;\Omega)$ satisfying $(\tau_{12}^s - \tau_{21}^s) = s$, $\mathbf{div}\,\boldsymbol{\tau}^s = 0$ in Ω, and $\|\boldsymbol{\tau}^s\|_{H(\mathbf{div}\,;\Omega)} \leq c\|s\|_{0,\Omega}$.*

Proof. For $s \in L^2(\Omega)$, let

$$\bar{s} := \frac{1}{|\Omega|}\int_\Omega s,$$

and $\lambda := s - \bar{s}$. We have that $\lambda \in L_0^2(\Omega) := \{u \in L^2(\Omega) : \int_\Omega u = 0\}$ and, clearly, $\|\lambda\|_{0,\Omega} \leq \|s\|_{0,\Omega}$. Then, there exists $v = (v_1, v_2) \in H_0^1(\Omega)^2$ such that $\operatorname{div} v = \lambda$ in Ω and $\|v\|_{1,\Omega} \leq \hat{c}\|\lambda\|_{0,\Omega}$ (cf. [12]). Now, we consider the following function

$$\varphi := v + \frac{\bar{s}}{2}\begin{pmatrix} x \\ y \end{pmatrix},$$

which satisfies $\operatorname{div}\varphi = s$ and $\|\varphi\|_{1,\Omega} \leq \|v\|_{1,\Omega} + \tilde{c}\|s\|_{0,\Omega}$. Next, we define

$$\boldsymbol{\tau}^s := -\operatorname{\mathsf{Curl}}\varphi = -\begin{pmatrix} \partial_2 v_1 & -\partial_1 v_1 - \frac{1}{2}\bar{s} \\ \partial_2 v_2 + \frac{1}{2}\bar{s} & -\partial_1 v_2 \end{pmatrix} \in L^2(\Omega)^{2\times 2}.$$

From this, we have that $\mathbf{div}\,\boldsymbol{\tau}^s = 0$, so that $\boldsymbol{\tau}^s \in H(\mathbf{div}\,;\Omega)$. Moreover,

$$\left(\tau_{12}^s - \tau_{21}^s\right) = \operatorname{div} v + \bar{s} = \lambda + \bar{s} = s$$

and it is easy to check that

$$\|\boldsymbol{\tau}^s\|_{H(\mathbf{div}\,;\Omega)} \leq c\|s\|_{0,\Omega}.$$

Thus, we end the proof. ∎

Now, we are in a position to prove an inf-sup condition for the bilinear form b.

Lemma 2.3. *There exists $C > 0$, independent of t, such that*

$$\sup_{\substack{(\boldsymbol{\tau},\xi)\in\mathbf{H}\\(\boldsymbol{\tau},\xi)\neq 0}} \frac{b((\boldsymbol{\tau},\xi),(\eta,s,v))}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} \geq C\|(\eta,s,v)\|_{\mathbf{Q}} \qquad \forall(\eta,s,v)\in\mathbf{Q}.$$

Proof. Let $(\eta,s,v)\in\mathbf{Q}$ and $\boldsymbol{\tau}^s$ as in Lemma 2.2. Then,

$$\sup_{\substack{(\boldsymbol{\tau},\xi)\in\mathbf{H}\\(\boldsymbol{\tau},\xi)\neq 0}} \frac{b((\boldsymbol{\tau},\xi),(\eta,s,v))}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} \geq \frac{b((\boldsymbol{\tau}^s,0),(\eta,s,v))|}{\|\boldsymbol{\tau}^s\|_{0,\Omega}+\|\mathbf{div}\,\boldsymbol{\tau}^s\|_{0,\Omega}} = \frac{\|s\|_{0,\Omega}^2}{\|\boldsymbol{\tau}^s\|_{0,\Omega}} \geq \frac{1}{c}\|s\|_{0,\Omega}. \tag{2.10}$$

Now, let $\widetilde{\boldsymbol{\tau}} := -\boldsymbol{\varepsilon}(z)$, where $z\in H_0^1(\Omega)^2$ is the unique solution of the following auxiliary problem:

$$-\mathbf{div}\,\boldsymbol{\varepsilon}(z)=\eta \quad \text{in } \Omega,$$
$$z=0 \quad \text{on } \partial\Omega.$$

Notice that this problem is well posed (as a consequence of Korn's inequality) and $\|\boldsymbol{\varepsilon}(z)\|_{0,\Omega}\leq \widetilde{C}\|\eta\|_{0,\Omega}$. Hence $\widetilde{\boldsymbol{\tau}}\in H(\mathbf{div}\,;\Omega)$, $\widetilde{\boldsymbol{\tau}}=\widetilde{\boldsymbol{\tau}}^{\mathrm{t}}$ and

$$\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}+\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}\leq(\widetilde{C}+1)\|\eta\|_{0,\Omega}.$$

Therefore,

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\tau},\xi)\in\mathbf{H}\\(\boldsymbol{\tau},\xi)\neq 0}} \frac{b((\boldsymbol{\tau},\xi),(\eta,s,v))}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} &\geq \frac{b((\widetilde{\boldsymbol{\tau}},0),(\eta,s,v))}{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}+\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}} \\ &= \frac{\|\eta\|_{0,\Omega}^2}{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}+\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}} \geq \frac{1}{\widetilde{C}+1}\|\eta\|_{0,\Omega}.\end{aligned} \tag{2.11}$$

Finally, let $\widetilde{\xi} := -\nabla\tilde{z}$, where $\tilde{z}\in H_0^1(\Omega)$ is the unique solution of the auxiliary problem:

$$-\Delta\tilde{z}=v \quad \text{in } \Omega,$$
$$\tilde{z}=0 \quad \text{on } \partial\Omega.$$

The same arguments as above allow us to prove that there exists $\hat{c}>0$, depending only on Ω, such that

$$\|\widetilde{\xi}\|_{0,\Omega}+\|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega}\leq\hat{c}\|v\|_{0,\Omega}.$$

Hence, it follows that

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\tau},\xi)\in\mathbf{H}\\(\boldsymbol{\tau},\xi)\neq 0}} \frac{b((\boldsymbol{\tau},\xi),(\eta,s,v))}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} &\geq \frac{b((\mathbf{0},\widetilde{\xi}),(\eta,s,v))}{(1+t)\|\widetilde{\xi}\|_{0,\Omega}+\|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega}} \\ &\geq \frac{1}{2(\|\widetilde{\xi}\|_{0,\Omega}+\|\mathrm{div}\,\widetilde{\xi}\|_{0,\Omega})}\left(\int_\Omega \eta\cdot\widetilde{\xi}+\|v\|_{0,\Omega}^2\right) \geq \frac{1}{2\hat{c}}\|v\|_{0,\Omega}-\frac{1}{2}\|\eta\|_{0,\Omega}.\end{aligned}$$

From this inequality and (2.11), it is immediate to show that

$$\sup_{\substack{(\boldsymbol{\tau},\xi)\in\mathbf{H}\\(\boldsymbol{\tau},\xi)\neq 0}} \frac{b((\boldsymbol{\tau},\xi),(\eta,s,v))}{\|(\boldsymbol{\tau},\xi)\|_{\mathbf{H}}} \geq \frac{1}{\hat{c}(\widetilde{C}+1)}\|v\|_{0,\Omega}.$$

Thus, the proof follows from this estimate, (2.10) and (2.11). ■

We are now in a position to state the main result of this section which yields the solvability of the continuous problem (2.6)–(2.7).

Theorem 2.4. *There exists a unique solution* $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ *to problem* (2.6)–(2.7) *and the following continuous dependence result holds:*

$$\|((\boldsymbol{\sigma},\gamma),(\beta,r,w))\|_{\mathbf{H}\times\mathbf{Q}} \leq C\|g\|_{0,\Omega},$$

where C *is independent of* t.

Proof. By virtue of Lemmas 2.1 and 2.3, the proof follows from a straightforward application of [13, Theorem II.1.1]. ■

Testing (2.6)–(2.7) with different functions, it is straightforward to show that $\beta \in H^1(\Omega)^2$, $w \in H^1(\Omega)$ and Eqs. (2.1)–(2.4) hold true. Therefore, we can apply the results from [4] to prove the following additional regularity result.

