Journal of Computational and Applied Mathematics 425 (2023) 115026

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A Morley-type virtual element approximation for a wind-driven ocean circulation model on polygonal meshes

D. Adak [a,b], D. Mora [a,c,*], A. Silgado [a]

[a] GIMNAP, Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile
[b] Department of Mechanical Engineering, Indian Institute of Technology Madras, Chennai 600036, India
[c] CI^{2}MA, Universidad de Concepción, Concepción, Chile

ARTICLE INFO

Article history:
Received 22 April 2022
Received in revised form 18 November 2022

MSC:
35Q35
65N12
65N15
65N30

Keywords:
Nonconforming virtual element
Ocean circulation model
Stream-function
Error estimates
Polygonal meshes
Velocity–vorticity recovery

ABSTRACT

In this work, we propose and analyze a Morley-type virtual element method to approximate the Stommel–Munk model in stream-function form. The discretization is based on the fully nonconforming virtual element approach presented in Antonietti et al., (2018) and Zhao et al., (2018). The analysis restricts to simply connected polygonal domains, not necessarily convex. Under standard assumptions on the computational domain we derive some inverse estimates, norm equivalence and approximation properties for an *enriching operator* E_h defined from the nonconforming space into its H^2-conforming counterpart. With the help of these tools we prove optimal error estimates for the stream-function in broken H^2-, H^1- and L^2-norms *under minimal regularity* condition on the weak solution. Employing postprocessing formulas and adequate polynomial projections we compute from the discrete stream-function further fields of interest, such as: the velocity and vorticity. Moreover, for these postprocessed variables we establish error estimates. Finally, we report practical numerical experiments on different families of polygonal meshes.

1. Introduction

The Stommel–Munk model in stream-function form is a linear fourth-order partial differential equation given by:

$$\epsilon_M \Delta^2 \psi - \epsilon_S \Delta \psi - \partial_x \psi = f \quad \text{in } \Omega, \tag{1.1}$$

with the boundary conditions:

$$\psi = \partial_{\mathbf{n}} \psi = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected domain with polygonal boundary $\partial\Omega$, $\partial_{\mathbf{n}}$ denotes the normal derivative, ψ is the stream-function of the horizontal velocity field $\mathbf{u}$ and f is the wind forcing term. In the model, the parameters ϵ_M and ϵ_S are the non-dimensional scale Munk and Stommel numbers, respectively, which are defined by:

$$\epsilon_M = \frac{A}{\beta L^3} \qquad \text{and} \qquad \epsilon_S = \frac{\gamma}{\beta L},$$

* Corresponding author at: GIMNAP, Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile.
E-mail addresses: dadak@ubiobio.cl (D. Adak), dmora@ubiobio.cl (D. Mora), alberth.silgado1701@alumnos.ubiobio.cl (A. Silgado).

https://doi.org/10.1016/j.cam.2022.115026
0377-0427/

where A is the eddy viscosity parametrization, L is the characteristic length scale, β is the coefficient multiplying the y-coordinate in the β-plane approximation and γ is the coefficient of the linear drag (or the Rayleigh friction), as might be generated by a bottom Ekman layer (for further details, see for instance [1–4]).

The Stommel–Munk model can be seen as a simplification of the Quasi-Geostrophic equations of the ocean (QGE) [4–6], both models are characterized by the presence of the biharmonic operator $\Delta^2\psi$, the rotational term $\partial_x\psi$, the source term f and they have the same boundary conditions (1.2), but the difference between these models lies in the presence of the nonlinear Jacobian operator in the QGE, whereas the linear Stommel–Munk model instead contains the Laplacian operator $\Delta\psi$. Despite the simplifications, the Stommel–Munk model turns out to be adequate to understand the large scale wind-driven ocean circulation at mid-latitudes due to the model preserves principal features of these currents (the wind forcing and the effects of rotation). The above fact converts the Stommel–Munk model in a standard problem in the geophysical fluid dynamics literature (see for instance [1,7,8]), for which different finite element discretizations have been studied, for instance, using the stream function-vorticity formulation (see [2,3,9]) and stream-function form. In particular, for the last formulation in [10] a B-spline based finite element discretization is introduced and error analysis for this scheme is developed in [11]. In [5] a discrete variational formulation based in C^0-discontinuous Galerkin method is provided and error estimate for the scheme is performed. In the present contribution, we develop and analyze a nonconforming Morley-type virtual element scheme to approximate the Stommel–Munk model formulated in terms of the stream-function. This formulation have outstanding characteristics, such as: there is only one scalar unknown in the system, the streamlines is one of the most useful tools in flow visualization. Moreover, in this work we propose to obtain two variables of great interest in oceanic fluid dynamics: the velocity and vorticity fields, from the discrete stream-function by using postprocessing formulas.

The development of adequate numerical schemes for discretizing PDEs on general polytopal meshes have undergone an intensive research in the past years. Different approaches have been proposed (see for instance [12] and the references therein) and among them we can find the Virtual Element Method (VEM), which since its introduction in the pioneering work [13] it has enjoyed a broad success in numerical modeling of scientific and engineering applications due to its elegant construction and promising results. A wide variety of problems have been addressed using the conforming VEM approach; see for instance [14–19], where second- and fourth-order problems have been analyzed. Moreover, in fluid mechanic the models studied, include: Stokes, Brinkman, Navier–Stokes flows and QGE; see for instance [20–27], where primal and mixed formulations have been considered.

On the other hand, the nonconforming VEM approach, also has presented a growing interest recently. Different schemes for several problems have been developed, for instance, second-order elliptic and fluid mechanic problems have been studied in [28–34]. Moreover, for fourth-order equations in [35] a H^2-nonconforming VEM for plate bending problems is analyzed, which the numerical solution turns to be C^0-continuous. Subsequently, in [36] a fully nonconforming VEM for biharmonic problems is developed. In this space the approximated solution does not require the global C^0-regularity. Besides, in [37] the authors, presented a VEM also for plate bending problems using the same degrees of freedom considered in [36]. However, the construction of the local virtual space have a different approach. The above fully nonconforming VEMs, in the lowest-order case ($k = 2$) can be seen as the extension of the Morley finite element [38] to polygonal meshes. For further nonconforming VEM involving fourth-order problems, see the Refs. [39–44].

In the present work, we propose and analyze a nonconforming Morley-type virtual element discretization for the Stommel–Munk model (cf. (1.1)–(1.2)) with applications in large scale wind-driven oceanic circulation. We consider an enhanced nonconforming virtual space based on the approach presented in [42] (see also [15,36,37]) to approximate the stream-function variable. This virtual element is characterized by not requiring any global C^0-regularity for the discrete solution and can be taken as a generalization of the classical Morley element to general polygonal meshes. Employing suitable projections operators, which are computable using only the degrees of freedom we construct the respective discrete bilinear forms and discrete load term. Then, we write a discrete virtual formulation and we prove its well-posedness by using the Lax–Milgram Theorem. We introduce an enriching operator from the enhanced nonconforming virtual space into its H^2-conforming counterpart (see [14]). For the enhanced H^2-conforming virtual space we recall its construction and we derive inverse inequalities and an equivalence between L^2- and ℓ^2-norms, which are key tools to establish some approximation properties involving the enriching operator, the bilinear form associated to the inner product H^2 and the consistency error. Then, with the help of these results we prove optimal error estimates for the stream-function in broken H^2-, H^1- and L^2-norms under the minimal regularity of the weak solution (cf. Theorem 2.2). Moreover, we propose to compute further variables of interest, such as: the velocity field and the fluid vorticity by a simple postprocess of the discrete VEM stream-function and using suitable projections, which are computable from the degrees of freedom. Finally, we point out that, the present contribution is a good stepping stones for the nonlinear one- and two-layers QGE (see [4,6,27,45]).

The remaining part of the manuscript is organized as follows: In Section 2 we introduce some notations that will be used throughout the paper and we write a weak formulation for the system (1.1)–(1.2). In Section 3 we introduce the fully nonconforming virtual element scheme of the weak formulation. In Section 4 we present some preliminary results including the construction of an enriching operator from the enhanced nonconforming virtual space into its H^2-conforming counterpart. Besides, we derive useful tools to establish the optimal error estimate in broken H^2-norm up to the regularity of the weak solution. Moreover, we obtain optimal error estimates in broken H^1- and L^2-norms by using duality arguments and under the same regularity of the continuous solution. In Section 5 we compute the velocity field and fluid vorticity by a simple postprocess of the discrete VEM stream-function. Finally, in Section 6 we report some numerical experiments exhibiting the behavior of our virtual scheme and confirming the our theoretical results.

The major contribution of the article can be summarized as follows:

- In this article, we extend the fully nonconforming virtual element approach [36,37] (see also [42]) to solve the fourth order Stommel–Munk model on polygonal meshes and we establish error estimates in broken H^2-, H^1- and L^2-norms *under the minimal regularity by introducing enriching operator* as mentioned in Section 4.2. Furthermore, the error estimations in broken L^2- and H^1-norms have been derived assuming the source term f belongs to $L^2(\Omega)$. Moreover, we have proposed novel strategies to compute the velocity and vorticity fields as a postprocess of the discrete stream-function using suitable polynomial projections.

2. The continuous formulation

2.1. Notations

From now on, we will follow the usual notation for Sobolev spaces, seminorms and norms [46]. We will denote a simply connected polygonal Lipschitz bounded domain of $\mathbb{R}^2$ by Ω and $\mathbf{n}=(n_i)_{1\leq i\leq 2}$ is the outward unit normal vector to the boundary $\partial\Omega$, while the vector $\mathbf{t}=(t_i)_{i=1,2}$ is the unit tangent to $\partial\Omega$ oriented such that $t_1=-n_2$, $t_2=n_1$. In addition, for any vector field $\mathbf{v}=(v_i)_{i=1,2}$ and any scalar function φ, we define the differential operators:

$$\mathrm{rot}\mathbf{v}:=\partial_1 v_2-\partial_2 v_1,\quad \nabla\varphi:=\begin{pmatrix}\partial_1\varphi\\ \partial_2\varphi\end{pmatrix}\quad\text{and}\quad \mathbf{curl}\,\varphi:=\begin{pmatrix}\partial_2\varphi\\ -\partial_1\varphi\end{pmatrix}.$$

Moreover, $\mathrm{D}^2\varphi:=(\partial_{ij}\varphi)_{i,j=1,2}$ denotes the Hessian matrix of φ.

In addition, in this work, c and C, with or without superscripts and subscripts, tildes or hats, will represent a strictly positive constant independent of the mesh parameter h, whose value can change in different occurrences.

2.2. Variational problem

Let $\mathcal{V}:=\{\varphi\in H^2(\Omega):\varphi=\partial_{\mathbf{n}}\varphi=0\quad\text{on}\quad\partial\Omega\}$. Then, we have that a variational formulation of problem (1.1)–(1.2) is given as follows: seek $\psi\in\mathcal{V}$, such that

$$A(\psi,\phi)=F(\phi)\qquad\forall\phi\in\mathcal{V},\tag{2.1}$$

where

$$A(\varphi,\phi):=\epsilon_M A_{\mathrm{D}}(\varphi,\phi)+\epsilon_S A_{\nabla}(\varphi,\phi)-A_{\mathrm{skew}}(\varphi,\phi)\qquad\forall\varphi,\phi\in\mathcal{V},\tag{2.2}$$

and the forms $A_{\mathrm{D}}(\cdot,\cdot)$, $A_{\nabla}(\cdot,\cdot)$, $A_{\mathrm{skew}}(\cdot,\cdot)$ and $F(\cdot)$ are defined by:

$$A_{\mathrm{D}}:\mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{R},\qquad A_{\mathrm{D}}(\varphi,\phi):=\int_\Omega \mathrm{D}^2\varphi:\mathrm{D}^2\phi\qquad\forall\varphi,\phi\in\mathcal{V},\tag{2.3}$$

$$A_{\nabla}:\mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{R},\qquad A_{\nabla}(\varphi,\phi):=\int_\Omega \nabla\varphi\cdot\nabla\phi\qquad\forall\varphi,\phi\in\mathcal{V},\tag{2.4}$$

$$A_{\mathrm{skew}}:\mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{R},\qquad A_{\mathrm{skew}}(\varphi,\phi):=\frac{1}{2}\int_\Omega\partial_x\varphi\phi-\frac{1}{2}\int_\Omega\partial_x\phi\,\varphi\qquad\forall\varphi,\phi\in\mathcal{V},\tag{2.5}$$

$$F:\mathcal{V}\longrightarrow\mathbb{R},\qquad F(\phi):=\int_\Omega f\phi\qquad\forall\phi\in\mathcal{V}.\tag{2.6}$$

Remark 2.1. We recall that the classical variational formulation of problem (1.1)–(1.2) is given by: seek $\psi\in\mathcal{V}$, such that

$$\epsilon_M A_{\mathrm{D}}(\psi,\phi)+\epsilon_S A_{\nabla}(\psi,\phi)-A_0(\psi,\phi)=F(\phi)\qquad\forall\phi\in\mathcal{V},$$

where

$$A_0(\psi,\phi):=\int_\Omega\partial_x\psi\,\phi.$$

We observe that the bilinear form $A_0(\cdot,\cdot)$ is equal to the skew-symmetric form $A_{\mathrm{skew}}(\cdot,\cdot)$ defined in (2.5). However, their discrete versions will lead to different bilinear forms, in general. Therefore, we point out that, our virtual method will be based on the weak formulation (2.1), keeping the skew-symmetric property for the bilinear form $A_{\mathrm{skew}}(\cdot,\cdot)$, which allows making the analysis of the scheme in a straightforward way.

We endow the space $\mathcal{V}$ with the norm $\|\varphi\|_{\mathcal{V}}:=(A_D(\varphi,\varphi))^{1/2}\quad\forall\varphi\in\mathcal{V}$, then the forms defined in (2.3)–(2.6) are continuous. More precisely, in the following lemma we summarize some properties for the forms defined in (2.2) and (2.6), which will be used to establish the well-posedness of problem (2.1).

Lemma 2.1. *For all $\varphi, \phi \in \mathcal{V}$, there exists a positive constants C_A, such that the forms $A(\cdot,\cdot)$ and $F(\cdot)$, defined in (2.2) and (2.6), respectively, satisfy the following properties:*

- $|A(\varphi, \phi)| \leq C_A \|\varphi\|_{\mathcal{V}} \|\phi\|_{\mathcal{V}}$;
- $A(\phi, \phi) \geq \epsilon_M \|\phi\|^2_{\mathcal{V}}$;
- $|F(\phi)| \leq \|F\|_{-2,\Omega} \|\phi\|_{\mathcal{V}}$.