Proposition 2.5. *Suppose that* Ω *is a convex polygon or a smoothly bounded domain in the plane and* $g \in L^2(\Omega)$. *Let* $((\boldsymbol{\sigma},\gamma),(\beta,r,w))$ *be the solution to problem* (2.6)–(2.7). *Then, there exists a constant* C, *independent of* t *and* g, *such that*

$$\|w\|_{2,\Omega} + \|\beta\|_{2,\Omega} + \|\gamma\|_{H(\mathrm{div};\Omega)} + t\|\gamma\|_{1,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega} + t\|\mathbf{div}\,\boldsymbol{\sigma}\|_{1,\Omega} + \|r\|_{1,\Omega} \leq C\|g\|_{0,\Omega}.$$

III. THE FINITE ELEMENT SCHEME

Let $\mathcal{T}_h$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T with mesh size $h := \max\{h_T : T \in \mathcal{T}_h\}$. In addition, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_k(S)$ the space of polynomials in two variables defined in S of total degree at most k. For each $T \in \mathcal{T}_h$ we define the local Raviart-Thomas space of order zero

$$RT_0(T) := \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\}.$$

On the other hand, for each triangle $T \in \mathcal{T}_h$, we denote by b_T the unique polynomial in $\mathbb{P}_3(T)$ that vanishes on ∂T and is normalized by $\int_T b_T = 1$. This cubic bubble function is extended by zero to $\Omega \setminus T$ and therefore it becomes an element of $H_0^1(\Omega)$. We define

$$B(\mathcal{T}_h) := \{\boldsymbol{\tau}_h \in H(\mathbf{div},\Omega) : (\tau_{i1h},\tau_{i2h}) \in Z(T),\ i = 1,2,\ \forall T \in \mathcal{T}_h\},$$

where

$$Z(T) := \operatorname{span}\{\operatorname{curl}(b_T),\ T \in \mathcal{T}_h\}.$$

Next, we define the following finite element subspaces:

$$H_h^{\sigma} := X_h \oplus B(\mathcal{T}_h),$$

where

$$X_h := \{\boldsymbol{\tau}_h \in H(\mathbf{div}\,, \Omega) : \ \boldsymbol{\tau}_h|_T \in [RT_0(T)^{\mathrm{t}}]^2 \ \forall T \in \mathcal{T}_h\}$$

is the global lowest-order Raviart-Thomas space,

$$\begin{aligned}
H_h^{\gamma} &:= \{\xi_h \in H(\operatorname{div}\,, \Omega) : \ \xi_h|_T \in RT_0(T) \ \forall T \in \mathcal{T}_h\},\\
Q_h^{w} &:= \{v_h \in L^2(\Omega) : \ v_h|_T \in \mathbb{P}_0(T) \ \forall T \in \mathcal{T}_h\},\\
Q_h^{\beta} &:= \{\eta_h \in L^2(\Omega)^2 : \ \eta_h|_T \in \mathbb{P}_0(T)^2 \ \forall T \in \mathcal{T}_h\},\\
Q_h^{r} &:= \{s_h \in H^1(\Omega) : \ s_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h\}.
\end{aligned}$$

At this point we recall that $H_h^{\sigma} \times Q_h^{\beta} \times Q_h^{r}$ corresponds to the PEERS finite elements introduced by Arnold, Brezzi and Douglas in [7].

Defining $\mathbf{H}_h := H_h^{\sigma} \times H_h^{\gamma}$ and $\mathbf{Q}_h := Q_h^{\beta} \times Q_h^{r} \times Q_h^{w}$, our mixed finite element scheme associated with the continuous formulation (2.6)–(2.7) reads as follows:

Find $((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *such that*

$$a((\boldsymbol{\sigma}_h, \gamma_h), (\boldsymbol{\tau}_h, \xi_h)) + b((\boldsymbol{\tau}_h, \xi_h), (\beta_h, r_h, w_h)) = 0 \qquad \forall (\boldsymbol{\tau}_h, \xi_h) \in \mathbf{H}_h, \tag{3.1}$$

$$b((\boldsymbol{\sigma}_h, \gamma_h), (\eta_h, s_h, v_h)) = F(\eta_h, s_h, v_h) \qquad \forall (\eta_h, s_h, v_h) \in \mathbf{Q}_h. \tag{3.2}$$

A. Convergence

Our next goal is to prove the corresponding discrete versions of Lemmas 2.1 and 2.3 and to use them to conclude the unique solvability and stability of problem (3.1)–(3.2). With this aim, we denote by $\mathbf{V}_h$ the so-called discrete kernel: $\mathbf{V}_h := \{(\boldsymbol{\tau}_h, \xi_h) \in \mathbf{H}_h : \ b((\boldsymbol{\tau}_h, \xi_h), (\eta_h, s_h, v_h)) = 0 \ \forall (\eta_h, s_h, v_h) \in \mathbf{Q}_h\}$; namely

$$\mathbf{V}_h = \left\{ (\boldsymbol{\tau}_h, \xi_h) \in \mathbf{H}_h : \ \int_\Omega \eta_h \cdot (\mathbf{div}\, \boldsymbol{\tau}_h + \xi_h) + \int_\Omega s_h(\tau_{12h} - \tau_{21h}) \right.$$
$$\left. + \int_\Omega v_h \operatorname{div} \xi_h = 0 \ \forall (\eta_h, s_h, v_h) \in \mathbf{Q}_h \right\}.$$

Let $(\boldsymbol{\tau}_h, \xi_h) \in \mathbf{V}_h$. Taking $(0, 0, v_h) \in \mathbf{Q}_h$ and using that $\operatorname{div} \xi_h|_T$ is a constant, we conclude that $\operatorname{div} \xi_h = 0$ in Ω. On the other hand, since $\mathbf{div}\, \boldsymbol{\tau}_h = 0$ in $\Omega \ \forall \boldsymbol{\tau}_h \in B(\mathcal{T}_h)$, we have that $\mathbf{div}\, \boldsymbol{\tau}_h|_T$ is a constant vector. Moreover, since $\operatorname{div} \xi_h = 0$, we have that $\xi_h|_T$ is also a constant vector. Therefore, by taking $(\eta_h, 0, 0) \in \mathbf{Q}_h$, we conclude that $\mathbf{div}\, \boldsymbol{\tau}_h + \xi_h = 0$ in Ω. Thus, we obtain that

$$\mathbf{V}_h = \left\{ (\boldsymbol{\tau}_h, \xi_h) \in \mathbf{H}_h : \ \xi_h + \mathbf{div}\, \boldsymbol{\tau}_h = 0 \text{ in } \Omega, \ \operatorname{div} \xi_h = 0 \text{ in } \Omega \right.$$
$$\left. \text{and } \int_\Omega s_h(\tau_{12h} - \tau_{21h}) = 0 \ \forall s_h \in Q_h^{r} \right\}.$$

Note that the third condition above does not guarantee the symmetry of the tensors in H_h^σ, as it was the case for the continuous kernel $\mathbf{V}$. Hence, we have that $\mathbf{V}_h$ is not included in $\mathbf{V}$. However, the proof of Lemma 2.1 can be repeated (since we have not used that $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}}$ in this proof) to obtain the following result:

Lemma 3.1. *There exists $\alpha^\star > 0$, independent of h and t, such that*

$$a((\boldsymbol{\tau}_h, \xi_h), (\boldsymbol{\tau}_h, \xi_h)) \geq \alpha^\star \|(\boldsymbol{\tau}_h, \xi_h)\|_{\mathbf{H}}^2 \qquad \forall (\boldsymbol{\tau}_h, \xi_h) \in \mathbf{V}_h.$$

We introduce the Raviart-Thomas interpolation operator $\mathcal{R} : H^1(\Omega)^2 \to H_h^\gamma$. Let us review some properties of this operator that we will use in the sequel (see, for instance, [12, 13]):

- Let $\mathcal{P}$ be the orthogonal projection from $L^2(\Omega)$ onto the finite element subspace Q_h^w. Then, for all $\xi \in H^1(\Omega)^2$, we have that

$$\operatorname{div} \mathcal{R}\xi = \mathcal{P}(\operatorname{div} \xi). \tag{3.3}$$

- There exists $C > 0$, independent of h, such that

$$\|\xi - \mathcal{R}\xi\|_{0,\Omega} \leq Ch\|\xi\|_{1,\Omega} \qquad \forall \xi \in H^1(\Omega)^2. \tag{3.4}$$

Now, let $\widetilde{\mathcal{R}} : H^1(\Omega)^{2\times 2} \to X_h$ be the operator defined on each row of $H^1(\Omega)^{2\times 2}$ by means of the Raviart-Thomas interpolation operator $\mathcal{R}$. Since $X_h \subset H_h^\sigma$, the operator $\widetilde{\mathcal{R}}$ can be seen as acting from $H^1(\Omega)^{2\times 2}$ into H_h^σ. The above properties of $\mathcal{R}$ lead to similar ones for $\widetilde{\mathcal{R}}$:

- Let $\widetilde{\mathcal{P}}$ be the orthogonal projection from $L^2(\Omega)^2$ onto the finite element subspace Q_h^β. Then, for all $\boldsymbol{\tau} \in H^1(\Omega)^{2\times 2}$, we have that

$$\mathbf{div}\, \widetilde{\mathcal{R}}\boldsymbol{\tau} = \widetilde{\mathcal{P}}(\mathbf{div}\, \boldsymbol{\tau}). \tag{3.5}$$

- There exists $C > 0$, independent of h, such that

$$\|\boldsymbol{\tau} - \widetilde{\mathcal{R}}\boldsymbol{\tau}\|_{0,\Omega} \leq Ch\|\boldsymbol{\tau}\|_{1,\Omega} \qquad \forall \boldsymbol{\tau} \in H^1(\Omega)^{2\times 2}. \tag{3.6}$$

Moreover, let $\Pi : L^2(\Omega) \to Q_h^r$ be the orthogonal projection. Then, it is well known that

$$\|s - \Pi s\|_{0,\Omega} \leq Ch\|s\|_{1,\Omega} \qquad \forall s \in H^1(\Omega). \tag{3.7}$$

The following lemma establishes the discrete analogue of Lemma 2.3.

Lemma 3.2. *There exists $C > 0$, independent of h and t, such that*

$$\sup_{\substack{(\boldsymbol{\tau}_h, \xi_h) \in \mathbf{H}_h \\ (\boldsymbol{\tau}_h, \xi_h) \neq 0}} \frac{b((\boldsymbol{\tau}_h, \xi_h), (\eta_h, s_h, v_h))}{\|(\boldsymbol{\tau}_h, \xi_h)\|_{\mathbf{H}}} \geq C\|(\eta_h, s_h, v_h)\|_{\mathbf{Q}} \qquad \forall (\eta_h, s_h, v_h) \in \mathbf{Q}_h.$$

Proof. Let $(\eta_h, s_h, v_h) \in \mathbf{Q}_h$. From Lemma 4.4 in [7], we know that there exists $\widetilde{\boldsymbol{\tau}}_h \in H_h^\sigma$ and $\tilde{c} > 0$ such that,

$$\frac{\int_\Omega \eta_h \cdot \mathbf{div}\, \widetilde{\boldsymbol{\tau}}_h + \int_\Omega s_h(\tilde{\tau}_{12h} - \tilde{\tau}_{21h})}{\|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} + \|\mathbf{div}\, \widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}} \geq \tilde{c}(\|\eta_h\|_{0,\Omega} + \|s_h\|_{0,\Omega}).$$

Hence, we have

$$\sup_{\substack{(\boldsymbol{\tau}_h,\xi_h)\in\mathbf{H}_h\\(\boldsymbol{\tau}_h,\xi_h)\neq 0}} \frac{b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}} \geq \frac{b((\widetilde{\boldsymbol{\tau}}_h,0),(\eta_h,s_h,v_h))}{\|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}+\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}}$$

$$\geq \tilde{c}(\|\eta_h\|_{0,\Omega}+\|s_h\|_{0,\Omega}).$$

Next, let z be the unique solution of the following problem:

$$-\Delta z = v_h \quad \text{in } \Omega,$$

$$z = 0 \quad \text{on } \partial\Omega.$$

Since $v_h \in L^2(\Omega)$ and Ω is a convex domain, a classical elliptic regularity result guarantees that $z \in H^2(\Omega)$ and that there exists $\bar{c} > 0$ such that $\|z\|_{2,\Omega} \leq \bar{c}\|v_h\|_{0,\Omega}$. Now, we define $\hat{\xi} := -\nabla z \in H^1(\Omega)^2$. We note that $\operatorname{div}\hat{\xi} = v_h$ in Ω and

$$\|\hat{\xi}\|_{1,\Omega} = \|\nabla z\|_{1,\Omega} \leq \|z\|_{2,\Omega} \leq \bar{c}\|v_h\|_{0,\Omega}.$$

Let $\hat{\xi}_h := \mathcal{R}\hat{\xi}$. From (3.3) and the fact that $\operatorname{div}\hat{\xi} = v_h$, we have that $\operatorname{div}\hat{\xi}_h = v_h$ in Ω. Hence, using the estimate (3.4), we deduce that

$$\|\hat{\xi}_h\|_{0,\Omega}+\|\operatorname{div}\hat{\xi}_h\|_{0,\Omega} \leq \|\hat{\xi}_h-\hat{\xi}\|_{0,\Omega}+\|\hat{\xi}\|_{0,\Omega}+\|\operatorname{div}\hat{\xi}\|_{0,\Omega}$$

$$\leq Ch\|\hat{\xi}\|_{1,\Omega}+\|\hat{\xi}\|_{1,\Omega} \leq \hat{C}\|v_h\|_{0,\Omega}.$$

Therefore, it follows that

$$\sup_{\substack{(\boldsymbol{\tau}_h,\xi_h)\in\mathbf{H}_h\\(\boldsymbol{\tau}_h,\xi_h)\neq 0}} \frac{b((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))}{\|(\boldsymbol{\tau}_h,\xi_h)\|_{\mathbf{H}}} \geq \frac{b((0,\hat{\xi}_h),(\eta_h,s_h,v_h))}{(1+t)\|\hat{\xi}_h\|_{0,\Omega}+\|\operatorname{div}\hat{\xi}_h\|_{0,\Omega}}$$

$$\geq \frac{1}{2(\|\hat{\xi}_h\|_{0,\Omega}+\|\operatorname{div}\hat{\xi}_h\|_{0,\Omega})}\left(\int_\Omega \eta_h\cdot\hat{\xi}_h+\int_\Omega v_h\operatorname{div}\hat{\xi}_h\right)$$

$$\geq \frac{1}{2(\|\hat{\xi}_h\|_{0,\Omega}+\|\operatorname{div}\hat{\xi}_h\|_{0,\Omega})}(\|v_h\|_{0,\Omega}^2-\|\eta_h\|_{0,\Omega}\|\hat{\xi}_h\|_{0,\Omega})$$

$$\geq \frac{1}{2\hat{C}}\|v_h\|_{0,\Omega}-\frac{1}{2}\|\eta_h\|_{0,\Omega}.$$

Thus, the same arguments used at the last step of the proof of Lemma 2.3, allow us to conclude the proof. ∎

We are now in a position to establish the unique solvability, and the convergence properties of the discrete problem (3.1)–(3.2).

Theorem 3.3. *There exists a unique* $((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h)) \in \mathbf{H}_h\times\mathbf{Q}_h$ *solution to problem* (3.1)–(3.2). *Moreover, there exists* $C > 0$, *independent of h and t, such that*

$$\|((\boldsymbol{\sigma},\gamma),(\beta,r,w))-((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))\|_{\mathbf{H}\times\mathbf{Q}}$$
$$\leq C \inf_{((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))\in\mathbf{H}_h\times\mathbf{Q}_h}\|((\boldsymbol{\sigma},\gamma),(\beta,r,w))-((\boldsymbol{\tau}_h,\xi_h),(\eta_h,s_h,v_h))\|_{\mathbf{H}\times\mathbf{Q}},$$

where $((\boldsymbol{\sigma},\gamma),(\beta,r,w)) \in \mathbf{H}\times\mathbf{Q}$ *is the unique solution to problem* (2.6)–(2.7).

Proof. It is a direct application of [13, Theorem II.2.1]. ■

B. Error Estimates

To establish the rate of convergence of the method, first we introduce some notation and prove some results that will be used in the sequel.

In the following, we indicate with e a general edge of the triangulation and with $\mathcal{E}_h$ the set of all such edges. Moreover, we indicate with h_e the length of $e \in \mathcal{E}_h$ and associate to each edge a unit normal vector n_e, chosen once and for all. Moreover, t_e denotes the tangent vector defined as the counterclockwise rotation of n_e by 90°. For each internal edge e of $\mathcal{E}_h$, we indicate with T^+ and T^- the two triangles of the mesh which have the edge e in common, where n_e corresponds to the outward normal for T^+ and the opposite for T^-. Then, given any piecewise regular (scalar or vector) function v in Ω, for each $e \in \mathcal{E}_h$ we define the jump on internal edges

$$[\![v]\!] := v^+|_e - v^-|_e,$$

where $v^\pm$ is the restriction of v to $T^\pm$. On boundary edges, the 'jump' $[\![v]\!]$ is simply given by the value of v on the edge. We introduce the following H^1-type discrete norm on piecewise constant vector functions:

$$\|q_h\|_{\star,h}^2 := \|q_h\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![q_h]\!]\|_{0,e}^2.$$

The following inf-sup condition holds true.