Theorem 2.1. *There exists a unique $\psi \in \mathcal{V}$ solution to problem (2.1), which satisfies the following continuous dependence on the data*

$$\|\psi\|_{\mathcal{V}} \leq C\|F\|_{-2,\Omega},$$

where C is a positive constant.

Proof. It is an immediate consequence of Lemma 2.1 and the Lax–Milgram Theorem. □

Now, we will state an additional regularity result for the solution of problem (2.1).

Theorem 2.2. *Let $\psi \in \mathcal{V}$ be the unique solution of problem (2.1). If $F \in H^{-1}(\Omega)$, then there exist $s \in (1/2, 1]$ and $C_{\mathrm{reg}} > 0$ such that $\psi \in H^{2+s}(\Omega)$ and*

$$\|\psi\|_{2+s,\Omega} \leq C_{\mathrm{reg}}\|F\|_{-1,\Omega}.$$

Proof. The proof follows from the classical regularity result for the biharmonic problem with homogeneous Dirichlet boundary conditions (see for instance [47]). □

3. Nonconforming virtual element discretization

In this section we will introduce a Morley-type VEM for the numerical approximation of problem (2.1) on general polygonal meshes, which is based on the fully nonconforming virtual element approach [36,37,42]. First, we introduce some notations to present the local and global nonconforming virtual space. Successively, we introduce some projectors on polynomial spaces to construct the discrete bilinear forms and discrete functional. Finally, we write the discrete problem and we establish its well-posedness by using the Lax–Milgram Theorem.

3.1. Notations and basic setting

Henceforth, we will denote by K a general polygon, by h_K and ∂K its diameter and boundary, respectively. Moreover, we will denote by N_K the number of vertices of K. Let $\{\mathscr{T}_h\}_{h>0}$ be a sequence of decompositions of Ω into general non-overlapping simple polygons K, where $h := \max_{K\in\mathscr{T}_h} h_K$. We will denote the set of the edges in $\mathscr{T}_h$ by $\mathscr{E}_h$, we decompose this set as $\mathscr{E}_h := \mathscr{E}_h^{\mathrm{int}} \cup \mathscr{E}_h^{\mathrm{bdry}}$, where $\mathscr{E}_h^{\mathrm{int}}$ and $\mathscr{E}_h^{\mathrm{bdry}}$ are the set of interior and boundary edges, respectively. Analogously, we will denote by $\mathscr{V}_h := \mathscr{V}_h^{\mathrm{int}} \cup \mathscr{V}_h^{\mathrm{bdry}}$ the set of the all vertices in $\mathscr{T}_h$, where $\mathscr{V}_h^{\mathrm{int}}$ and $\mathscr{V}_h^{\mathrm{bdry}}$ are the set of interior and boundary vertices, respectively.

Additionally, for each $K \in \mathscr{T}_h$, we denote by $\mathbf{n}_K$ its unit outward normal vector and by $\mathbf{t}_K$ its tangential vector along the boundary ∂K. Besides, we will use the notation $\mathbf{n}_e$ and $\mathbf{t}_e$ for a unit normal and tangential vector of an edge $e \in \mathscr{E}_h$, respectively.

For any subset $\mathcal{D} \subset \mathbb{R}^2$ and each integer $\ell \geq 0$ we denote by $\mathbb{P}_\ell(\mathcal{D})$ the space of polynomials of degree up to ℓ defined on $\mathcal{D}$. Moreover, we define the piecewise ℓ-order polynomial space by:

$$\mathbb{P}_\ell(\mathscr{T}_h) := \{q \in L^2(\Omega) : q|_K \in \mathbb{P}_\ell(K) \quad \forall K \in \mathscr{T}_h\}.$$

Next, for any integer number $t > 0$, we introduce the following broken Sobolev space

$$H^t(\mathscr{T}_h) := \{\phi \in L^2(\Omega) : \phi|_K \in H^t(K) \quad \forall K \in \mathscr{T}_h\}$$

endowed with the following broken seminorm

$$|\phi|_{t,h} := \Big(\sum_{K\in\mathscr{T}_h} |\phi|^2_{t,K}\Big)^{1/2}. \tag{3.1}$$

Now, we will define the jump operator denoted by $[\![\cdot]\!]$, as follow: for each function $\phi \in H^2(\mathscr{T}_h)$ and for an internal edge $e \in \mathscr{E}_h^{\mathrm{int}}$, we define $[\![\phi]\!] := \phi^+ - \phi^-$, where $\phi^\pm$ denotes the trace of $\phi|_{K^\pm}$, with $e \subseteq \partial K^+ \cap \partial K^-$. For a boundary edge $e \in \mathscr{E}_h^{\mathrm{bdry}}$, the operator jump is define as: $[\![\phi]\!] := \phi|_e$.

We introduce a subspace of $H^2(\mathscr{T}_h)$ with certain continuity, given by:

$$H^{2,\mathrm{NC}}(\mathscr{T}_h) := \Big\{ \phi_h \in H^2(\mathscr{T}_h) : \phi_h \text{ continuous at internal vertices}, \\ \phi_h(\mathbf{v}_i) = 0 \quad \forall \mathbf{v}_i \in \mathscr{V}_h^{\mathrm{bdry}}, \qquad \int_e [\![\partial_{\mathbf{n}_e}\phi_h]\!] = 0 \quad \forall e \in \mathscr{E}_h \Big\}. \tag{3.2}$$

For the theoretical analysis, we suppose that $\mathscr{T}_h$ satisfies the following assumptions: there exists a real number $\rho > 0$ such that, every $K \in \mathscr{T}_h$, we have

$\mathbf{A_1}$: K is star-shaped with respect to every point of a ball of radius $\geq \rho h_K$;
$\mathbf{A_2}$: the ratio between the shortest edge and the diameter h_K of K is larger than ρ.

We decompose the continuous forms defined in (2.2)–(2.5) as follows:

$$A_{\mathrm{D}}(\varphi,\phi) = \sum_{K\in\mathscr{T}_h} A_{\mathrm{D}}^K(\varphi,\phi) := \sum_{K\in\mathscr{T}_h} \int_K \mathrm{D}^2\varphi : \mathrm{D}^2\phi \qquad \forall\varphi,\phi\in\mathcal{V},$$

$$A_{\nabla}(\varphi,\phi) = \sum_{K\in\mathscr{T}_h} A_{\nabla}^K(\varphi,\phi) := \sum_{K\in\mathscr{T}_h} \int_K \nabla\varphi\cdot\nabla\phi \qquad \forall\varphi,\phi\in\mathcal{V},$$

$$A_{\mathrm{skew}}(\varphi,\phi) = \sum_{K\in\mathscr{T}_h} A_{\mathrm{skew}}^K(\varphi,\phi) := \sum_{K\in\mathscr{T}_h} \frac{1}{2}\int_K \partial_x\varphi\,\phi - \frac{1}{2}\int_K \partial_x\phi\,\varphi \qquad \forall\varphi,\phi\in\mathcal{V}.$$

Also, we split

$$A(\varphi,\phi) = \sum_{K\in\mathscr{T}_h} A^K(\varphi,\phi) := \sum_{K\in\mathscr{T}_h} (\epsilon_M A_{\mathrm{D}}^K(\varphi,\phi) + \epsilon_S A_{\nabla}^K(\varphi,\phi) - A_{\mathrm{skew}}^K(\varphi,\phi)) \qquad \forall\varphi,\phi\in\mathcal{V}.$$

3.2. Local and global nonconforming virtual element spaces

For every polygon $K \in \mathscr{T}_h$, we introduce the following preliminary local virtual space (for further details see [42, Section 3.4] and [15,36,37]):

$$\widetilde{\mathcal{V}}_h(K) := \left\{\phi_h \in H^2(K) : \Delta^2\phi_h \in \mathbb{P}_2(K),\ \ \phi_h|_e \in \mathbb{P}_2(e),\ \ \Delta\phi_h|_e \in \mathbb{P}_0(e) \quad \forall e\subseteq\partial K\right\}.$$

Next, for a given $\phi_h \in \widetilde{\mathcal{V}}_h(K)$, we introduce the following set of linear operators (which will be degrees of freedom after of the enhancement technique):

- $\mathbf{D_1}$: the values of $\phi_h(\mathbf{v}_i)$ for all vertex $\mathbf{v}_i$ of the polygon K;
- $\mathbf{D_2}$: the moments

$$\int_e \partial_{\mathbf{n}_e}\phi_h \qquad \forall \text{ edge } e\subseteq\partial K.$$

For each polygon K, we define the following projector

$$\begin{aligned} \Pi_K^{\mathrm{D}} : \widetilde{\mathcal{V}}_h(K) &\longrightarrow \mathbb{P}_2(K) \subseteq \widetilde{\mathcal{V}}_h(K),\\ \phi_h &\longmapsto \Pi_K^{\mathrm{D}}\phi_h, \end{aligned}$$

where $\Pi_K^{\mathrm{D}}\phi_h$ is the solution of the local problems:

$$\begin{aligned} A_{\mathrm{D}}^K(\Pi_K^{\mathrm{D}}\phi_h, q) &= A_{\mathrm{D}}^K(\phi_h, q) \qquad \forall q\in\mathbb{P}_2(K),\\ \widehat{\Pi_K^{\mathrm{D}}\phi_h} &= \widehat{\phi_h} \qquad \int_{\partial K}\nabla\Pi_K^{\mathrm{D}}\phi_h = \int_{\partial K}\nabla\phi_h, \end{aligned}$$

and the operator $\widehat{(\cdot)}$ is defined as follows:

$$\widehat{\varphi_h} := \frac{1}{N_K}\sum_{i=1}^{N_K}\varphi_h(\mathbf{v}_i), \tag{3.3}$$

and $\mathbf{v}_i$, $1\leq i\leq N_K$ are the vertices of K.

Moreover, as stated by the following lemma, the polynomial projection Π_K^{D} is computable using the sets $\mathbf{D_1}$ and $\mathbf{D_2}$ (for more details see [37,42]).

Lemma 3.1. *The operator $\Pi_K^{\mathrm{D}} : \widetilde{\mathcal{V}}_h(K) \longrightarrow \mathbb{P}_2(K)$ is explicitly computable for every $\phi_h \in \widetilde{\mathcal{V}}_h(K)$, using only the information of the linear operators $\mathbf{D_1}$ and $\mathbf{D_2}$.*

Now, for each $K \in \mathscr{T}_h$ we introduce the enhanced fully nonconforming virtual space:

$$\mathcal{V}_h(K) := \left\{\phi_h \in \widetilde{\mathcal{V}}_h(K) : \int_e(\phi_h - \Pi_K^{\mathrm{D}}\phi_h) = 0 \quad \forall e\subseteq\partial K,\ \ \int_K p(\phi_h - \Pi_K^{\mathrm{D}}\phi_h) = 0 \quad \forall p\in\mathbb{P}_2(K)\right\}. \tag{3.4}$$

The following result summarize the main properties of the local virtual space $\mathcal{V}_h(K)$. The proof can be obtained following the arguments in [15,36,37,42].

Proposition 3.1. *For each polygon K, the space $\mathcal{V}_h(K)$ defined in (3.4) satisfies the following properties:*

- $\mathbb{P}_2(K) \subseteq \mathcal{V}_h(K)$.
- *The sets of linear operators* $\mathbf{D_1}$ *and* $\mathbf{D_2}$ *constitutes a set of degrees of freedom for* $\mathcal{V}_h(K)$.
- *The operator* $\Pi_K^{\mathrm{D}} : \mathcal{V}_h(K) \longrightarrow \mathbb{P}_2(K)$ *is computable using the degrees of freedom* $\mathbf{D_1}$ *and* $\mathbf{D_2}$.

Now, for every decomposition $\mathscr{T}_h$ of Ω into polygons K, we introduce the fully nonconforming global virtual space to the numerical approximation of problem (2.1) as follows:

$$\mathcal{V}_h := \left\{\phi_h \in H^{2,\mathrm{NC}}(\mathscr{T}_h) : \phi_h|_K \in \mathcal{V}_h(K) \quad \forall K \in \mathscr{T}_h\right\}, \tag{3.5}$$

where the space $H^{2,\mathrm{NC}}(\mathscr{T}_h)$ is defined in (3.2).

It is observed that $\mathcal{V}_h \subseteq H^{2,\mathrm{NC}}(\mathscr{T}_h)$ but $\mathcal{V}_h \nsubseteq \mathcal{V}$. Furthermore, we have that the nonconforming virtual element does not require the C^0-continuity over Ω.

3.3. Polynomial projection operators

In this subsection, we introduce further polynomial projections, which will be useful to build the respective discrete forms.

First, for all $m \in \mathbb{N} \cup \{0\}$, we consider the usual $L^2(K)$-projection onto the polynomial space $\mathbb{P}_m(K)$: for each $\phi \in L^2(K)$, the function $\Pi_K^m \phi \in \mathbb{P}_m(K)$ is defined as the unique function satisfying

$$\left(q, \phi - \Pi_K^m \phi\right)_{0,K} = 0 \qquad \forall q \in \mathbb{P}_m(K). \tag{3.6}$$

Lemma 3.2. *Let Π_K^2 be the operator defined in (3.6), with $m = 2$. Then, for each $\phi_h \in \mathcal{V}_h(K)$ we have that the polynomial functions $\Pi_K^2\phi_h$ and $\Pi_K^2(\partial_x\phi_h)$ are computable using only the information of the degrees freedom* $\mathbf{D_1}$ *and* $\mathbf{D_2}$.

Proof. Let $\phi_h \in \mathcal{V}_h(K)$. Then the function $\Pi_K^2\phi_h$ is easily obtained from the definition of the space $\mathcal{V}_h(K)$ (cf. (3.4)). On the other hand, using integration by parts and the definition of $\Pi_K^2\phi_h$, for all $q \in \mathbb{P}_2(K)$, we have

$$\int_K \partial_x\phi_h q = -\int_K \phi_h \partial_x q + \int_{\partial K} \phi_h q \mathbf{n}_K^x = -\int_K (\Pi_K^2\phi_h)\partial_x q + \int_{\partial K} \phi_h q \mathbf{n}_K^x,$$

where $\mathbf{n}_K^x$ is the first component of normal vector $\mathbf{n}_K$. We notice that the first term on the right hand side of the above equality depends only on $\Pi_K^2\phi_h$, hence computable using the degrees of freedom (see Proposition 3.1). The boundary integral is computable using $\mathbf{D_1}$ and the moments of $\Pi_K^{\mathrm{D}}\phi_h$ on the each edge $e \subseteq \partial K$ (cf. (3.4)). $\square$

Next, for each polygon K, we define the projector $\Pi_K^{\nabla} : \mathcal{V}_h(K) \longrightarrow \mathbb{P}_2(K) \subseteq \mathcal{V}_h(K)$, as the solution of the local problems:

$$A_{\nabla}^K(\Pi_K^{\nabla}\phi_h, q) = A_{\nabla}^K(\phi_h, q) \quad \forall q \in \mathbb{P}_2(K),$$
$$\widehat{\Pi_K^{\nabla}\phi_h} = \widehat{\phi_h},$$

and the operator $\widehat{(\cdot)}$ is defined in (3.3).

The following result establishes that the polynomial projection Π_K^{∇} is computable from the sets $\mathbf{D_1}$ and $\mathbf{D_2}$. The result follows the same arguments used in the proof of Lemma 3.2.

Lemma 3.3. *The operator $\Pi_K^{\nabla} : \mathcal{V}_h(K) \longrightarrow \mathbb{P}_2(K)$ is explicitly computable for every $\phi_h \in \mathcal{V}_h(K)$, using only the information of the linear operators* $\mathbf{D_1}$ *and* $\mathbf{D_2}$.

3.4. Construction of the discrete forms

In the present section, we will build the discrete version of the continuous local forms defined in (2.3)–(2.6) by using the operators introduced in the above subsection.

First, let $\mathcal{S}_{\mathrm{D}}^K(\cdot,\cdot)$ and $\mathcal{S}_{\nabla}^K(\cdot,\cdot)$ be any symmetric positive definite bilinear forms to be chosen as to satisfy:

$$\begin{aligned} &c_0 A_{\mathrm{D}}^K(\phi_h,\phi_h) \le \mathcal{S}_{\mathrm{D}}^K(\phi_h,\phi_h) \le c_1 A_{\mathrm{D}}^K(\phi_h,\phi_h) \qquad \forall \phi_h \in \mathcal{V}_h(K), \text{ with } \Pi_K^{\mathrm{D}}\phi_h = 0,\\ &c_2 A_{\nabla}^K(\phi_h,\phi_h) \le \mathcal{S}_{\nabla}^K(\phi_h,\phi_h) \le c_3 A_{\nabla}^K(\phi_h,\phi_h) \qquad \forall \phi_h \in \mathcal{V}_h(K), \text{ with } \Pi_K^{\nabla}\phi_h = 0, \end{aligned} \tag{3.7}$$

with c_0, c_1, c_2 and c_3 positive constants independent of h and K. A classical choice for the bilinear forms $\mathcal{S}_{\mathrm{D}}^K(\cdot,\cdot)$ and $\mathcal{S}_{\nabla}^K(\cdot,\cdot)$ satisfying (3.7) is given by the Euclidean scalar product associated to the degrees of freedom scaled appropriately (see [28,36,37]). More precisely, we choose the following representation:

$$\mathcal{S}_{\mathrm{D}}^K(\varphi_h,\phi_h) := h_K^{-2}\sum_{i=1}^{N_{\mathrm{dof}}^K} \mathrm{dof}_i(\varphi)\mathrm{dof}_i(\phi) \qquad \text{and} \qquad \mathcal{S}_{\nabla}^K(\varphi_h,\phi_h) := \sum_{i=1}^{N_{\mathrm{dof}}^K} \mathrm{dof}_i(\varphi)\mathrm{dof}_i(\phi),$$

for all $\varphi_h, \phi_h \in \mathcal{V}_h(K)$, where N^K_{dof} denote the number of degrees freedom of $\mathcal{V}^h_k(K)$ and dof_i is the operator that to each smooth enough function ϕ associates the ith local degree of freedom $\text{dof}_i(\phi)$, with $1 \le i \le N^K_{\text{dof}}$.

Thus, we define the following global form $A^h : \mathcal{V}_h \times \mathcal{V}_h \longrightarrow \mathbb{R}$, given by:

$$A^h(\varphi_h, \phi_h) = \sum_{K\in\mathscr{T}_h} A^{h,K}(\varphi_h, \phi_h) = \sum_{K\in\mathscr{T}_h} (\epsilon_M A^{h,K}_{\text{D}}(\varphi_h, \phi_h) + \epsilon_M A^{h,K}_{\nabla}(\varphi_h, \phi_h) - A^{h,K}_{\text{skew}}(\varphi_h, \phi_h)), \tag{3.8}$$

where the discrete local bilinear forms, $A^{h,K}_{\text{D}} : \mathcal{V}_h(K) \times \mathcal{V}_h(K) \longrightarrow \mathbb{R}$, $A^{h,K}_{\nabla} : \mathcal{V}_h(K) \times \mathcal{V}_h(K) \longrightarrow \mathbb{R}$ and $A^{h,K}_{\text{skew}} : \mathcal{V}_h(K) \times \mathcal{V}_h(K) \longrightarrow \mathbb{R}$, approximating the continuous bilinear forms $A^K_{\text{D}}(\cdot,\cdot)$, $A^K_{\nabla}(\cdot,\cdot)$ and $A^K_{\text{skew}}(\cdot,\cdot)$ are given by

$$A^{h,K}_{\text{D}}(\varphi_h, \phi_h) := A^K_{\text{D}}\big(\Pi^{\text{D}}_K\varphi_h, \Pi^{\text{D}}_K\phi_h\big) + \mathcal{S}^K_{\text{D}}\big((I-\Pi^{\text{D}}_K)\varphi_h, (I-\Pi^{\text{D}}_K)\phi_h\big), \tag{3.9}$$

$$A^{h,K}_{\nabla}(\varphi_h, \phi_h) := A^K_{\nabla}\big(\Pi^{\nabla}_K\varphi_h, \Pi^{\nabla}_K\phi_h\big) + \mathcal{S}^K_{\nabla}\big((I-\Pi^{\nabla}_K)\varphi_h, (I-\Pi^{\nabla}_K)\phi_h\big), \tag{3.10}$$

$$A^{h,K}_{\text{skew}}(\varphi_h, \phi_h) := \frac{1}{2}\int_K \Pi^2_K(\partial_x\varphi_h)\,\Pi^2_K\phi_h - \frac{1}{2}\int_K \Pi^2_K(\partial_x\phi_h)\,\Pi^2_K\varphi_h. \tag{3.11}$$

The following result establishes the usual consistency and stability properties for the discrete local forms.

Proposition 3.2. *The local bilinear forms $A^K_{\text{D}}(\cdot,\cdot)$, $A^K_{\nabla}(\cdot,\cdot)$, $A^K(\cdot,\cdot)$, $A^{h,K}_{\text{D}}(\cdot,\cdot)$, $A^{h,K}_{\nabla}(\cdot,\cdot)$ and $A^{h,K}(\cdot,\cdot)$ on each element K satisfy*

- *Consistency: for all $h > 0$ and for all $K \in \mathscr{T}_h$, we have that*

$$A^{h,K}(q, \phi_h) = A^K(q, \phi_h) \qquad \forall q \in \mathbb{P}_2(K), \qquad \forall \phi_h \in \mathcal{V}_h(K), \tag{3.12}$$

- *Stability and boundedness: There exist positive constants α_i, $i = 1, \ldots, 4$, independent of K, such that:*

$$\alpha_1 A^K_{\text{D}}(\phi_h, \phi_h) \le A^{h,K}_{\text{D}}(\phi_h, \phi_h) \le \alpha_2 A^K_{\text{D}}(\phi_h, \phi_h) \qquad \forall \phi_h \in \mathcal{V}_h(K), \tag{3.13}$$

$$\alpha_3 A^K_{\nabla}(\phi_h, \phi_h) \le A^{h,K}_{\nabla}(\phi_h, \phi_h) \le \alpha_4 A^K_{\nabla}(\phi_h, \phi_h) \qquad \forall \phi_h \in \mathcal{V}_h(K). \tag{3.14}$$

Proof. The proof follows standard arguments in the VEM literature (see [13,36,37]). □

Finally, we consider the following approximation for the functional defined in (2.6):

$$F^h(\phi_h) := \sum_{K\in\mathscr{T}_h} F^{h,K}(\phi_h) \qquad \forall \phi_h \in \mathcal{V}_h, \tag{3.15}$$

where the local functional $F^{h,K}(\cdot)$ is defined by

$$F^{h,K}(\phi_h) := \int_K \Pi^2_K f\,\phi_h \equiv \int_K f\,\Pi^2_K\phi_h \qquad \forall \phi_h \in \mathcal{V}_h(K).$$

For the continuous bilinear forms $A_\star(\cdot,\cdot)$, with $\star \in \{\text{D}, \nabla, \text{skew}\}$, we adopt the following notation:

$$A_\star(\varphi_h, \phi_h) := \sum_{K\in\mathscr{T}_h} A^K_\star(\varphi_h, \phi_h) \qquad \forall \varphi_h, \phi_h \in \mathcal{V} + \mathcal{V}_h. \tag{3.16}$$

We also adopt the same notation by the bilinear form $A(\cdot,\cdot)$ and the functional $F(\cdot)$.

3.5. Discrete problem and its well-posedness

In this subsection, we present the discrete virtual element formulation and we establish its well-posedness by using the Lax–Milgram Theorem.

The fully nonconforming virtual element problem reads as: seek $\psi_h \in \mathcal{V}_h$, such that

$$A^h(\psi_h, \phi_h) = F^h(\phi_h) \qquad \forall \phi_h \in \mathcal{V}_h, \tag{3.17}$$

where $A^h(\cdot,\cdot)$ is the discrete bilinear forms defined in (3.8) and $F^h(\cdot)$ is the discrete functional introduced in (3.15).

The following lemma establishes properties for the application $|\cdot|_{2,h}$ defined in (3.1), with $t = 2$.

Lemma 3.4. *For all $\phi_h \in \mathcal{V}_h$, the following inequality holds:*

$$\|\phi_h\|_{0,\Omega} + |\phi_h|_{1,h} \le C|\phi_h|_{2,h},$$

where C is a positive constant, independent of h. Moreover, we have that $|\cdot|_{2,h}$ is a norm on the space $\mathcal{V}_h$.

Proof. The proof is established in [37, Lemma 5.1]. □

The following result establishes some properties for the discrete forms defined in the last subsection, which will be used to conclude the well-posedness of the discrete problem (3.17). The proof follows from the definition of the respective forms.

Lemma 3.5. *For all $\varphi_h, \phi_h \in \mathcal{V}_h$, there exist positive constants $C_{A^h}, \widetilde{\alpha}, C_{F^h}$, independent of h, such that the forms defined in (3.11), (3.8) and (3.15) satisfy the following properties:*

$$|A^h(\varphi_h, \phi_h)| \leq C_{A^h} |\varphi_h|_{2,h} |\phi_h|_{2,h}, \qquad A^h(\phi_h, \phi_h) \geq \widetilde{\alpha} |\phi_h|^2_{2,h}, \tag{3.18}$$

$$A^{h,K}_{\mathrm{skew}}(\phi_h, \phi_h) = 0, \qquad |F^h(\phi_h)| \leq C_{F^h} \|f\|_{0,\Omega} |\phi_h|_{2,h}. \tag{3.19}$$

We have the following result of existence and uniqueness.

Theorem 3.1. *The discrete problem (3.17) admits a unique solution $\psi_h \in \mathcal{V}_h$, which satisfies the following continuous dependence on the data*

$$|\psi_h|_{2,h} \leq C \|f\|_{0,\Omega},$$

where the positive constant C is independent of h.

Proof. It is an immediate consequence of Lemma 3.5 and the Lax–Milgram Theorem. □

4. Convergence analysis

In this section we will establish error estimates for the nonconforming VEM presented in Section 3.5. First, we present some preliminary results useful for the analysis. Successively, we introduce an *enriching operator* E_h from the nonconforming space $\mathcal{V}_h$ into its conforming counterpart. Then, we derive some approximation properties involving this operator and the bilinear form $A_{\mathrm{D}}(\cdot,\cdot)$ (cf. (2.3)). By using the above tools we establish an error estimate in broken H^2-norm *under minimal regularity condition on the stream-function* ψ (cf. Theorem 2.2). Finally, by using duality arguments we derive error estimates in broken H^1- and L^2-norms under the same regularity of the weak solution and assuming the source term f belongs to $L^2(\Omega)$.

4.1. Preliminary results

We start recalling an important approximation result for polynomials on star-shaped domains (see, for instance [42,48]).

Proposition 4.1. *Assume that $\mathbf{A_1}$ is satisfied. Then, for every $\phi \in H^{2+t}(K)$, with $t \in [0,1]$, there exist $\phi_\pi \in \mathbb{P}_2(K)$ and $C > 0$, independent of h, such that*

$$\|\phi - \phi_\pi\|_{\ell,K} \leq C h_K^{2+t-\ell} |\phi|_{2+t,K}, \qquad \ell = 0, 1, 2.$$

We have the following approximation result in the virtual space $\mathcal{V}_h$ (see [36,37,42]).

Proposition 4.2. *Assume that $\mathbf{A_1} - \mathbf{A_2}$ are satisfied. Then, for each $\phi \in H^{2+t}(\Omega)$, with $t \in [0,1]$, there exist $\phi_I \in \mathcal{V}_h$ and $C > 0$, independent of h, such that*

$$\|\phi - \phi_I\|_{\ell,K} \leq C h_K^{2+t-\ell} |\phi|_{2+t,K}, \qquad \ell = 0, 1, 2.$$

We have the following estimation involving the continuous and discrete functionals.

Proposition 4.3. *Let $f \in L^2(\Omega)$ and let $F(\cdot)$ and $F^h(\cdot)$ be the functionals defined in* (2.6) and (3.15)*, respectively. Then under assumption $\mathbf{A_1}$, we have the following estimate:*

$$\|F - F^h\|_{\mathcal{V}_h'} := \sup_{\substack{\phi_h \in \mathcal{V}_h \\ \phi_h \neq 0}} \frac{|F(\phi_h) - F^h(\phi_h)|}{|\phi_h|_{2,h}} \leq C h^2 \|f\|_{0,\Omega}.$$

Proof. The proof follows from the definition of the functionals $F(\cdot)$ and $F^h(\cdot)$, together with approximation property of the projector Π^2_K. □

We finish this subsection presenting some technical lemmas, which will be useful in the next sections. Proof of this results can be obtained following arguments of [40,48,49].