Lemma 3.4. *There exists $C > 0$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h\in X_h\\ \boldsymbol{\tau}_h\neq 0}} \frac{\int_\Omega q_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h\|_{0,\Omega}} \geq C\|q_h\|_{\star,h} \qquad \forall q_h \in Q_h^\beta.$$

Proof. The proof of the above inf-sup condition is simple; thus, we give only a brief sketch. Given $q_h \in Q_h^\beta$, let $\boldsymbol{\tau}_h^1 \in X_h$ with degrees of freedom $\boldsymbol{\tau}_h^1 n_e := h_e^{-1}[\![q_h]\!]$ for all $e \in \mathcal{E}_h$. An element-wise integration by parts and the definition of the jump operator yield

$$\int_\Omega q_h \cdot \mathbf{div}\, \boldsymbol{\tau}_h^1 = \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![q_h]\!]\|_{0,e}^2.$$

Moreover, by a scaling argument, we have that $\|\boldsymbol{\tau}_h^1\|_{0,\Omega} \leq c_1\|q_h\|_{\star,h}$. On the other hand, by repeating the arguments used to prove the standard inf-sup condition for Raviart-Thomas elements, we have that there exists $\boldsymbol{\tau}_h^2 \in X_h$ such that $\mathbf{div}\, \boldsymbol{\tau}_h^2 = q_h$ and $\|\boldsymbol{\tau}_h^2\|_{0,\Omega} \leq c_2\|q_h\|_{0,\Omega}$. This allows us to end the proof. ■

To establish the rate of convergence of the method, we will use Theorem 3.3 and the following result.

Proposition 3.5. *Let $((\boldsymbol{\sigma}, \gamma), (\beta, r, w)) \in \mathbf{H} \times \mathbf{Q}$ be the unique solution of problem* (2.6)–(2.7). *Then, there exist $\boldsymbol{\sigma}_I \in X_h$ such that*

$$\int_\Omega (\mathbf{div}\, \boldsymbol{\sigma}_I + \mathcal{R}\gamma) \cdot q_h = 0 \qquad \forall q_h \in Q_h^\beta. \tag{3.8}$$

Moreover, for all $\epsilon \in (0,1)$*, there exists* $C_\epsilon > 0$ *such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{0,\Omega} \le C_\epsilon h \left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right] \|g\|_{0,\Omega}. \tag{3.9}$$

Proof. Let $(\boldsymbol{\sigma}_I, p_h) \in X_h \times Q_h^\beta$ be the solution of the following discrete mixed problem:

$$\int_\Omega \boldsymbol{\sigma}_I : \boldsymbol{\tau}_h + \int_\Omega \mathbf{div}\, \boldsymbol{\tau}_h \cdot p_h = \int_\Omega \widetilde{\mathcal{R}}\boldsymbol{\sigma} : \boldsymbol{\tau}_h \qquad \forall \boldsymbol{\tau}_h \in X_h, \tag{3.10}$$

$$\int_\Omega \mathbf{div}\, \boldsymbol{\sigma}_I \cdot q_h = -\int_\Omega \mathcal{R}\gamma \cdot q_h \qquad \forall q_h \in Q_h^\beta. \tag{3.11}$$

By using standard results for mixed problems, we know that there exists a unique solution to the above problem. Moreover, using (3.5) and the fact that $\mathbf{div}\, \boldsymbol{\sigma} + \gamma = 0$ in Ω, it is easy to obtain from (3.10) and (3.11) that

$$\|\boldsymbol{\sigma}_I - \widetilde{\mathcal{R}}\boldsymbol{\sigma}\|_{0,\Omega}^2 = \int_\Omega (\mathcal{R}\gamma - \gamma) \cdot p_h. \tag{3.12}$$

On the other hand, from Lemma 3.4 and (3.10), we obtain that

$$C\|p_h\|_{\star,h} \le \sup_{\substack{\boldsymbol{\tau}_h \in X_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{\int_\Omega p_h \cdot \operatorname{div} \boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h\|_{0,\Omega}} = \sup_{\substack{\boldsymbol{\tau}_h \in X_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{\int_\Omega (\widetilde{\mathcal{R}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_I) : \boldsymbol{\tau}_h}{\|\boldsymbol{\tau}_h\|_{0,\Omega}} \le \|\widetilde{\mathcal{R}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{0,\Omega}. \tag{3.13}$$

Now, since $p_h \in Q_h^\beta$, on each element T we can write $p_h|_T = \nabla(\phi_h|_T)$, with ϕ_h being a piecewise linear discontinuous function. Moreover, ϕ_h can be chosen such that $\int_T \phi_h = 0$ for all $T \in \mathcal{T}_h$; therefore, we have that

$$\|\phi_h\|_{0,T} \le Ch\|\nabla\phi_h\|_{0,T} = Ch\|p_h\|_{0,T} \qquad \forall T \in \mathcal{T}_h. \tag{3.14}$$

From (3.12), first integrating by parts, then using that $(\gamma - \mathcal{R}\gamma) \cdot n_e$ is single valued and has zero average on each edge, we obtain

$$\begin{aligned}
\|\boldsymbol{\sigma}_I - \widetilde{\mathcal{R}}\boldsymbol{\sigma}\|_{0,\Omega}^2 &= \int_\Omega (\mathcal{R}\gamma - \gamma) \cdot p_h = \sum_{T \in \mathcal{T}_h} \int_T (\mathcal{R}\gamma - \gamma) \cdot \nabla\phi_h \\
&= \sum_{T \in \mathcal{T}_h} \left[\int_T \operatorname{div}(\gamma - \mathcal{R}\gamma)\phi_h + \int_{\partial T} (\mathcal{R}\gamma - \gamma) \cdot n_T \phi_h\right] \\
&= \underbrace{\sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(\gamma - \mathcal{R}\gamma)\phi_h}_{E_1} + \underbrace{\sum_{e \in \mathcal{E}_h} \int_e (\mathcal{R}\gamma - \gamma) \cdot n_e ([\![\phi_h]\!] - \overline{[\![\phi_h]\!]})}_{E_2},
\end{aligned} \tag{3.15}$$

where $\overline{[\![\phi_h]\!]} \in \mathbb{R}$ denotes the average of $[\![\phi_h]\!]$ on the edge $e \in \mathcal{E}_h$.

Our next goal is to bound the two terms on the right hand side above. For the first one, we recall that $\operatorname{div} \mathcal{R}\gamma = \mathcal{P}(\operatorname{div}\gamma) = -\mathcal{P}g$, and use (3.14) and (3.13) to obtain

$$E_1 \le C \sum_{T \in \mathcal{T}_h} \|g\|_{0,T}\, \|\phi_h\|_{0,T} \le Ch\|g\|_{0,\Omega}\, \|\boldsymbol{\sigma}_I - \widetilde{\mathcal{R}}\boldsymbol{\sigma}\|_{0,\Omega}. \tag{3.16}$$

To bound the second term, first we note that, for all $e \in \mathcal{E}_h$, by using an inverse inequality and standard approximation results, it follows that

$$\|[\![\phi_h]\!] - \overline{[\![\phi_h]\!]}\|_{L^\infty(e)} \leq C h_e^{-1/2} \|[\![\phi_h]\!] - \overline{[\![\phi_h]\!]}\|_{0,e} \leq C h_e^{1/2} \left\| \frac{\partial [\![\phi_h]\!]}{\partial t_e} \right\|_{0,e}$$
$$= C h_e^{1/2} \|[\![\nabla \phi_h \cdot t_e]\!]\|_{0,e} = C h_e^{1/2} \|[\![p_h \cdot t_e]\!]\|_{0,e}.$$

Therefore, since $\mathcal{R}\gamma \cdot n_e$ is constant on each edge,

$$\begin{aligned}
E_2 &= \sum_{e \in \mathcal{E}_h} \int_e \gamma \cdot n_e ([\![\phi_h]\!] - \overline{[\![\phi_h]\!]}) \\
&\leq C \sum_{e \in \mathcal{E}_h} \|\gamma \cdot n_e\|_{L^1(e)} \|[\![\phi_h]\!] - \overline{[\![\phi_h]\!]}\|_{L^\infty(e)} \\
&\leq C \sum_{e \in \mathcal{E}_h} \|\gamma \cdot n_e\|_{L^1(e)} h_e^{1/2} \|[\![p_h \cdot t_e]\!]\|_{0,e} \\
&\leq C \sum_{T \in \mathcal{T}_h} h_T \|\gamma \cdot n_T\|_{L^1(\partial T)} \sum_{e \in \partial T} h_e^{-1/2} \|[\![p_h]\!]\|_{0,e}.
\end{aligned} \tag{3.17}$$