Lemma 4.1. *There exists $\widetilde{C} > 0$, independent of h_K, such that*

$$\|q\|_{0,K} \leq \widetilde{C} h_K^{-i} \|q\|_{-i,K} \qquad \forall q \in \mathbb{P}_\ell(K), \quad \ell \geq 0, \quad i = 1, 2.$$

Lemma 4.2. *If the assumption* $\mathbf{A_1}$ *is satisfied, for each* $\varepsilon > 0$, *there exist positive constants* C, C_ε, *independent of* h_K, *such that*

$$\|\varphi\|_{0,\partial K} \le C_1(\varepsilon h_K^{1/2}|\varphi|_{1,K} + C_\varepsilon h_K^{-1/2}\|\varphi\|_{0,K}) \qquad \forall \varphi \in H^1(K),$$
$$|\varphi|_{1,K} \le C_2(\varepsilon h_K|\varphi|_{2,K} + C_\varepsilon h_K^{-1}\|\varphi\|_{0,K}) \qquad \forall \varphi \in H^2(K).$$

Lemma 4.3. *The projection* $\widetilde{\Pi}_e^0 : L^2(K) \longrightarrow \mathbb{P}_0(e)$ *defined by the following average* $\widetilde{\Pi}_e^0\varphi := \dfrac{1}{h_e}\int_e \varphi ds$, *satisfies*

$$\|\varphi - \widetilde{\Pi}_e^0\varphi\|_{0,e} \le C h_K^{1/2}|\varphi|_{1,K} \qquad \forall \varphi \in H^1(K).$$

4.2. Enriching operator

In this subsection, we will focus on proposing and analyzing an enriching operator E_h from the enhanced nonconforming space $\mathcal{V}_h$ into its H^2-conforming counterpart. Following the ideas of [40], we can construct an enriching operator $E_h : \mathcal{V}_h \longrightarrow \mathcal{V}_h^{\mathrm{C}}$, where $\mathcal{V}_h^{\mathrm{C}}$ is the enhanced H^2-conforming virtual element space considered in [14]. The construction is based on the degrees of freedom of $\mathcal{V}_h^{\mathrm{C}}$.

For the sake of completeness, we will recall the construction of the virtual enhanced H^2-conforming space of lowest order and the enriching operator E_h.

Conforming virtual local space. For every polygon $K \in \mathscr{T}_h$, we introduce the following preliminary finite dimensional space [14]:

$$\widetilde{\mathcal{V}}_h^{\mathrm{C}}(K) := \big\{\phi_h \in H^2(K) : \Delta^2\phi_h \in \mathbb{P}_2(K), \phi_h|_{\partial K} \in C^0(\partial K), \phi_h|_e \in \mathbb{P}_3(e)\ \forall e \subseteq \partial K,$$
$$\nabla\phi_h|_{\partial K} \in [C^0(\partial K)]^2, \partial_{\mathbf{n}_e}\phi_h|_e \in \mathbb{P}_1(e)\ \forall e \subseteq \partial K\big\},$$

Next, for a given $\phi_h \in \widetilde{\mathcal{V}}_h^{\mathrm{C}}(K)$, we introduce two sets $\mathscr{D}_{\mathbf{1}}^{\mathbf{v}}$ and $\mathscr{D}_{\mathbf{2}}^{\nabla}$ of linear operators from the local virtual space $\widetilde{\mathcal{V}}_h^{\mathrm{C}}(K)$ into $\mathbb{R}$:

- $\mathscr{D}_{\mathbf{1}}^{\mathbf{v}}$: the values of $\phi_h(\mathbf{v})$ for all vertex $\mathbf{v} \in \partial K$,
- $\mathscr{D}_{\mathbf{2}}^{\nabla}$: the values of $h_{\mathbf{v}}\nabla\phi_h(\mathbf{v})$ for all vertex $\mathbf{v} \in \partial K$,

where $h_{\mathbf{v}}$ is a characteristic length attached to each vertex $\mathbf{v}$, for instance to the maximum diameter of the elements with $\mathbf{v}$ as a vertex. Now, we consider the operator $\Pi_K^{\mathrm{D,C}} : \widetilde{\mathcal{V}}_h^{\mathrm{C}}(K) \longrightarrow \mathbb{P}_2(K) \subseteq \widetilde{\mathcal{V}}_h^{\mathrm{C}}(K)$ associated to the conforming approach, which is computable using the sets $\mathscr{D}_{\mathbf{1}}^{\mathbf{v}}$ and $\mathscr{D}_{\mathbf{2}}^{\nabla}$ (for more details see [14, Lemma 2.1]).

Next, for each $K \in \mathscr{T}_h$, we introduce the conforming local enhanced virtual space as follows:

$$\mathcal{V}_h^{\mathrm{C}}(K) := \left\{\phi_h \in \widetilde{\mathcal{V}}_h^{\mathrm{C}}(K) : (\phi_h - \Pi_K^{\mathrm{D,C}}\phi_h, q)_{0,K} = 0 \quad \forall q \in \mathbb{P}_2(K)\right\}. \tag{4.1}$$

In this space the sets $\mathscr{D}_{\mathbf{1}}^{\mathbf{v}}$ and $\mathscr{D}_{\mathbf{2}}^{\nabla}$ constitutes a set of degrees of freedom.

Conforming virtual global space. For every decomposition $\mathscr{T}_h$ of Ω into polygons K, we define the conforming virtual spaces $\mathcal{V}_h^{\mathrm{C}}$:

$$\mathcal{V}_h^{\mathrm{C}} := \left\{\phi_h \in \mathcal{V} :\ \phi_h|_K \in \mathcal{V}_h^{\mathrm{C}}(K) \qquad \forall K \in \mathscr{T}_h\right\}. \tag{4.2}$$

For a vertex $\mathbf{v} \in \mathscr{V}_h$, we denote by $\omega(\mathbf{v})$ the union of all elements in $\mathscr{T}_h$, sharing the vertex $\mathbf{v}$ and by $N(\mathbf{v})$ the number of elements of $\omega(\mathbf{v})$.

For any $\varphi_h \in \mathcal{V}_h$, we introduce the piecewise L^2-projection Π^2, as follows:

$$\Pi^2\varphi_h|_K = \Pi_K^2(\varphi_h|_K),$$

where Π_K^2 is the L^2-projection from $\mathcal{V}_h(K)$ onto $\mathbb{P}_2(K)$ (cf. Lemma 3.2) and $\mathcal{V}_h(K)$ is the local nonconforming virtual space defined in (3.4).

For each function $\varphi_h \in \mathcal{V}_h$, the function $E_h\varphi_h \in \mathcal{V}_h^{\mathrm{C}}$ in the conforming counterpart will be constructed as follows:

$$E_h(\varphi_h)(x) = \sum_{i=1}^{N_{\mathrm{dof}}^{\mathrm{C}}} \mathscr{D}_i(E_h(\varphi_h))\chi_i(x),$$

where the functions $\{\chi_i\}_{i=1}^{N_{\mathrm{dof}}^{\mathrm{C}}}$ are the set of shape basis functions associated to space $\mathcal{V}_h^{\mathrm{C}}$ and $N_{\mathrm{dof}}^{\mathrm{C}} := \dim(\mathcal{V}_h^{\mathrm{C}})$. More precisely, the values of degrees of freedom for the enriching operator are determined as follows:

1. For the values at interior vertices $\mathbf{v} \in \mathscr{V}_h^{\mathrm{int}}$, we consider:

$$\mathscr{D}_{\mathbf{1}}^{\mathbf{v}}(E_h\varphi_h) := \frac{1}{N(\mathbf{v})}\sum_{\widetilde{K}\in\omega(\mathbf{v})} \Pi^2\varphi_h|_{\widetilde{K}}(\mathbf{v}).$$

2. For the gradient values at interior vertices $\mathbf{v} \in \mathscr{V}_h^{\text{int}}$, we consider:

$$\mathscr{D}_{\mathbf{1}}^{\nabla}(E_h\varphi_h) := \frac{1}{N(\mathbf{v})} \sum_{\widetilde{K} \in \omega(\mathbf{v})} h_{\mathbf{v}} \nabla(\Pi^2 \varphi_h|_{\widetilde{K}})(\mathbf{v}).$$

We will denote by $\boldsymbol{\chi}(\cdot) := \{\boldsymbol{\chi}_{\mathbf{v}}, \boldsymbol{\chi}_{\nabla}\}$ the degrees of freedom vector corresponding to the H^2-conforming virtual element space $\mathcal{V}_h^{\mathrm{C}}(K)$, with $\boldsymbol{\chi}_{\mathbf{v}}$ collecting the degrees of freedom in $\mathscr{D}_{\mathbf{1}}^{\mathbf{v}}$ and $\boldsymbol{\chi}_{\nabla}$ the degrees of freedom in $\mathscr{D}_{\mathbf{2}}^{\nabla}$.

In what follows, we will derive some approximation properties for the operator E_h. To do that, first we will establish two technical tools: inverse inequalities for the enhanced H^2-conforming virtual space $\mathcal{V}_h^{\mathrm{C}}(K)$ defined in (4.2) and a norm equivalence between the degrees of freedom vector $\boldsymbol{\chi}$ and L^2-norm.

In order to establish the results mentioned above, first we will consider three preliminary lemmas. We start with an H^2-orthogonal decomposition.

Lemma 4.4. *Any function $\phi \in H^2(K)$ admits the decomposition $\phi = \phi_1 + \phi_2$, where*

1. *$\phi_1 \in H^2(K)$, $\phi_1|_{\partial K} = \phi|_{\partial K}$, $\partial_{\mathbf{n}_K}\phi_1 = \partial_{\mathbf{n}_K}\phi$ and $\Delta^2\phi_1 = 0 \quad in \quad K$.*
2. *$\phi_2 \in H_0^2(K)$, $\Delta^2\phi_2 = \Delta^2\phi \quad in \quad K$.*

Moreover, this decomposition is H^2-orthogonal in the sense that

$$|\phi|_{2,K}^2 = |\phi_1|_{2,K}^2 + |\phi_2|_{2,K}^2.$$

Proof. Let $\phi \in H^2(K)$, then we can choose ϕ_2 as the H^2-projection of ϕ to $H_0^2(K)$, i.e., we define $\phi_2 \in H_0^2(K)$ as the unique solution of the following local problem:

$$\int_K \mathrm{D}^2\phi_2 : \mathrm{D}^2\varphi = \int_K \mathrm{D}^2\phi : \mathrm{D}^2\varphi \qquad \forall \varphi \in H_0^2(K).$$

Thus, we define $\phi_1 := (\phi - \phi_2) \in H^2(K)$. We notice that by construction the functions ϕ_1 and ϕ_2 satisfy the properties of lemma. $\square$

For the functions ϕ_1 and ϕ_2 of the above decomposition, we will derive useful inequalities to establish an inverse estimate in the H^2-conforming virtual space $\mathcal{V}_h^{\mathrm{C}}(K)$. For the biharmonic part, we have the next inequality.

Lemma 4.5. *For any $\varepsilon > 0$, there exist positive constants C, C_ε, independent of h_K, such that*

$$|\phi_1|_{2,K} \leq C(\varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K}).$$

Proof. Let $\phi \in \mathcal{V}_h^{\mathrm{C}}(K)$ and $\phi_1 \in H^2(K)$ such that Lemma 4.4 holds true. Then, we define the space

$$S_{\phi_1}(K) := \left\{\varphi \in H^2(K) : \varphi|_{\partial K} = \phi_1|_{\partial K}, \quad \partial_{\mathbf{n}_K}\varphi = \partial_{\mathbf{n}_K}\phi_1\right\}.$$

For each $\varphi \in S_{\phi_1}(K)$ we have that $\varphi - \phi_1 \in H_0^2(K)$. Then, since $\Delta^2\phi_1 = 0 \quad \text{in} \quad K$, applying integration by part we get

$$\int_K \mathrm{D}^2\phi_1 : \mathrm{D}^2(\varphi - \phi_1) = 0,$$

which implies

$$|\varphi|_{2,K}^2 = |\phi_1|_{2,K}^2 + |\varphi - \phi_1|_{2,K}^2.$$

Therefore,

$$|\phi_1|_{2,K} \leq |\varphi|_{2,K} \qquad \forall \varphi \in S_{\phi_1}(K). \tag{4.3}$$

Now, for every $K \in \mathscr{T}_h$, let $\mathcal{T}_K$ be the sub-triangulation obtained by connecting each vertex of K with the center of the ball with respect to which K is starred (cf. Assumption $\mathbf{A_1}$). Then, on each triangle of $\mathcal{T}_K$ we consider the reduced Hsieh–Clough–Tocher element (HCT) defined in [50]. Thus, for $\phi \in \mathcal{V}_h^{\mathrm{C}}(K)$ we choose the interpolant $I_K\phi$ in the HCT element, for which it is fulfilled that

$$I_K\phi|_{\partial K} = \phi|_{\partial K} = \phi_1|_{\partial K} \quad \text{and} \quad \partial_{\mathbf{n}_K}(I_K\phi) = \partial_{\mathbf{n}_K}\phi = \partial_{\mathbf{n}_K}\phi_1.$$

Hence, $I_K\phi \in S_{\phi_1}(K)$ and by the definition of $I_K\phi$, we also have the following estimate

$$\begin{aligned}\|I_K\phi\|_{0,K} &\leq C(h_K^{1/2}\|I_K\phi\|_{0,\partial K} + h_K^{3/2}\|\partial_{\mathbf{n}_K}(I_K\phi)\|_{0,\partial K}) \\ &= C(h_K^{1/2}\|\phi\|_{0,\partial K} + h_K^{3/2}\|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K}).\end{aligned} \tag{4.4}$$

Then, taking $\varphi = I_K\phi \in S_{\phi_1}(K)$ in (4.3) and using the inverse inequality for polynomials (cf. Lemma 4.1) and estimate (4.4), we obtain

$$|\phi_1|_{2,K} \le |I_K\phi|_{2,K} \le Ch_K^{-2}\|I_K\phi\|_{0,K} \le C(h_K^{-3/2}\|\phi\|_{0,\partial K} + h_K^{-1/2}\|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K}). \tag{4.5}$$

Next, we will estimate the two terms on the right hand side of (4.5). Indeed, from Lemma 4.2, for every $\varepsilon > 0$, there exist $C, C_\varepsilon > 0$, independent of h, such that

$$h_K^{-3/2}\|\phi\|_{0,\partial K} \le Ch_K^{-3/2}(\varepsilon h_K^{1/2}|\phi|_{1,K} + C_\varepsilon h_K^{-1/2}\|\phi\|_{0,K}) \le C(\varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K}). \tag{4.6}$$

Now, for the second term in (4.5) we notice that

$$\|\nabla\phi\|_{0,\partial K}^2 = \int_{\partial K}\nabla\phi\cdot\nabla\phi = \|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K}^2 + \|\partial_{\mathbf{t}_K}\phi\|_{0,\partial K}^2 \le (\|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K} + \|\partial_{\mathbf{t}_K}\phi\|_{0,\partial K})^2. \tag{4.7}$$

From the above identity and Lemma 4.2, for every $\varepsilon > 0$, there exist $C, C_\varepsilon > 0$, independent of h, such that

$$h_K^{-1/2}\|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K} \le C(\varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K}). \tag{4.8}$$

Then, the desired result follows inserting the estimates (4.6) and (4.8) in (4.5). $\square$

For the function ϕ_2 of the decomposition in Lemma 4.4, we have the following result.

Lemma 4.6. *For any $\varepsilon > 0$, there exists positive constants C, C_ε, independent of h_K, such that*

$$|\phi_2|_{2,K} \le C(\varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K}).$$

Proof. Let $\phi \in \mathcal{V}_h^{\mathrm{C}}(K)$ and $\phi_2 \in H^2(K)$ such that Lemma 4.4 holds true. Then, since $\phi_2 \in H_0^2(K)$ and $\Delta^2\phi_2 = \Delta^2\phi \in \mathbb{P}_2(K)$ in K, we use an integration by part, the Cauchy–Schwarz and inverse inequalities for polynomials (cf. Lemma 4.1) to obtain

$$|\phi_2|_{2,K}^2 = \int_K \phi_2\Delta^2\phi_2 \le \|\Delta^2\phi_2\|_{0,K}\|\phi_2\|_{0,K} \le Ch_K^{-2}\|\Delta^2\phi_2\|_{-2,K}\|\phi_2\|_{0,K} \le Ch_K^{-2}|\phi_2|_{2,K}\|\phi_2\|_{0,K}.$$

From the above estimate, Lemma 4.4 and the triangle inequality, we get

$$|\phi_2|_{2,K} \le Ch_K^{-2}\|\phi_2\|_{0,K} = Ch_K^{-2}\|\phi - \phi_1\|_{0,K} \le Ch_K^{-2}\|\phi\|_{0,K} + Ch_K^{-2}\|\phi_1\|_{0,K}. \tag{4.9}$$

In what follows, we will establish estimates for the second term on the right hand side in (4.9). Applying the Poincaré–Friedrichs inequality for H^2 functions, Cauchy–Schwarz inequality and using (4.7), we get

$$\begin{aligned}\|\phi_1\|_{0,K} &\le C\left(h_K^2|\phi_1|_{2,K} + \left|\int_{\partial K}\phi_1\right| + h_K\left|\int_{\partial K}\nabla\phi_1\right|\right)\\ &\le C(h_K^2|\phi_1|_{2,K} + h_K^{1/2}\|\phi_1\|_{0,\partial K} + h_K^{3/2}(\|\partial_{\mathbf{n}_K}\phi_1\|_{0,\partial K} + \|\partial_{\mathbf{t}_K}\phi_1\|_{0,\partial K})).\end{aligned} \tag{4.10}$$

Now, we observe that $\phi_1|_e = \phi|_e \in \mathbb{P}_3(e) \quad \forall e \subseteq \partial K$. Then, by using standard inverse estimate for polynomials in one variable, we have

$$\|\partial_{\mathbf{t}_K}\phi_1\|_{0,\partial K} \le Ch_K^{-1}\|\phi_1\|_{0,\partial K}.$$

Thus, inserting the above estimation in (4.10) we get

$$\|\phi_1\|_{0,K} \le C(h_K^2|\phi_1|_{2,K} + h_K^{1/2}\|\phi\|_{0,\partial K} + h_K^{3/2}\|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K}),$$

where we have used the fact that $\phi_1|_{\partial K} = \phi|_{\partial K}$ and $\partial_{\mathbf{n}_K}\phi_1 = \partial_{\mathbf{n}_K}\phi$. Now, employing the above estimates and both inequalities of Lemma 4.2 we get

$$\begin{aligned}\|\phi_1\|_{0,K} &\le (h_K^2|\phi_1|_{2,K} + h_K^{3/2}\|\partial_{\mathbf{n}_K}\phi\|_{0,\partial K} + h_K^{1/2}\|\phi\|_{0,\partial K})\\ &\le C(h_K^2|\phi_1|_{2,K} + \varepsilon h_K^2|\phi|_{2,K} + C_\varepsilon\|\phi\|_{0,K}),\end{aligned}$$

which implies along with Lemma 4.5 that

$$\begin{aligned}h_K^{-2}\|\phi_1\|_{0,K} &\le C(|\phi_1|_{2,K} + \varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K})\\ &\le C(\varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K}).\end{aligned} \tag{4.11}$$

From the estimates (4.9) and (4.11) for any $\varepsilon > 0$, there exists positive constants C, C_ε, independent of h_K, such that

$$|\phi_2|_{2,K} \le C(h_K^{-2}\|\phi\|_{0,K} + Ch_K^{-2}\|\phi_1\|_{0,K}) \le C(\varepsilon|\phi|_{2,K} + C_\varepsilon h_K^{-2}\|\phi\|_{0,K}).$$

The proof is complete. $\square$

We have the following inverse inequalities for the H^2-conforming space $\mathcal{V}_h^{\mathrm{C}}$ defined in (4.1).

Lemma 4.7. *For any $\phi_h \in \mathcal{V}_h^C(K)$, there exists a positive constant C, independent of h_K, such that*

$$|\phi_h|_{2,K} \leq Ch_K^{-2}\|\phi_h\|_{0,K} \quad \text{and} \quad |\phi_h|_{1,K} \leq Ch_K^{-1}\|\phi_h\|_{0,K}. \tag{4.12}$$

Proof. Let $\phi_h \in \mathcal{V}_h^C(K)$ and $\phi_{h,1}, \phi_{h,2}$ such that Lemma 4.4 holds true. Then, employing the triangle inequality together with Lemmas 4.5 and 4.6, we have

$$|\phi_h|_{2,K} \leq |\phi_{h,1}|_{2,K} + |\phi_{h,2}|_{2,K} \leq C(\varepsilon|\phi_h|_{2,K} + C_\varepsilon h_K^{-2}\|\phi_h\|_{0,K}),$$

where C and C_ε are independent of h_K. Then, choosing ε small enough and absorbing the term $C\varepsilon|\phi_h|_{2,K}$ on the left hand side of the above estimate we obtain the first inverse inequality in (4.12). The second inequality in (4.12) is an immediate consequence of the first estimate and Lemma 4.2. □

Now we will establish a norm equivalence between the degrees of freedom vector $\boldsymbol{\chi}$ and the L^2-norm.

Lemma 4.8. *For any $\phi_h \in \mathcal{V}_h^C(K)$, there exist positive constants C_1 and C_2, independent of h_K, such that*

$$C_1 h_K\|\boldsymbol{\chi}(\phi_h)\|_{\ell^2} \leq \|\phi_h\|_{0,K} \leq C_2 h_K\|\boldsymbol{\chi}(\phi_h)\|_{\ell^2}.$$

Proof. Let $\phi_h \in \mathcal{V}_h^C(K)$. Then, for the lower bound, we have the $\phi_h|_{\partial K}$ and $\partial_{\mathbf{n}_K}\phi_h$ are polynomial functions on each edge of ∂K. Therefore, using standard scaling argument we have

$$h_K\|\boldsymbol{\chi}(\phi_h)\|_{\ell^2} \leq C(h_K^{1/2}\|\phi_h\|_{0,\partial K} + h_K^{3/2}\|\partial_{\mathbf{n}_K}\phi_h\|_{0,\partial K}).$$

Now, following similar arguments to those used in Lemmas 4.5 and 4.6, we have

$$h_K\|\boldsymbol{\chi}(\phi_h)\|_{\ell^2} \leq C(h_K^2|\phi_h|_{2,K} + \|\phi_h\|_{0,K}).$$

Therefore, applying the first the inverse inequality in (4.12) we get

$$h_K\|\boldsymbol{\chi}(\phi_h)\|_{\ell^2} \leq C\|\phi_h\|_{0,K}.$$

To obtain the upper bound we proceed as in [40, Lemma 3.6]. □

Employing the above lemmas, we can establish the following approximation properties for the enriching operator E_h, which will play a important role to obtain a priori error estimate of our scheme under minimal regularity condition on the exact solution.

Lemma 4.9. *For all $\varphi_h \in \mathcal{V}_h$, there exists $C > 0$, independent of h, such that*

$$\|\varphi_h - E_h\varphi_h\|_{0,\Omega} + h|\varphi_h - E_h\varphi_h|_{1,h} + h^2|E_h\varphi_h|_{2,\Omega} \leq Ch^2|\varphi_h|_{2,h}.$$

Proof. First we will proof $\|\varphi_h - E_h\varphi_h\|_{0,\Omega} \leq Ch^2|\varphi_h|_{2,h}$. Indeed, for all $\varphi_h \in \mathcal{V}_h$, the function $(\Pi^2\varphi_h - E_h\varphi_h)|_K \in \mathcal{V}_h^C(K)$. Then, by using the triangle inequality, the Bramble–Hilbert Lemma and Lemma 4.8, we get

$$\begin{aligned}\|\varphi_h - E_h\varphi_h\|_{0,K} &\leq \|\varphi_h - \Pi_K^2\varphi_h\|_{0,K} + \|\Pi_K^2\varphi_h - E_h\varphi_h\|_{0,K}\\ &\leq Ch_K^2|\varphi_h|_{2,K} + h_K\|\boldsymbol{\chi}(\Pi_K^2\varphi_h - E_h\varphi_h)\|_{\ell^2}.\end{aligned} \tag{4.13}$$

Now, by using the argument employed in [40, Lemma 4.2], we have that

$$\|\boldsymbol{\chi}(\Pi_K^2\varphi_h - E_h\varphi_h)\|_{\ell^2} \leq h_K|\varphi_h|_{2,\omega(K)},$$

where $\omega(K)$ denote the union of all elements in $\mathscr{T}_h$ sharing a vertex or an edge with K.

Then, inserting the above estimate in (4.13), we obtain

$$\|\varphi_h - E_h\varphi_h\|_{0,K} \leq Ch_K^2|\varphi_h|_{2,\omega(K)} \qquad \forall \varphi_h \in \mathcal{V}_h. \tag{4.14}$$

Next, let $\varphi_\pi \in \mathbb{P}_2(K)$ be the polynomial such that Proposition 4.1 holds true with respect to φ_h. Then, using triangle and inverse inequalities for polynomial and for H^2-conforming space (cf. Lemma 4.7) together with Lemma 4.8, we have

$$\begin{aligned}|\varphi_h - E_h\varphi_h|_{2,K} &\leq |\varphi_h - \varphi_\pi|_{2,K} + |\Pi_K^2(\varphi_\pi - \varphi_h)|_{2,K} + |\Pi_K^2\varphi_h - E_h\varphi_h|_{2,K}\\ &\leq C(|\varphi_h|_{2,K} + h_K^{-2}\|\varphi_\pi - \varphi_h\|_{0,K} + h_K^{-2}\|\Pi_K^2\varphi_h - E_h\varphi_h\|_{0,K})\\ &\leq C(|\varphi_h|_{2,K} + h_K^{-2}h_K^2|\varphi_h|_{2,K} + h_K^{-2}h_K^2|\varphi_h|_{2,\omega(K)})\\ &\leq C|\varphi_h|_{2,\omega(K)}.\end{aligned} \tag{4.15}$$

Thus, summing on each $K \in \mathscr{T}_h$ in (4.14) and (4.15), and using triangle inequality we obtain

$$\|\varphi_h - E_h\varphi_h\|_{0,\Omega} \leq Ch^2|\varphi_h|_{2,h} \quad \text{and} \quad |E_h\varphi_h|_{2,\Omega} \leq C|\varphi_h|_{2,h}. \tag{4.16}$$

On the other hand, using the second inequality in Lemma 4.2 and (4.14), for $(\varphi_h - E_h\varphi_h)|_K \in H^2(K)$, there exists a constant $C > 0$, independent to h_K, such that

$$
\begin{aligned}
|\varphi_h - E_h\varphi_h|_{1,K} &\le C(h_K|\varphi_h - E_h\varphi_h|_{2,K} + h_K^{-1}\|\varphi_h - E_h\varphi_h\|_{0,K}) \\
&\le Ch_K|\varphi_h|_{2,\omega(K)}.
\end{aligned}
$$

Then, from the above inequality, we obtain

$$
|\varphi_h - E_h\varphi_h|_{1,h} \le Ch|\varphi_h|_{2,h}. \tag{4.17}
$$

The proof of the theorem follows from (4.16) and (4.17). □

Now, using Lemma 4.9 we will establish two estimations involving the bilinear form $A_{\mathrm{D}}(\cdot,\cdot)$ defined in (2.3) keeping the notation (3.16).