On the other hand, the arguments from [13, Section III.3.3] can be used to prove that, for all $p > 2$, there exist $C_p > 0$ such that

$$\|\gamma \cdot n_T\|_{L^1(\partial T)} \leq C_p \big(h_T^\epsilon \|\gamma\|_{L^p(T)} + h_T \|\operatorname{div} \gamma\|_{0,T}\big), \tag{3.18}$$

where $\epsilon := (1 - 2/p) \in (0, 1)$. Moreover, from the Sobolev embedding theorem (see, for instance, [14]), it follows that $H^\epsilon(T) \hookrightarrow L^p(T)$ for all $T \in \mathcal{T}_h$. Due to the shape regularity of the mesh, a standard scaling immediately yields the bound

$$\|\gamma\|_{L^p(T)}^2 \leq C\big(h^{-2\epsilon} \|\gamma\|_{0,T}^2 + |\gamma|_{H^\epsilon(T)}^2\big). \tag{3.19}$$

Thus, first using (3.17) and (3.18), then the Cauchy-Schwarz inequality and finally (3.19), we obtain

$$\begin{aligned}
E_2 &\leq C_p \sum_{T \in \mathcal{T}_h} \big(h_T^{1+\epsilon} \|\gamma\|_{L^p(T)} + h_T^2 \|\operatorname{div} \gamma\|_{0,T}\big) \sum_{e \in \partial T} h_e^{-1/2} \|[\![p_h]\!]\|_{0,e} \\
&\leq C_p \left(\sum_{T \in \mathcal{T}_h} h_T^{2+2\epsilon} \|\gamma\|_{L^p(T)}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![p_h]\!]\|_{0,e}^2 \right)^{1/2} \\
&\quad + C_p h^2 \|\operatorname{div} \gamma\|_{0,\Omega} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![p_h]\!]\|_{0,e}^2 \right)^{1/2}
\end{aligned}$$

$$\leq C_p \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\gamma\|_{0,T}^2 + h_T^{2+2\epsilon} |\gamma|_{H^\epsilon(T)}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![p_h]\!]\|_{0,e}^2 \right)^{1/2}$$

$$+ C_p h^2 \|\operatorname{div} \gamma\|_{0,\Omega} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![p_h]\!]\|_{0,e}^2 \right)^{1/2}.$$

Using the above estimate and recalling (3.13), we obtain that, for all $\epsilon \in (0, 1)$,

$$\begin{aligned} E_2 &\leq C_\epsilon h (\|\gamma\|_{0,\Omega} + h^\epsilon |\gamma|_{H^\epsilon(\Omega)} + h\|g\|_{0,\Omega}) \|\boldsymbol{\sigma}_I - \widetilde{\mathcal{R}}\boldsymbol{\sigma}\|_{0,\Omega} \\ &\leq C_\epsilon h \left[1 + \left(\frac{h}{t} \right)^\epsilon \right] \|g\|_{0,\Omega} \, \|\boldsymbol{\sigma}_I - \widetilde{\mathcal{R}}\boldsymbol{\sigma}\|_{0,\Omega}, \end{aligned} \tag{3.20}$$

where in the last step we have used that, since $\|\gamma\|_{0,\Omega} + t\|\gamma\|_{1,\Omega} \leq C\|g\|_{0,\Omega}$ (cf. Proposition 2.5), $t^\epsilon \|\gamma\|_{H^\epsilon(\Omega)} \leq C\|g\|_{0,\Omega}$ for all $\epsilon \in [0, 1]$ (see for instance [12, Theorem I.1.4]).

Finally, the proof follows by substituting (3.16) and (3.20) into (3.15) and using the triangular inequality, (3.6) and Proposition 2.5. ∎

To prove the rate of convergence of the method, we will also use the following result.

Lemma 3.6. *There exists $C > 0$, independent of h, such that*

$$\|\xi_h - \widetilde{\mathcal{P}}\xi_h\|_{0,\Omega} \leq Ch\|\operatorname{div} \xi_h\|_{0,\Omega} \qquad \forall \xi_h \in H_h^\gamma.$$

Proof. Let $\xi_h \in H_h^\gamma$. Then, by using standard error estimates,

$$\|\xi_h - \widetilde{\mathcal{P}}\xi_h\|_{0,\Omega}^2 = \sum_{T \in \mathcal{T}_h} \|\xi_h - \widetilde{\mathcal{P}}\xi_h\|_{0,T}^2 \leq C_1 \sum_{T \in \mathcal{T}_h} h_T^2 |\xi_h|_{1,T}^2 \leq Ch^2 \|\operatorname{div} \xi_h\|_{0,\Omega}^2.$$

where we have used that, for Raviart-Thomas elements, $|\xi_h|_{1,T}^2 = \frac{1}{2}\|\operatorname{div} \xi_h\|_{0,T}^2$. ∎

The following theorem provides the rate of convergence of our mixed finite element scheme.

Theorem 3.7. *Let $((\boldsymbol{\sigma}, \gamma), (\beta, r, w)) \in \mathbf{H} \times \mathbf{Q}$ and $((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solutions to the continuous and discrete problems* (2.6)–(2.7) *and* (3.1)–(3.2)*, respectively. If $g \in H^1(\Omega)$, then, for all $\epsilon \in (0, 1)$, there exists $C_\epsilon > 0$ such that*

$$\|((\boldsymbol{\sigma}, \gamma), (\beta, r, w)) - ((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h))\|_{\mathbf{H}\times\mathbf{Q}} \leq C_\epsilon h \left[1 + \left(\frac{h}{t} \right)^\epsilon \right] \|g\|_{1,\Omega}.$$

Proof. Let $\boldsymbol{\sigma}_I \in X_h$ be as in Proposition 3.5. According to Theorem 3.3, we have that

$$\begin{aligned} &\|((\boldsymbol{\sigma}, \gamma), (\beta, r, w)) - ((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h))\|_{\mathbf{H}\times\mathbf{Q}} \\ &\qquad \leq C\|((\boldsymbol{\sigma}, \gamma), (\beta, r, w)) - ((\boldsymbol{\sigma}_I, \mathcal{R}\gamma), (\widetilde{\mathcal{P}}\beta, \Pi r, \mathcal{P}w))\|_{\mathbf{H}\times\mathbf{Q}} \\ &\qquad = C(\|(\boldsymbol{\sigma}, \gamma) - (\boldsymbol{\sigma}_I, \mathcal{R}\gamma)\|_{\mathbf{H}} + \|(\beta, r, w) - (\widetilde{\mathcal{P}}\beta, \Pi r, \mathcal{P}w)\|_{\mathbf{Q}}). \end{aligned}$$

The second term is easily bounded by using standard error estimates for $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, (3.7) and Proposition 2.5:

$$\|(\beta, r, w) - (\widetilde{\mathcal{P}}\beta, \Pi r, \mathcal{P}w)\|_{\mathbf{Q}} \leq Ch\|g\|_{0,\Omega}. \tag{3.21}$$

For the first term, we write

$$\begin{aligned}
\|(\boldsymbol{\sigma}, \gamma) - (\boldsymbol{\sigma}_I, \mathcal{R}\gamma)\|_{\mathbf{H}} &= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{0,\Omega} + \|\mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_I) + (\gamma - \mathcal{R}\gamma)\|_{0,\Omega} \\
&\quad + t\|\gamma - \mathcal{R}\gamma\|_{0,\Omega} + \|\mathrm{div}\,(\gamma - \mathcal{R}\gamma)\|_{0,\Omega} \\
&\leq C_\epsilon h \left[1 + \left(\frac{h}{t}\right)^\epsilon\right] \|g\|_{0,\Omega} + Ch\|g\|_{0,\Omega} \\
&\quad + Cht\|\gamma\|_{1,\Omega} + Ch\|g\|_{1,\Omega},
\end{aligned} \tag{3.22}$$

where we have used (3.9) for the first term and standard error estimates for Raviart-Thomas elements for the third and the fourth. Regarding the second term, the estimate follows from the fact that $\mathbf{div}\,\boldsymbol{\sigma} + \gamma = 0$ in Ω and, hence,

$$\|\mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_I) + (\gamma - \mathcal{R}\gamma)\|_{0,\Omega} = \|\mathbf{div}\,\boldsymbol{\sigma}_I + \mathcal{R}\gamma\|_{0,\Omega} = \|\mathcal{R}\gamma - \widetilde{\mathcal{P}}(\mathcal{R}\gamma)\|_{0,\Omega},$$

the latter because of (3.8). Finally, from Lemma 3.6, we have

$$\|\mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_I) + (\gamma - \mathcal{R}\gamma)\|_{0,\Omega} \leq Ch\|g\|_{0,\Omega}.$$

Thus, the proof follows from (3.21), (3.22) and Proposition 2.5. ∎

Remark 3.1. The error estimate from the previous theorem is optimal with respect to the mesh size h and only involves the problem data. It is not thoroughly independent of the thickness t, but this dependence is very mild, since the estimate holds for any $\epsilon > 0$. On the other hand, the term $\|g\|_{1,\Omega}$ could actually be replaced by $(\sum_{T\in\mathcal{T}} \|g\|_{1,T}^2)^{1/2}$. In fact, this norm is only used to derive (3.22) and this relies on a Raviart-Thomas interpolation error estimate which holds true element by element. Therefore, the theorem holds for piecewise smooth loads, too.