Lemma 4.10. *Let $\varphi \in H^{2+t}(\Omega)$, with $t \in [0,1]$. Then, for all $\phi_h \in \mathcal{V}_h$ we have*

$$
A_{\mathrm{D}}(\varphi, \phi_h - E_h\phi_h) \le Ch^t\|\varphi\|_{2+t,\Omega}|\phi_h|_{2,h}.
$$

Proof. Following similar arguments in [51, Section 4.1], it is enough to prove the estimation for $t = 0$ and $t = 1$. Indeed, let $\varphi \in H^2(\Omega)$, then for any $\phi_h \in \mathcal{V}_h$, by using the Cauchy–Schwarz inequality and Lemma 4.9, we have

$$
A_{\mathrm{D}}(\varphi, \phi_h - E_h\phi_h) = \sum_{K\in\mathcal{T}_h} A_{\mathrm{D}}^K(\varphi, \phi_h - E_h\phi_h) \le C\|\varphi\|_{2,\Omega}|\phi_h|_{2,h}. \tag{4.18}
$$

Now, let $\varphi \in H^3(\Omega)$. Then, for all $\phi_h \in \mathcal{V}_h$, by using integration by part (see [37]), we have that

$$
\begin{aligned}
A_{\mathrm{D}}(\varphi, \phi_h - E_h\phi_h) &= -\sum_{K\in\mathcal{T}_h}\int_K \nabla(\Delta\varphi)\cdot\nabla(\phi_h - E_h\phi_h) + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\Delta\varphi - \frac{\partial^2\varphi}{\partial\mathbf{t}_K^2}\right)\frac{\partial(\phi_h - E_h\phi_h)}{\partial\mathbf{n}_K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial^2\varphi}{\partial\mathbf{n}_K\partial\mathbf{t}_K}\frac{\partial(\phi_h - E_h\phi_h)}{\partial\mathbf{t}_K} =: T_1 + T_2 + T_3.
\end{aligned} \tag{4.19}
$$

Next, we will bound the terms T_1, T_2 and T_3. Indeed, for the term T_1, we use the Cauchy–Schwarz inequality and Lemma 4.9, we have

$$
T_1 \le \left(\sum_{K\in\mathcal{T}_h}\|\nabla(\Delta\varphi)\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\|\nabla(\phi_h - E_h\phi_h)\|_{0,K}^2\right)^{1/2} \le \|\varphi\|_{3,\Omega}|\phi_h - E_h\phi_h|_{1,h} \le Ch\|\varphi\|_{3,\Omega}|\phi_h|_{2,h}. \tag{4.20}
$$

Now, we will bound the terms T_2 and T_3. For convenience, we set

$$
\zeta_2 := \left(\Delta\varphi - \frac{\partial^2\varphi}{\partial\mathbf{t}_e^2}\right) \qquad \text{and} \qquad \zeta_3 := \frac{\partial^2\varphi}{\partial\mathbf{n}_e\partial\mathbf{t}_e}.
$$

By using the fact that $E_h\phi_h \in \mathcal{V}_h^{\mathrm{C}} \subset C^1(\bar{\Omega}) \cap \mathcal{V}$ and the definition of the space $\mathcal{V}_h$, we have:

$$
\int_e p_0[\![\nabla(\phi_h - E_h\phi_h)\cdot\mathbf{n}_e]\!] = 0 \qquad \forall p_0 \in \mathbb{P}_0(e). \tag{4.21}
$$

Then, for the term T_2 from (4.21), with $p_0 = \widetilde{\Pi}_e^0\zeta_2 \in \mathbb{P}_0(e)$ (cf. Lemma 4.3), we obtain

$$
\begin{aligned}
T_2 &= \sum_{K\in\mathcal{T}_h}\int_{\partial K}\zeta_2\,\nabla(\phi_h - E_h\phi_h)\cdot\mathbf{n}_K = \sum_{e\in\mathcal{E}_h}\int_e(\zeta_2 - \widetilde{\Pi}_e^0\zeta_2)[\![\nabla(\phi_h - E_h\phi_h)\cdot\mathbf{n}_e]\!] \\
&\le \left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|\zeta_2 - \widetilde{\Pi}_e^0\zeta_2\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}|e|\|[\![\nabla(\phi_h - E_h\phi_h)\cdot\mathbf{n}_e]\!]\|_{0,e}^2\right)^{1/2} \\
&\le Ch\|\varphi\|_{3,\Omega}|\phi_h|_{2,h},
\end{aligned} \tag{4.22}
$$

where we have used the trace inequality (cf. Lemma 4.2) and Lemma 4.9.

Since $\phi_h - E_h\phi_h$ is continuous at internal vertices and vanishes at boundary vertices. Then, for all $p_0 \in \mathbb{P}_0(e)$, we obtain

$$
\int_e p_0\left[\!\!\left[\frac{\partial(\phi_h - E_h\phi_h)}{\partial\mathbf{t}_e}\right]\!\!\right] = -\int_e\frac{\partial p_0}{\partial\mathbf{t}_e}[\![\phi_h - E_h\phi_h]\!] + ([\![\phi_h - E_h\phi_h]\!]p_0)(\mathbf{v}_2) - ([\![\phi_h - E_h\phi_h]\!]p_0)(\mathbf{v}_1) = 0,
$$

where we have used the fact that the jump $[\![\phi_h - E_h\phi_h]\!]$ is zero when evaluated at the endpoints $\mathbf{v}_1$ and $\mathbf{v}_2$ of edge $e \in \mathscr{E}_h^{\text{int}}$. Thus, taking $p_0 = \widetilde{\Pi}_e^0\zeta_3 \in \mathbb{P}_0(e)$ in the above identity, we have that

$$T_3 = \sum_{e\in\mathscr{E}_h}\int_e \zeta_3 \left[\!\!\left[\frac{\partial(\phi_h - E_h\phi_h)}{\partial \mathbf{t}_e}\right]\!\!\right] = \sum_{e\in\mathscr{E}_h}\int_e (\zeta_3 - \widetilde{\Pi}_e^0\zeta_3)\left[\!\!\left[\frac{\partial(\phi_h - E_h\phi_h)}{\partial \mathbf{t}_e}\right]\!\!\right]$$
$$\le \left(\sum_{e\in\mathscr{E}_h}|e|^{-1}\|\zeta_3 - \widetilde{\Pi}_e^0\zeta_3\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h}|e|\,\|[\![\nabla(\phi_h - E_h\phi_h)\cdot\mathbf{t}_K]\!]\|_{0,e}^2\right)^{1/2}.$$

Now, employing the same arguments used to obtain the estimation (4.22), we get

$$T_3 \le Ch\|\varphi\|_{3,\Omega}|\phi_h|_{2,h}. \tag{4.23}$$

Inserting (4.20), (4.22) and (4.23) in (4.19), we obtain

$$A_{\mathrm{D}}(\varphi, \phi_h - E_h\phi_h) \le Ch\|\varphi\|_{3,\Omega}|\phi_h|_{2,h} \qquad \forall\varphi \in H^3(\Omega). \tag{4.24}$$

Then, from (4.18), (4.24) and the real method of interpolation (see [51, Equation (4.2)] and [46]), we have the desired result. □

Furthermore, for the bilinear form $A_{\mathrm{D}}(\cdot,\cdot)$ we have the following result.

Lemma 4.11. *For $\varphi \in H^{2+t}(\Omega)$ and $\chi \in H^{2+t}(\Omega)\cap\mathcal{V}$, with $t \in [0,1]$, it holds:*

$$A_{\mathrm{D}}(\varphi, \chi - \chi_I) \le Ch^{2t}\|\varphi\|_{2+t,\Omega}\|\chi\|_{2+t,\Omega},$$

where $\chi_I \in \mathcal{V}_h$ is the interpolant of χ in the virtual space $\mathcal{V}_h$ (cf. Proposition 4.2).

Proof. Let $\varphi \in H^2(\Omega)$ and $\chi \in \mathcal{V}$. Then, by using Proposition 4.2 we have that

$$A_{\mathrm{D}}(\varphi, \chi - \chi_I) = \sum_{K\in\mathscr{T}_h} A_{\mathrm{D}}^K(\varphi, \chi - \chi_I) \le C\|\varphi\|_{2,\Omega}|\chi - \chi_I|_{2,h} \le C\|\varphi\|_{2,\Omega}\|\chi\|_{2,\Omega}.$$

Now, for $\varphi \in H^3(\Omega)$ and $\chi \in H^3(\Omega)\cap\mathcal{V}$, the proof follows from the same arguments used in Lemma 4.10, setting $\chi \in C^1(\bar{\Omega})\cap\mathcal{V}$ and $\chi_I \in \mathcal{V}_h$ instead of $E_h\phi_h$ and ϕ_h, respectively, and employing Proposition 4.2. Indeed, using an integration by part as in (4.19), we have that

$$A_{\mathrm{D}}(\varphi, \chi - \chi_I) = -\sum_{K\in\mathscr{T}_h}\int_K \nabla(\Delta\varphi)\cdot\nabla(\chi - \chi_I) + \sum_{K\in\mathscr{T}_h}\int_{\partial K}\left(\Delta\varphi - \frac{\partial^2\varphi}{\partial\mathbf{t}_K^2}\right)\frac{\partial(\chi - \chi_I)}{\partial\mathbf{n}_K}$$
$$+ \sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\partial^2\varphi}{\partial\mathbf{n}_K\partial\mathbf{t}_K}\frac{\partial(\chi - \chi_I)}{\partial\mathbf{t}_K} =: T_1^A + T_2^A + T_3^A.$$

For the term T_1^A, we use the Cauchy–Schwarz inequality and Proposition 4.2, to obtain

$$T_1^A \le Ch^2\|\varphi\|_{3,\Omega}\|\chi\|_{3,\Omega},$$

while the terms T_2^A and T_3^A are bounded as in Lemma 4.10 and using Proposition 4.2, as follows:

$$T_2^A + T_3^A \le Ch^2\|\varphi\|_{3,\Omega}\|\chi\|_{3,\Omega}.$$

The proof follows by combining the above estimates and the real method of interpolation (see [51, Equation (4.2)] and [46]). □

4.3. A priori estimation

In this subsection we will establish an error estimate in broken H^2-norm under minimal regularity condition on the exact stream-function ψ, i.e., $\psi \in H^{2+s}(\Omega)$, with $s \in (1/2, 1]$ (cf. Theorem 2.2).

First, we start noticing that for all $\phi_h \in \mathcal{V}_h$ the consistency error (also called nonconformity error) is given by:

$$\mathcal{N}_h(\psi, \phi_h) := A(\psi, \phi_h) - F(\phi_h), \tag{4.25}$$

where $\psi \in H^{2+s}(\Omega)\cap\mathcal{V}$ is the solution of problem (2.1). Moreover, we have the following estimation for the consistency error $\mathcal{N}_h(\psi,\cdot)$ defined above.

Lemma 4.12. *Let ψ be the solution of problem (2.1). Then, for all $\phi_h \in \mathcal{V}_h$, there exists a constant $C > 0$, independent to h, such that*

$$\mathcal{N}_h(\psi, \phi_h) \leq Ch^s(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega})|\phi_h|_{2,h},$$

where $\mathcal{N}_h(\psi, \cdot)$ is the consistency error defined by the relation (4.25) and $s \in (1/2, 1]$ is such that $\psi \in H^{2+s}(\Omega) \cap \mathcal{V}$ (cf. Theorem 2.2).

Proof. For all $\phi_h \in \mathcal{V}_h$, we have that $E_h\phi_h \in \mathcal{V}_h^C \subset \mathcal{V}$. Then, taking $E_h\phi_h$ as test function in (2.1), we obtain

$$A(\psi, E_h\phi_h) = F(E_h\phi_h). \tag{4.26}$$

Thus, from (4.25) and (4.26), we get

$$\begin{aligned}
\mathcal{N}_h(\psi, \phi_h) &:= A(\psi, \phi_h) - F(\phi_h) = A(\psi, \phi_h) - F(\phi_h - E_h\phi_h) - F(E_h\phi_h) \\
&= \epsilon_M A_{\mathrm{D}}(\psi, \phi_h - E_h\phi_h) + \epsilon_S A_{\nabla}(\psi, \phi_h - E_h\phi_h) \\
&\quad - A_{\mathrm{skew}}(\psi, \phi_h - E_h\phi_h) - F(\phi_h - E_h\phi_h).
\end{aligned}$$

From the above identity, the Cauchy–Schwarz inequality, continuity of the forms $A_{\nabla}(\cdot,\cdot)$, $A_{\mathrm{skew}}(\cdot,\cdot)$ and $F(\cdot)$, Lemmas 4.10 and 4.9, we get

$$\begin{aligned}
\mathcal{N}_h(\psi, \phi_h) &\leq C\epsilon_M h^s \|\psi\|_{2+s,\Omega}|\phi_h|_{2,h} + \epsilon_S|\psi|_{1,\Omega}|\phi_h - E_h\phi_h|_{1,h} \\
&\quad + C(|\phi_h - E_h\phi_h|_{1,h} + \|\phi_h - E_h\phi_h\|_{0,\Omega})\|\psi\|_{2,\Omega} + C\|f\|_{0,\Omega}\|\phi_h - E_h\phi_h\|_{0,\Omega} \\
&\leq Ch^s(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega})|\phi_h|_{2,h}.
\end{aligned}$$

The proof is complete. $\square$

We have the following Strang-type result.

Lemma 4.13. *Under the mesh assumptions $\mathbf{A_1} - \mathbf{A_2}$. Let ψ and ψ_h be the unique solutions to problems (2.1) and (3.17), respectively. Then, for each approximation ψ_I of ψ in $\mathcal{V}_h$ and for every approximation ψ_π of ψ in $\mathbb{P}_2(\mathscr{T}_h)$, there exists a positive constant C, independent of h, such that*

$$|\psi - \psi_h|_{2,h} \leq C\Big(|\psi - \psi_I|_{2,h} + |\psi - \psi_\pi|_{2,h} + \|F - F^h\|_{\mathcal{V}_h'} + \sup_{\substack{\phi_h \in \mathcal{V}_h \\ \phi_h \neq 0}} \frac{\mathcal{N}_h(\psi, \phi_h)}{|\phi_h|_{2,h}}\Big),$$

where $\mathcal{N}_h(\psi, \phi_h)$ is the consistency error defined in (4.25).

Proof. Let $\psi_I \in \mathcal{V}_h$ be the interpolant of ψ such that Proposition 4.2 holds true. We set $\delta_h := (\psi_h - \psi_I) \in \mathcal{V}_h$. Then,

$$|\psi - \psi_h|_{2,h} \leq |\psi - \psi_I|_{2,h} + |\delta_h|_{2,h}. \tag{4.27}$$

By using the property (3.18) and the consistency of bilinear form $A^{h,K}(\cdot,\cdot)$ (cf. (3.12)), we have

$$\begin{aligned}
\widetilde{\alpha}|\delta_h|_{2,h}^2 &\leq A^h(\delta_h, \delta_h) = A^h(\psi_h, \delta_h) - A^h(\psi_I, \delta_h) \\
&= F^h(\delta_h) - F(\delta_h) - \mathcal{N}_h(\psi, \delta_h) - \sum_{K \in \mathscr{T}_h} A^{h,K}(\psi_I - \psi_\pi, \delta_h) + \sum_{K \in \mathscr{T}_h} A^K(\psi - \psi_\pi, \delta_h).
\end{aligned}$$

From the above it follows that

$$|\delta_h|_{2,h} \leq C\Big(|\psi - \psi_I|_{2,h} + |\psi - \psi_\pi|_{2,h} + \|F - F^h\|_{\mathcal{V}_h'} + \sup_{\substack{\phi_h \in \mathcal{V}_h \\ \phi_h \neq 0}} \frac{\mathcal{N}_h(\psi, \phi_h)}{|\phi_h|_{2,h}}\Big). \tag{4.28}$$

Thus, from (4.27) and (4.28), we conclude the proof. $\square$

The following theorem provides the rate of convergence of our virtual element scheme.