C. A Postprocessing of Transverse Displacement and Rotations

In this section we present an element-wise postprocessing procedure, which allows us to build piecewise linear transverse displacement and rotations with improved approximation properties. With this aim, we introduce another H^1-type discrete norm for all sufficiently regular (scalar or vector) functions v.

$$\|v\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![v]\!]\|_{0,e}^2.$$

Given the discrete solution $((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, we define a postprocessed transverse displacement $w_h^\star \in L^2(\Omega)$ as follows. For all $T \in \mathcal{T}_h$ let $w_h^\star \in \mathbb{P}_1(T)$ be such that

$$\mathcal{P}w_h^\star = w_h, \tag{3.23}$$

$$\nabla w_h^\star = \widetilde{\mathcal{P}}(\beta_h + t^2\kappa^{-1}\gamma_h). \tag{3.24}$$

It is immediate to check that $w_h^\star$ is well defined and unique.

We start proving the following preliminary result.

Lemma 3.9. *There holds*

$$\|\mathcal{P}w - w_h\|_{1,h} \leq Ch\left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right]\|g\|_{1,\Omega}.$$

Proof. To prove the result we will apply the following inf-sup condition, whose proof we do not include since is very similar to that of Lemma 3.4: for all $v_h \in Q_h^w$, there exists $\xi_h \in H_h^\gamma$ such that

$$\int_\Omega v_h \operatorname{div} \xi_h = \|v_h\|_{1,h}^2 \quad \text{and} \quad \|\xi_h\|_{0,\Omega} \leq C\|v_h\|_{1,h}. \tag{3.25}$$

Taking $v_h = (w_h - \mathcal{P}w)$, noting that $\operatorname{div} \xi_h$ is piecewise constant and finally using (2.6) and (3.1), we obtain

$$\begin{aligned}\|\mathcal{P}w - w_h\|_{1,h}^2 &= \int_\Omega (\mathcal{P}w - w_h)\operatorname{div} \xi_h = \int_\Omega (w - w_h)\operatorname{div} \xi_h \\ &= \int_\Omega (\beta - \beta_h)\xi_h + t^2\kappa^{-1}\int_\Omega (\gamma - \gamma_h)\xi_h.\end{aligned}$$

The proof follows from the above equation by using a Cauchy-Schwarz inequality, recalling Theorem 3.7 and using (3.25). ∎

Now we are in a position to prove an improved convergence result for the postprocessed transverse displacement.

Proposition 3.10. *There holds*

$$\left\|w - w_h^\star\right\|_{1,h} \leq Ch\left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right]\|g\|_{1,\Omega}.$$

Proof. Let $\widetilde{w}_h$ and $\widetilde{w}$ be such that

$$w_h^\star = w_h + \widetilde{w}_h \quad \text{and} \quad w = \mathcal{P}w + \widetilde{w}. \tag{3.26}$$

Since $\widetilde{w}_h$ and $\widetilde{w}$ have zero average on each element T, by applying a scaled trace inequality we have that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\widetilde{w}_h - \widetilde{w}]\!]\|_{0,e}^2 &\leq \sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|\widetilde{w}_h - \widetilde{w}\|_{0,T}^2 + |\widetilde{w}_h - \widetilde{w}|_{1,T}^2\right) \\ &\leq C\sum_{T\in\mathcal{T}_h}\|\nabla(\widetilde{w}_h - \widetilde{w})\|_{0,T}^2.\end{aligned} \tag{3.27}$$

We now observe that, due to (3.26), there hold $\nabla w_h^\star|_T = \nabla \widetilde{w}_h|_T$ and $\nabla w|_T = \nabla \widetilde{w}|_T$ for all $T \in \mathcal{T}_h$. Therefore, first we use (3.23) and (2.3) and then standard properties of the projector $\widetilde{\mathcal{P}}$, to obtain for all $T \in \mathcal{T}_h$

$$\begin{aligned}
\|\nabla(\widetilde{w}_h - \widetilde{w})\|_{0,T}^2 &= \|\widetilde{\mathcal{P}}(\beta_h + t^2\kappa^{-1}\gamma_h) - (\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 \\
&\le \|\widetilde{\mathcal{P}}(\beta_h + t^2\kappa^{-1}\gamma_h) - \widetilde{\mathcal{P}}(\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 \\
&\quad + \|\widetilde{\mathcal{P}}(\beta + t^2\kappa^{-1}\gamma) - (\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 \\
&\le \|(\beta_h + t^2\kappa^{-1}\gamma_h) - (\beta + t^2\kappa^{-1}\gamma)\|_{0,T}^2 + Ch_T^2|(\beta + t^2\kappa^{-1}\gamma)|_{1,T}^2 \\
&\le C\big(\|\beta_h - \beta\|_{0,T}^2 + t^4\|\gamma_h - \gamma\|_{0,T}^2 + h_T^2|\beta|_{1,T}^2 + h_T^2 t^4|\gamma|_{1,T}^2\big).
\end{aligned}$$

The above estimate, combined with Theorem 3.7 and Proposition 2.5, immediately yield

$$\sum_{T\in\mathcal{T}_h} \|\nabla(\widetilde{w}_h - \widetilde{w})\|_{0,T}^2 \le Ch^2\left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right]^2 \|g\|_{1,\Omega}^2. \tag{3.28}$$

From (3.27), (3.28) and the definition of $\|\cdot\|_{1,h}$, we finally obtain

$$\|\widetilde{w}_h - \widetilde{w}\|_{1,h} \le Ch\left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right]\|g\|_{1,\Omega},$$

which combined with Lemma 3.9 and a triangle inequality lead to

$$\left\|w - w_h^\star\right\|_{1,h} \le \|\mathcal{P}w - w_h\|_{1,h} + \|\widetilde{w}_h - \widetilde{w}\|_{1,h} \le Ch\left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right]\|g\|_{1,\Omega}.$$

Thus we conclude the proof. ∎

We define also a postprocessed rotation field $\beta_h^\star \in L^2(\Omega)^2$ as follows: For all $T \in \mathcal{T}_h$, let $\beta_h^\star \in \mathbb{P}_1(T)^2$ be such that

$$\widetilde{\mathcal{P}}\beta_h^\star = \beta_h, \ \ \nabla\beta_h^\star = \widehat{\mathcal{P}}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + r_h\mathbf{J}),$$

where $\widehat{\mathcal{P}}$ is the L^2 projection onto the space of piecewise constant $\mathbb{R}^{2\times 2}$ tensor fields. It is immediate to check that $\beta_h^\star$ is well defined and unique. Moreover, the following result can be proved by following the same lines as above.

Proposition 3.11. *There holds*

$$\left\|\beta - \beta_h^\star\right\|_{1,h} \le Ch\left[1 + \left(\frac{h}{t}\right)^{\epsilon}\right]\|g\|_{1,\Omega}.$$

Finally note that both postprocessing procedures are fully local and therefore have a negligible computational cost.

Remark 3.2. Although the main purpose of this scheme is to compute a better approximation of the bending moments and the shear stress, using this postprocessing, a piecewise linear approximation of transverse displacement and rotations converging in an H^1-type norm can be recovered. Note in particular that, from the definition of the norm $\|\cdot\|_{1,h}$ and the fact that the jumps of w and β are null, it follows that at the limit for $h \to 0$ the postprocessed discrete functions will also be continuous.

D. Hybridization of the Discrete Problem

Similarly as in [7] the solution of the discrete problem (3.1)–(3.2) can be computed by solving an equivalent linear system significantly smaller and positive definite. We will show briefly such construction. We start introducing the following 'broken' spaces:

$$
\begin{aligned}
\widetilde{X}_h &:= \{\boldsymbol{\tau}_h \in L^2(\Omega)^{2\times 2} : \ \boldsymbol{\tau}_h|_T \in [RT_0(T)^{\mathrm{t}}]^2 \ \forall T \in \mathcal{T}_h\},\\
\widetilde{H}_h^{\gamma} &:= \{\xi_h \in L^2(\Omega)^2 : \ \xi_h|_T \in RT_0(T) \ \forall T \in \mathcal{T}_h\},\\
\widetilde{H}_h^{\sigma} &:= \widetilde{X}_h \oplus B(\mathcal{T}_h),\\
\widetilde{\mathbf{H}}_h &:= \widetilde{H}_h^{\sigma} \times \widetilde{H}_h^{\gamma}.
\end{aligned}
$$

Note that no inter-element continuity is required for the above spaces. Furthermore, we introduce a space of Lagrange multipliers that we use to enforce the continuity condition on the solution. This is a discrete space of piecewise constant (vector) functions defined on the set $\mathcal{E}_h^{\mathrm{int}}$ of internal edges of the triangulation:

$$
\Xi_h := \left\{(k_h, d_h) : \ \mathcal{E}_h^{\mathrm{int}} \to \mathbb{R}^2 \times \mathbb{R}\right\}.
$$

Since such functions (k_h, d_h) are constant on each edge, they can be identified with the collection of its values (k_e, d_e) on the internal edges $e \in \mathcal{E}_h^{\mathrm{int}}$. We also introduce the bilinear form $\vartheta : \widetilde{\mathbf{H}}_h \times \Xi_h \to \mathbb{R}$ defined by

$$
\vartheta((\boldsymbol{\tau}_h, \xi_h), (k_h, d_h)) := \sum_{e \in \mathcal{E}_h^{\mathrm{int}}} \left(k_e \cdot \int_e [\![\boldsymbol{\tau}_h n_e]\!] + d_e \int_e [\![\xi_h \cdot n_e]\!]\right),
$$

where $[\![\cdot]\!]$ is the jump operator defined above. It is easy to check that the original discrete problem (3.1)–(3.2) is equivalent to the following one:

Find $((\boldsymbol{\sigma}_h, \gamma_h), (\beta_h, r_h, w_h), (x_h, z_h)) \in \widetilde{\mathbf{H}}_h \times \mathbf{Q}_h \times \Xi_h$ *such that*

$$
a((\boldsymbol{\sigma}_h, \gamma_h), (\boldsymbol{\tau}_h, \xi_h)) + b((\boldsymbol{\tau}_h, \xi_h), (\beta_h, r_h, w_h)) + \vartheta((\boldsymbol{\tau}_h, \xi_h), (x_h, z_h)) = 0 \qquad \forall (\boldsymbol{\tau}_h, \xi_h) \in \widetilde{\mathbf{H}}_h, \tag{3.29}
$$

$$
b((\boldsymbol{\sigma}_h, \gamma_h), (\eta_h, 0, v_h)) = F(\eta_h, 0, v_h) \qquad \forall (\eta_h, v_h) \in Q_h^{\beta} \times Q_h^{w}, \tag{3.30}
$$

$$
b((\boldsymbol{\sigma}_h, \gamma_h), (0, s_h, 0)) = F(0, s_h, 0) \qquad \forall s_h \in Q_h^{r}, \tag{3.31}
$$

$$
\vartheta((\boldsymbol{\sigma}_h, \gamma_h), (k_h, d_h)) = 0 \qquad \forall (k_h, d_h) \in \Xi_h. \tag{3.32}
$$

The advantage is that most variables in the above system can be eliminated with the following static condensation procedure:

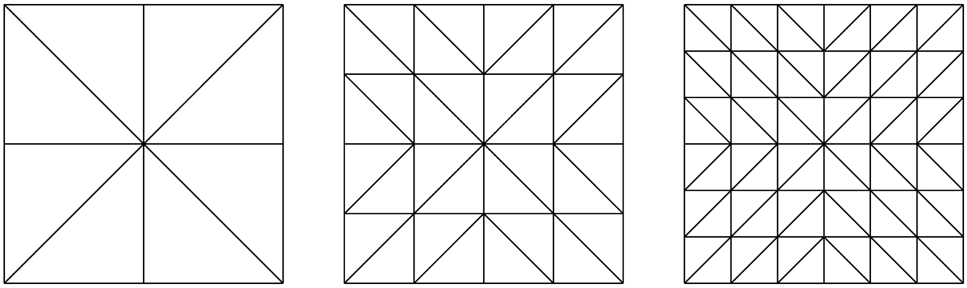

FIG. 1. Square plate: uniform meshes.

1. Due to the full inter-element discontinuity of the functions in $\widetilde{\mathbf{H}}_h$, from (3.29) one can compute $(\boldsymbol{\sigma}_h, \gamma_h)$ as a function of the remaining variables by solving an 11×11 system for each element of the mesh.
2. Substituting $(\boldsymbol{\sigma}_h, \gamma_h)$ as a function of the remaining variables in (3.30), one calculates (β_h, w_h) as a function of F (namely of g), r_h and (x_h, z_h). Note again that, due to the inter-element discontinuity of the functions in $Q_h^{\beta} \times Q_h^{w}$, such operation reduces to solving a 3×3 linear system for each element of the mesh.
3. Substituting back the result (β_h, w_h) of item 2 into item 1, allows computing also $(\boldsymbol{\sigma}_h, \gamma_h)$ as a function of F, r_h and (x_h, z_h). Therefore, Eqs. (3.31)–(3.32) now become a linear system that can be solved for r_h and (x_h, z_h), which constitutes the main bulk of the computations.

It can be checked that the final system obtained in item 3 above is symmetric and positive definite. The dimension of such a system corresponds to that of the space $Q_h^r \times \Xi_h$, i.e., the number of vertices plus three times the number of internal edges. Therefore, the size of the final system is similar to that of more standard finite elements. For instance, the well known low-order Durán-Liberman element adopts a total of three degrees of freedom per vertex plus one degree of freedom per edge.

IV. NUMERICAL RESULTS

This numerical method has been implemented in a MATLAB code. We report in this section some numerical experiments which confirm the theoretical results proved above.

We have taken as a test problem an isotropic and homogeneous plate $\Omega := (0,1) \times (0,1)$ clamped on its whole boundary, for which the analytical solution is explicitly known (see [15]). We analyze the convergence properties of the method by considering different uniform meshes as those shown in Fig. 1, and keeping the thickness fixed to $t = 0.001$.

Choosing the following transverse load,

$$\begin{aligned} g(x,y) = & \frac{E}{12(1-\nu^2)} \\ & \times \{12y(y-1)(5x^2-5x+1)[2y^2(y-1)^2 + x(x-1)(5y^2-5y+1)] \\ & + 12x(x-1)(5y^2-5y+1)[2x^2(x-1)^2 + y(y-1)(5x^2-5x+1)]\}, \end{aligned}$$

TABLE I. Errors and experimental rates of convergence for $\boldsymbol{\sigma}$, $(\mathbf{div}\,\boldsymbol{\sigma}+\gamma)$, γ, r, β and w.

N	err($\boldsymbol{\sigma}$)	rc($\boldsymbol{\sigma}$)	err($\boldsymbol{\sigma},\gamma$)	rc($\boldsymbol{\sigma},\gamma$)	err(γ)	rc(γ)
609	0.40270e-04	—	0.29609e-03	—	0.31715e-02	—
2497	0.19649e-04	1.054	0.14805e-03	1.018	0.15876e-02	1.016
10113	0.09760e-04	1.019	0.07404e-03	1.009	0.07942e-02	1.008
40705	0.04868e-04	1.008	0.03702e-03	1.004	0.03971e-02	1.004
163329	0.02431e-04	1.004	0.01851e-03	1.002	0.01986e-02	1.002
N	err(r)	rc(r)	err(β)	rc(β)	err(w)	rc(w)
609	0.87462e-04	—	0.39713e-04	—	0.66226e-05	—
2497	0.39217e-04	1.178	0.18189e-04	1.147	0.27707e-05	1.280
10113	0.15009e-04	1.398	0.08884e-04	1.043	0.13136e-05	1.086
40705	0.05491e-04	1.457	0.04416e-04	1.013	0.06478e-05	1.025
163329	0.01991e-04	1.466	0.02205e-04	1.004	0.03228e-05	1.007

the exact solution of problem (2.6)–(2.7) is given by

$$
\begin{aligned}
w(x,y) &= \frac{1}{3}x^3(x-1)^3y^3(y-1)^3 - \frac{2t^2}{5(1-\nu)}[y^3(y-1)^3x(x-1)(5x^2-5x+1) \\
&\quad + x^3(x-1)^3y(y-1)(5y^2-5y+1)], \\
\beta_1(x,y) &= y^3(y-1)^3x^2(x-1)^2(2x-1), \\
\beta_2(x,y) &= x^3(x-1)^3y^2(y-1)^2(2y-1).
\end{aligned}
$$

The material constants have been chosen $E=1$ and $\nu=0.30$ and the shear correction factor has been taken $k=5/6$.