Theorem 4.1. *Under the mesh assumption $\mathbf{A_1} - \mathbf{A_2}$. Let ψ and ψ_h be the unique solutions of problem (2.1) and problem (3.17), respectively. Then, there exists a positive constant C, independent of h, such that*

$$|\psi - \psi_h|_{2,h} \leq Ch^s(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega}),$$

where $s \in (1/2, 1]$ is such that $\psi \in H^{2+s}(\Omega) \cap \mathcal{V}$ (cf. Theorem 2.2).

Proof. The proof follows combining Lemma 4.13, Propositions 4.1, 4.2, 4.3 and Lemma 4.12. $\square$

4.4. Error estimate in H^1- and L^2

In this section we establish error estimates in broken H^1- and L^2-norms for the stream-function using duality arguments, under same regularity of the weak solution ψ and of the source term f stated in Theorem 4.1.

Theorem 4.2. *Under the mesh assumption* $\mathbf{A_1} - \mathbf{A_2}$. *Let ψ and ψ_h be the unique solutions of problems (2.1) and (3.17), respectively. Then, there exists a positive constant C, independent of h, such that*

$$\|\psi - \psi_h\|_{0,\Omega} + |\psi - \psi_h|_{1,h} \leq Ch^{2s}(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega}), \tag{4.29}$$

where $s \in (1/2, 1]$ is such that $\psi \in H^{2+s}(\Omega) \cap \mathcal{V}$ (cf. Theorem 2.2).

Proof. In order to prove the H^1 estimate in (4.29), let $\psi_I \in \mathcal{V}_h$ be the interpolant of ψ such that Proposition 4.2 holds true. We set $\delta_h := (\psi_h - \psi_I) \in \mathcal{V}_h$. Then, we write

$$\psi_h - \psi = (\psi_h - \psi_I) + (\psi_I - \psi) = (\psi_I - \psi) + (\delta_h - E_h\delta_h) + E_h\delta_h.$$

Thus, by using the triangle inequality together Proposition 4.2, Lemma 4.9 and Theorem 4.1, we have

$$\begin{aligned} |\psi - \psi_h|_{1,h} &\leq |\psi - \psi_I|_{1,h} + |\delta_h - E_h\delta_h|_{1,h} + |E_h\delta_h|_{1,h} \\ &\leq Ch^{2s}\|\psi\|_{2+s,\Omega} + \|\nabla E_h\delta_h\|_{0,\Omega}. \end{aligned} \tag{4.30}$$

In what follows, we will estimate the term $\|\nabla E_h\delta_h\|_{0,\Omega}$. To do that, we consider the following auxiliary problem: seek $\phi \in \mathcal{V}$, such that

$$A(w, \phi) = \int_\Omega \nabla(E_h\delta_h) \cdot \nabla w \qquad \forall w \in \mathcal{V}, \tag{4.31}$$

where the bilinear form $A(\cdot, \cdot)$ is defined in (2.2). From Theorem 2.2, we have that $\phi \in H^{2+s}(\Omega) \cap \mathcal{V}$ and

$$\|\phi\|_{2+s,\Omega} \leq C\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.32}$$

where $C > 0$ is a constant independent of h.

Then, taking $w = E_h\delta_h \in \mathcal{V}_h^C \subset \mathcal{V}$ as test function, adding and subtracting δ_h in problem (4.31), we obtain

$$\|\nabla E_h\delta_h\|_{0,\Omega}^2 = A(E_h\delta_h, \phi) = A(E_h\delta_h - \delta_h, \phi) + A(\delta_h, \phi) =: T_1 + T_2. \tag{4.33}$$

We will estimate the terms T_1 and T_2 in the above identity. Indeed, for the T_1, we use the definition of bilinear form $A(\cdot, \cdot)$, Proposition 4.2, together with Lemma 4.10, Theorem 4.1 and the triangle inequality, to obtain

$$\begin{aligned} T_1 &:= \epsilon_M A_{\mathrm{D}}(E_h\delta_h - \delta_h, \phi) + \epsilon_S A_\nabla(E_h\delta_h - \delta_h, \phi) - A_{\mathrm{skew}}(E_h\delta_h - \delta_h, \phi) \\ &\leq Ch^s\|\phi\|_{2+s,\Omega}|\delta_h|_{2,h} + |E_h\delta_h - \delta_h|_{1,h}|\phi|_{1,\Omega} \\ &\quad + C(|E_h\delta_h - \delta_h|_{1,h}\|\phi\|_{0,\Omega} + |\phi|_{1,\Omega}\|E_h\delta_h - \delta_h\|_{0,\Omega}) \\ &\leq Ch^s h^s(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega})\|\phi\|_{2+s,\Omega}. \end{aligned}$$

Then, from the above estimate and (4.32) we obtain

$$T_1 \leq Ch^{2s}(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega})\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.34}$$

To bound the term T_2, we consider $\phi_I \in \mathcal{V}_h$ the interpolant of ϕ such that Proposition 4.2 holds true. Then, rewriting $\delta_h = (\psi_h - \psi) + (\psi - \psi_I)$, adding and subtracting ϕ_I, and using the bilinearity of form $A(\cdot, \cdot)$, we obtain

$$\begin{aligned} T_2 &:= A(\delta_h, \phi) = A(\psi - \psi_I, \phi) + A(\psi_h - \psi, \phi - \phi_I) + A(\psi_h - \psi, \phi_I) \\ &= T_2^a + T_2^b + T_2^c. \end{aligned} \tag{4.35}$$

Now, we will estimate each term in (4.35). Indeed, we use again the definition of bilinear form $A(\cdot, \cdot)$, Proposition 4.2, Lemma 4.11 and Theorem 4.1, to obtain

$$\begin{aligned} T_2^a &:= \epsilon_M A_{\mathrm{D}}(\psi - \psi_I, \phi) + \epsilon_S A_\nabla(\psi - \psi_I, \phi) - A_{\mathrm{skew}}(\psi - \psi_I, \phi) \\ &\leq Ch^{2s}\|\phi\|_{2+s,\Omega}\|\psi\|_{2+s,\Omega} + |\psi_I - \psi|_{1,h}|\phi|_{1,\Omega} \\ &\quad + C(|\phi|_{1,\Omega}\|\psi_I - \psi\|_{0,\Omega} + \|\phi\|_{0,\Omega}|\psi_I - \psi|_{1,h}) \\ &\leq Ch^{2s}\|\psi\|_{2+s,\Omega}\|\nabla E_h\delta_h\|_{0,\Omega}. \end{aligned} \tag{4.36}$$

For T_2^b, we use the continuity of bilinear form $A(\cdot, \cdot)$, Proposition 4.2, Theorem 4.1 and (4.32), to get

$$T_2^b := A(\psi_h - \psi, \phi - \phi_I) \leq Ch^{2s}(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega})\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.37}$$

Finally, we will bound the term T_2^c in (4.35), as follow: we use the bilinearity of form $A(\cdot,\cdot)$, add and subtract adequate terms, and we use the fact that $A^h(\psi_h,\phi_I)=F^h(\phi_I)$ and $A(\psi,\phi)=F(\phi)$, to get

$$\begin{aligned} T_2^c &:= A(\psi_h-\psi,\phi_I)=A(\psi_h,\phi_I)-A(\psi,\phi_I)\\ &=(A(\psi_h,\phi_I)-A^h(\psi_h,\phi_I))+(F^h(\phi_I)-F(\phi_I))+F(\phi_I-\phi)+A(\psi,\phi-\phi_I). \end{aligned} \tag{4.38}$$

Next, from (4.36), continuity of functional $F(\cdot)$ and Proposition 4.2, we have

$$A(\psi,\phi-\phi_I)+F(\phi_I-\phi)\le Ch^{2s}(\|\psi\|_{2+s,\Omega}+\|f\|_{0,\Omega})\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.39}$$

By using the definition of the functionals $F(\cdot)$ and $F^h(\cdot)$ (cf. (2.6) and (3.15)), approximation properties of the projector Π_K^2, the Hölder and triangle inequalities together with Proposition 4.2, we have

$$F^h(\phi_I)-F(\phi_I)\le\sum_{K\in\mathscr{T}_h}\|f\|_{0,K}\|\phi_I-\Pi_K^2\phi_I\|_{0,K}\le Ch^{2s}\|f\|_{0,\Omega}\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.40}$$

The last term in (4.38) is bounded as follow: let ψ_π,ϕ_π be the approximations of ψ and ϕ in $\mathbb{P}_2(\mathscr{T}_h)$, such that Proposition 4.1 hold true. Then, adding and subtracting these terms and by using the consistency of bilinear form $A(\cdot,\cdot)$ (cf. (3.12)), we have:

$$\begin{aligned} A(\psi_h,\phi_I)-A^h(\psi_h,\phi_I) &= \sum_{K\in\mathscr{T}_h}[A^K(\psi_h-\psi_\pi,\phi_I-\phi_\pi)-A^{h,K}(\psi_h-\psi_\pi,\phi_I-\phi_\pi)]\\ &\le Ch^{2s}(\|\psi\|_{2+s,\Omega}+\|f\|_{0,\Omega})\|\nabla E_h\delta_h\|_{0,\Omega}, \end{aligned} \tag{4.41}$$

where we have used the continuity of bilinear forms $A^{h,K}(\cdot,\cdot)$, $A^K(\cdot,\cdot)$ together with Propositions 4.1 and 4.2, Theorem 4.1 and estimate (4.32). Thus, from (4.39)–(4.41), we obtain

$$T_2^c\le Ch^{2s}(\|\psi\|_{2+s,\Omega}+\|f\|_{0,\Omega})\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.42}$$

Then, inserting the estimates (4.36), (4.37) and (4.42) in (4.35), we have that

$$T_2\le Ch^{2s}(\|\psi\|_{2+s,\Omega}+\|f\|_{0,\Omega})\|\nabla E_h\delta_h\|_{0,\Omega}. \tag{4.43}$$

Therefore, from (4.33), (4.34) and (4.43), we get

$$\|\nabla E_h\delta_h\|_{0,\Omega}\le Ch^{2s}(\|\psi\|_{2+s,\Omega}+\|f\|_{0,\Omega}). \tag{4.44}$$

Thus, the H^1 estimate in (4.29) follows from (4.30) and (4.44).

On the other hand, the L^2 estimate in (4.29) follows from the triangle inequality, Proposition 4.2, Lemma 4.9 and Theorem 4.1. In fact,

$$\begin{aligned} \|\psi-\psi_h\|_{0,\Omega} &\le \|\psi-\psi_I\|_{0,\Omega}+\|\delta_h-E_h\delta_h\|_{0,\Omega}+\|E_h\delta_h\|_{0,\Omega}\\ &\le Ch^{2+s}\|\psi\|_{2+s,\Omega}+Ch^2(|\psi_h-\psi|_{2,h}+|\psi-\psi_I|_{2,h})+C|E_h\delta_h|_{1,\Omega}\\ &\le Ch^{2s}(\|\psi\|_{2+s,\Omega}+\|f\|_{0,\Omega}), \end{aligned}$$

where we have used norm equivalence in $\mathcal{V}$ and estimate (4.44). The proof is complete. □

5. Computing further fields of interest

In this section we compute discrete velocity and vorticity fields using the discrete stream-function obtained with the nonconforming virtual scheme (3.17) and suitable projections, which are computable from the degrees of freedom $\mathbf{D_1}$ and $\mathbf{D_2}$. Moreover, we establish error estimates for these postprocessed variables, which are fields of great importance in oceanic fluid dynamics [1–3,7,9].

5.1. Computing the velocity field

We begin with the horizontal fluid velocity. First, we notice that if $\psi\in\mathcal{V}$ is the unique solution of the weak formulation (2.1), then

$$\mathbf{u}=\mathbf{curl}\,\psi. \tag{5.1}$$

At the discrete level, we compute the velocity as a post-processing of the discrete stream-function ψ_h as follow: if ψ_h is the unique solution of (3.17). Then, the function

$$\mathbf{u}_h:=\boldsymbol{\Pi}^1\mathbf{curl}\,\psi_h \tag{5.2}$$

is a computable approximation of the velocity, where we have used the notation

$$(\boldsymbol{\Pi}^1\mathbf{w})|_K=\boldsymbol{\Pi}_K^1(\mathbf{w}|_K)\qquad\forall\mathbf{w}\in[L^2(\Omega)]^2\quad\text{and}\quad\forall K\in\mathscr{T}_h.$$

We observe that the function (5.2) is computable using the degrees of freedom $\mathbf{D_1}$ and $\mathbf{D_2}$ introduced in Section 3.2. Indeed, applying integration by parts, for all $\mathbf{q} \in [\mathbb{P}_1(K)]^2$ we have

$$\int_K \mathbf{curl}\,\psi_h \cdot \mathbf{q} = \int_K \psi_h \mathrm{rot}\mathbf{q} - \int_{\partial K} \phi_h(\mathbf{q}\cdot\mathbf{t}_K) = \mathrm{rot}\mathbf{q}\int_K (\Pi_K^2\psi_h) - \int_{\partial K}\phi_h(\mathbf{q}\cdot\mathbf{t}_K).$$

Clearly, both terms above are computable using the sets $\mathbf{D_1}$ and $\mathbf{D_2}$.

The following result establishes the order of convergence between the exact and the discrete velocity:

Theorem 5.1. *Assume that the hypotheses of Theorem 4.1 hold true, then there exists a positive constant C, independent of h, such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} + h^s|\mathbf{u}-\mathbf{u}_h|_{1,h} \le Ch^{2s}(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega}),$$

where $s \in (1/2, 1]$ is such that $\psi \in H^{2+s}(\Omega)\cap\mathcal{V}$ (cf. Theorem 2.2).

Proof. From (5.1) and (5.2), the triangle inequality, stability and approximation properties of projector $\boldsymbol{\Pi}_K^1$, we have:

$$\begin{aligned}
|\mathbf{u}-\mathbf{u}_h|_{1,h}^2 &= \sum_{K\in\mathscr{T}_h} |\mathbf{curl}\,\psi - \boldsymbol{\Pi}_K^1\mathbf{curl}\,\psi_h|_{1,K}^2 \\
&\le C\left(\sum_{K\in\mathscr{T}_h} h_K^{2s}\|\mathbf{curl}\,\psi\|_{1+s,K}^2 + \sum_{K\in\mathscr{T}_h}|\boldsymbol{\Pi}_K^1\mathbf{curl}\,(\psi-\psi_h)|_{1,K}^2\right) \\
&\le Ch^{2s}\left(\|\psi\|_{2+s,\Omega}^2 + \|f\|_{0,\Omega}^2\right),
\end{aligned}$$

where we have used Theorem 4.1.