In what follows, N denotes the number of degrees of freedom, which according to Section D corresponds to $N:=\dim(Q_h^r)+\dim(\Xi_h)$. Moreover, we define the individual errors by:

$$
\begin{aligned}
\mathrm{err}(\boldsymbol{\sigma}) &:= \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}, & \mathrm{err}(\boldsymbol{\sigma},\gamma) &:= \|(\mathbf{div}\,\boldsymbol{\sigma}+\gamma)-(\mathbf{div}\,\boldsymbol{\sigma}_h+\gamma_h)\|_{0,\Omega}, \\
\mathrm{err}(r) &:= \|r-r_h\|_{0,\Omega}, & \mathrm{err}(\gamma) &:= t\|\gamma-\gamma_h\|_{0,\Omega}+\|\mathrm{div}\,(\gamma-\gamma_h)\|_{0,\Omega}, \\
\mathrm{err}(\beta) &:= \|\beta-\beta_h\|_{0,\Omega}, & \mathrm{err}(w) &:= \|w-w_h\|_{0,\Omega},
\end{aligned}
$$

where $((\boldsymbol{\sigma},\gamma),(\beta,r,w))\in\mathbf{H}\times\mathbf{Q}$ and $((\boldsymbol{\sigma}_h,\gamma_h),(\beta_h,r_h,w_h))\in\mathbf{H}_h\times\mathbf{Q}_h$ are the solutions to problems (2.6)–(2.7) and (3.1)–(3.2), respectively.

We have also computed experimental rates of convergence for each individual error as follows:

$$
\mathrm{rc}(\cdot) := -2\frac{\log(\mathrm{err}(\cdot)/\mathrm{err}'(\cdot))}{\log(N/N')},
$$

where N and N' denote the degrees of freedom of two consecutive triangulations with respective errors err and err$'$.

Table I shows the convergence history of the mixed finite element scheme (3.1)–(3.2) applied to our test problem.

We observe from these tables that a clear rate of convergence $O(h)$ is attained for all quantities. Actually, the computation of r seems to be superconvergent.

Figures 2–5 show the profiles of all the computed quantities obtained with the finest mesh $(N=163329)$.

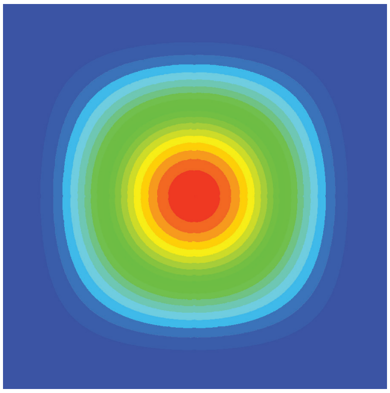

FIG. 2. Transverse displacement w_h. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

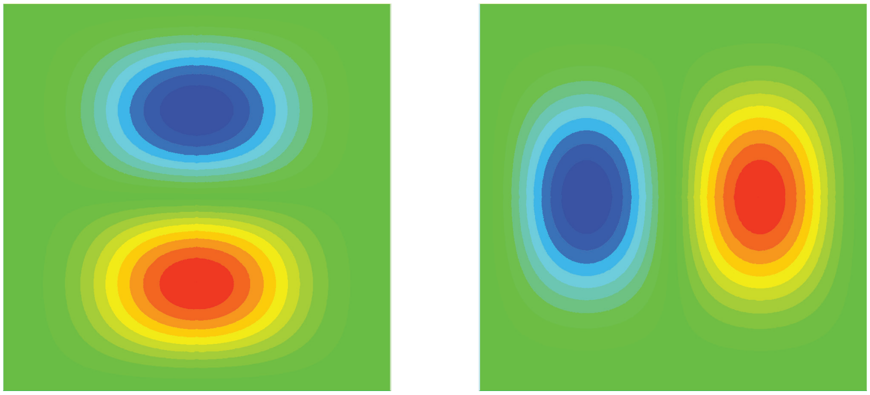

FIG. 3. Rotations β_{1h} (left) and β_{2h} (right). [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

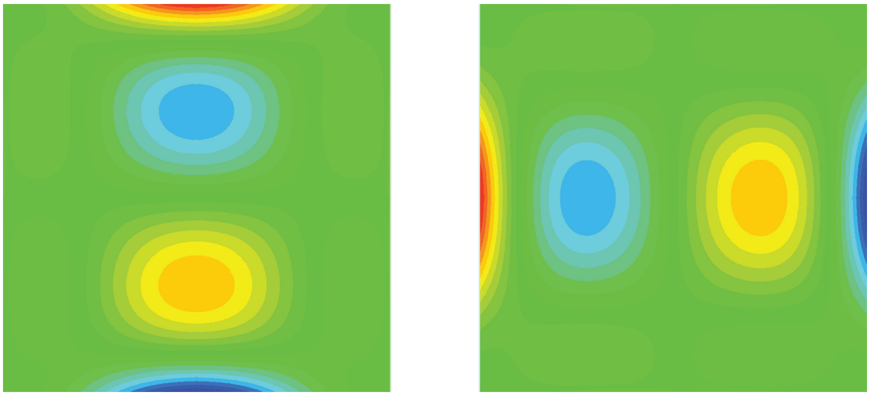

FIG. 4. Shear stress γ_{1h} (left) and γ_{2h} (right). [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

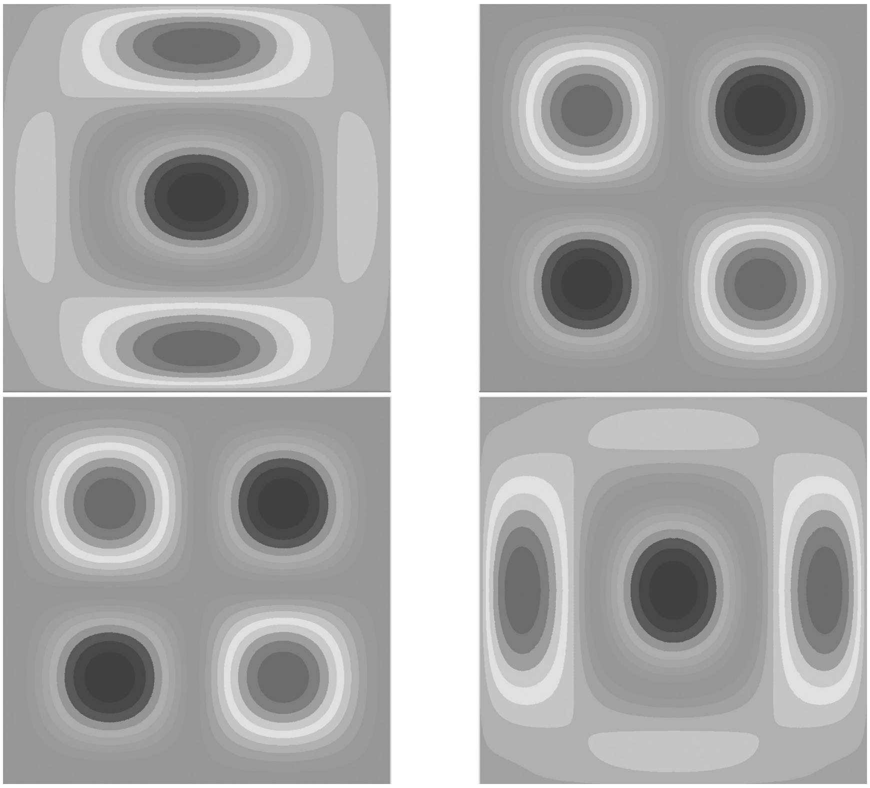

FIG. 5. Bending moments $\boldsymbol{\sigma}_{11h}$ (top-left), $\boldsymbol{\sigma}_{12h}$ (top-right), $\boldsymbol{\sigma}_{21h}$ (bottom-left), and $\boldsymbol{\sigma}_{22h}$ (bottom-right).

V. CONCLUSIONS

We have introduced a finite element method to solve the bending problem for a Reissner-Mindlin plate. The method is based on a dual mixed variational formulation, in which the unknowns are both stresses and displacements. In addition, the symmetry of the bending moment tensor is imposed in a weak sense.

The discretization scheme uses PEERS finite elements for the bending moments and the corresponding Lagrange multiplier to recover the symmetry. Shear stresses are discretized by lowest-order Raviart-Thomas elements, while the kinematic variables are approximated by piecewise constant functions.

Despite the high number of the involved degrees of freedom, the actual scheme implementation can be efficiently made by using a hybridization procedure. Therefore, the resulting approach has a computational cost which is comparable with those of other low-order schemes.

Error estimates are derived for the bending moment $\boldsymbol{\sigma}$ and the shear stress γ, both in $H(\mathrm{div})$. For these estimates to hold, an additional piecewise smoothness assumption is needed for the load $g = -\mathrm{div}\,\gamma$, which is the only data of the problem. We note that standard methods based on a transverse displacement and rotation discretization only lead to approximations of the shear stress in weaker norms.

The method is proved to be practically locking-free, without the need of any reduction operator. In fact, the obtained error estimates only depend on norms of the solution which can be a priori bounded in terms of the data g. These error estimates are not fully independent of the plate thickness t, since they involve a term $(\frac{h}{t})^{\epsilon}$. However the exponent ϵ can be arbitrarily small, so that the dependence on t is actually very mild. This is confirmed by the numerical experiments.

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