The proof of the L^2 estimate is obtained repeating the above arguments and using Theorem 4.2. □

5.2. Computing the vorticity field

Now, we will present an strategy to compute the fluid vorticity ω as a postprocess from the discrete stream-function ψ_h of the VEM (3.17) by using the projection Π_K^0 defined in (3.6), with $m = 0$.

We recall that the vorticity $\omega = \mathrm{rot}\mathbf{u}$, then using the identity $\mathbf{u} = \mathbf{curl}\,\psi$, we get

$$\omega = \mathrm{rot}\mathbf{u} = \mathrm{rot}(\mathbf{curl}\,\psi) = -\Delta\psi. \tag{5.3}$$

We compute a discrete vorticity as follows: if $\psi_h \in \mathcal{V}_h$ is the unique solution of (3.17), then the function

$$\omega_h := -\Pi^0(\Delta\psi_h) \tag{5.4}$$

is an approximation of the fluid vorticity, where we have used the notation

$$(\Pi^0 v)|_K = \Pi_K^0(v|_K) \qquad \forall v \in L^2(\Omega) \quad \text{and} \quad \forall K \in \mathscr{T}_h.$$

We observe that the function defined (5.4) is fully computable using directly the degree of freedom $\mathbf{D_2}$. Indeed, by using the definition of Π_K^0 and integration by parts, we obtain

$$\Pi_0^K\Delta\psi_h = \frac{1}{|K|}\int_{\partial K}\partial_{\mathbf{n}_K}\psi_h,$$

where $|K|$ denotes the area of polygon K.

We have the following convergence result for the discrete vorticity.

Theorem 5.2. *Assume that the hypotheses of Theorem 4.1 hold true, then there exists a positive constant C, independent of h, such that*

$$\|\omega-\omega_h\|_{0,\Omega} \le Ch^s(\|\psi\|_{2+s,\Omega} + \|f\|_{0,\Omega}),$$

where $s \in (1/2, 1]$ is such that $\psi \in H^{2+s}(\Omega)\cap\mathcal{V}$ (cf. Theorem 2.2).

Proof. The proof follows from (5.3), (5.4) and the same arguments used in Theorem 5.1. □

Remark 5.1. We note that following the same arguments as in this section we can recover the *potential vorticity* variable, given by (see [4, Equation (6)]):

$$q := -\frac{U}{\beta L^2}\Delta\psi + y,$$

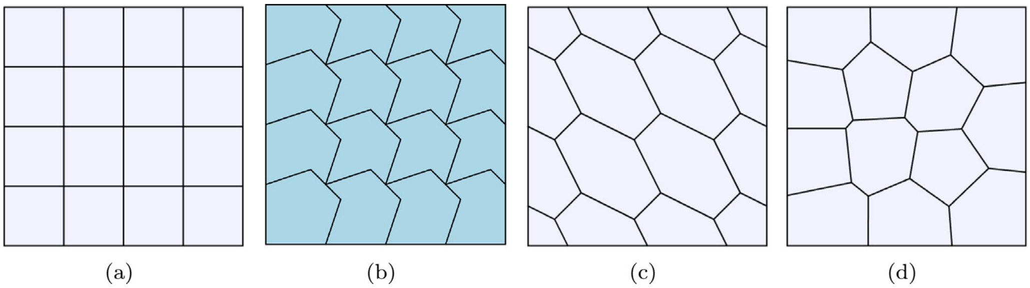


Fig. 1. Domain discretized with different meshes: (a) Square, (b) non-convex mesh, (c) uniform polygon and (d) Voronoi mesh.

where L is the characteristic length scale, β is the coefficient multiplying the y-coordinate in the β-plane approximation and U is the Sverdrup velocity (for further details see for instance [4]). A result analogous to Theorem 5.2 can be proven in this case.

6. Numerical results

In this section, we would like to discuss three numerical experiments to justify our theoretical estimates derived in Sections 4 and 5. Numerical experiments are performed over different type of polygonal meshes such as square, non-convex mesh, uniform polygon, and Voronoi mesh (see Fig. 1). For all test cases, errors are computed in broken H^2-, H^1- and L^2-norms and the Munk and Stommel parameters are chosen as $\epsilon_M = 6 \times 10^{-5}$ and $\epsilon_S = 0.05$, respectively, for the first and third test, while for the second example we have set $\epsilon_M = \epsilon_S = 1$. In the first numerical test, we have considered a solution with a boundary layer on the left hand side. In second example, we consider a non-convex L-shaped domain to justify theoretical rate of convergence in different norms. In the third test we investigate the behavior of our scheme considering a realistic problem with the wind forcing term. In addition, by using post-processing technique, we have computed discrete *velocity* and *vorticity* fields from discrete stream-function ψ_h as described in Section 5. We compute the errors for stream-function ψ, velocity $\mathbf{u}$, and vorticity ω fields in different computable norms as follows

- $\mathcal{E}_i(\psi) := |\psi - \Pi^{\mathrm{D}}\psi_h|_{i,h} \qquad \forall i \in \{0, 1, 2\}$;
- $\mathcal{E}_i(\mathbf{u}) := |\mathbf{u} - \mathbf{u}_h|_{i,h} = |\mathbf{curl}\,\psi - \boldsymbol{\Pi}^1\mathbf{curl}\,\psi_h|_{i,h} \qquad \forall i \in \{0, 1\}$;
- $\mathcal{E}_0(\omega) := \|\omega - \omega_h\|_{0,\Omega} = \|\Delta\psi - \Pi^0(\Delta\psi_h)\|_{0,\Omega}$.

Further, we introduce, the notation $\mathcal{R}_i(\eta)$, $i \in \{0, 1, 2\}$ to denote the rate of convergence in broken H^2-, H^1- and L^2-norms, where $\eta \in \{\psi, \mathbf{u}, \omega\}$.

6.1. Test 1. Western boundary layer

Inspired by [3,4], we have examined western boundary layer model problem on square domain $\Omega := (0, 1)^2$. The analytical solution is given by

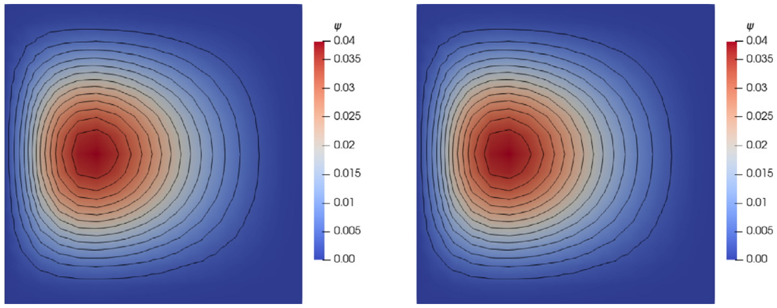
$$\psi(x, y) = \frac{1}{\pi^2}\left((1 - x)(1 - e^{-5x})\sin(\pi y)\right)^2.$$

Further, the right hand side force function f is computed using (1.1).

In Fig. 2, we have posted the discrete stream-function ψ_h and exact stream-function ψ and it is noticed that a thin boundary layer appeared near $x = 0$, corresponding to a western boundary layer. In Fig. 3, the approximated and exact vorticity of the above mentioned problem are displayed. The rate of convergence of stream-function ψ_h is displayed in Fig. 4. Using post-processing technique, we approximate corresponding velocity $\mathbf{u}$, and vorticity field ω in Fig. 5. In continuation, we would like to highlight that the presence of small coefficients Munk and Stommel parameters affect the decay of errors in different norms for stream-function as well as velocity and vorticity field for coarse meshes which eventually reduce the rate of convergence as shown in Fig. 4 and in Fig. 5, respectively. However, for finer mesh, experimental order of convergence matches with the theoretical order of convergence.

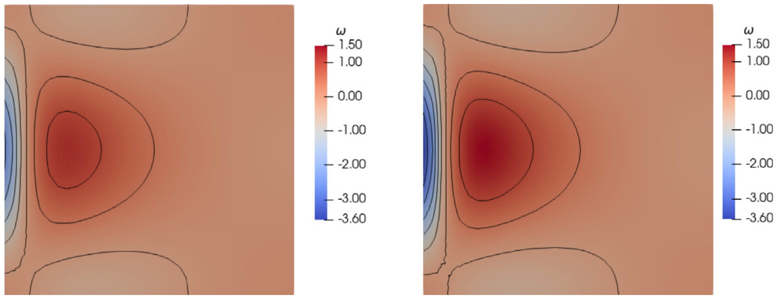
6.2. Test 2. L-shaped domain with exact solution

In this example we solve the Stommel–Munk model (1.1) on an L-shaped domain: $\Omega := (-1, 1)^2 \setminus ([0, 1) \times (-1, 0])$. For the experiment, we have considered a triangular mesh with coefficients $\epsilon_M = 1$ and $\epsilon_S = 1$ and we take the right hand



(a) Numerical solution (b) Exact solution

Fig. 2. Test 1. Numerical approximation of stream function of western boundary layer problem. Computational domain is discretized with square mesh with mesh size $h = 1/64$.



(a) Numerical solution (b) Exact solution

Fig. 3. Test 1. Numerical approximation of vorticity of western boundary layer problem. Computational domain is discretized with square mesh with mesh size $h = 1/64$.

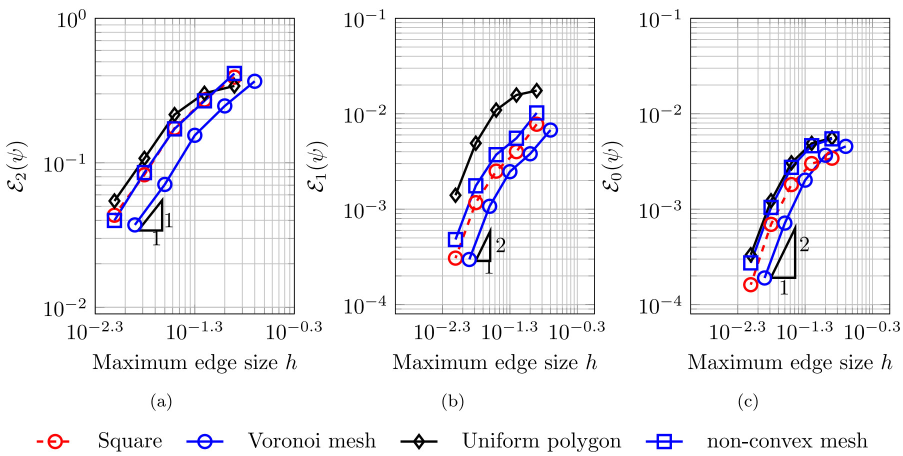


Fig. 4. Test 1: Convergence of the stream-function ψ in broken H^2-, H^1- and L^2-norms with mesh refinement for different types of discretization.

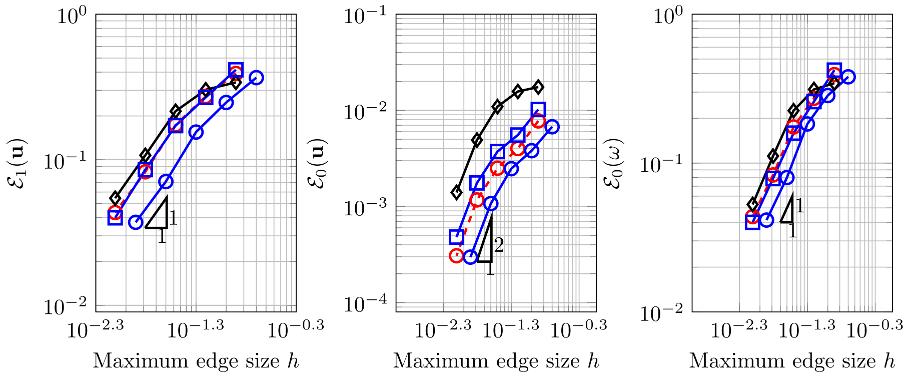
side term and nonhomogeneous Dirichlet boundary conditions in such a way that the exact solution in polar coordinates is given by

$$\psi(r, \theta) = r^{5/3} \sin\left(\frac{5\theta}{3}\right).$$

The analytical solution ψ is singular at the re-entrant corner of the computational domain Ω. Further, we have $\psi \in H^{\frac{8}{3}-\epsilon}(\Omega)$ for $\epsilon > 0$. From the analysis and according to the regularity it is predicted that the order of convergence for stream-function ψ is $\mathcal{O}(h^{2/3})$ in broken H^2-norm and which is clearly observed in Table 1. In the same table it can be

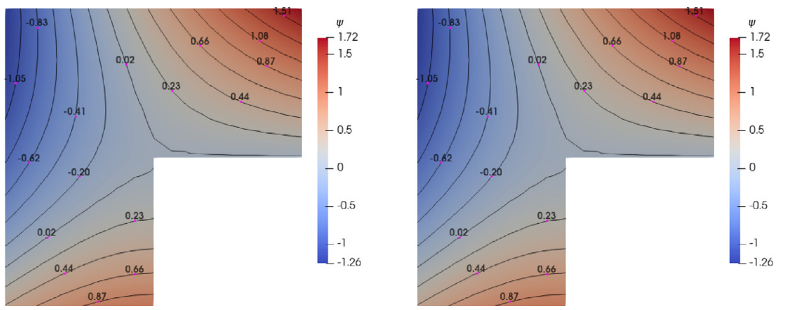
Table 1
Test 2. Errors for the stream-function in broken H^2-, H^1- and L^2-norms obtained with $\epsilon_M = 1$ and $\epsilon_S = 1$.

h	$\mathcal{E}_2(\psi)$	$\mathcal{R}_2(\psi)$	$\mathcal{E}_1(\psi)$	$\mathcal{R}_1(\psi)$	$\mathcal{E}_0(\psi)$	$\mathcal{R}_0(\psi)$	$\mathcal{E}_1(\mathbf{u})$	$\mathcal{R}_1(\mathbf{u})$	$\mathcal{E}_0(\mathbf{u})$	$\mathcal{R}_0(\mathbf{u})$	$\mathcal{E}_0(\omega)$	$\mathcal{R}_0(\omega)$
1/2	5.5569e−1	–	6.4046e−2	–	1.2167e−2	–	5.4887e−1	–	6.4153e−2	–	2.4074e−1	–
1/4	3.7465e−1	0.56	2.5424e−2	1.33	3.6058e−3	1.75	3.7410e−1	0.55	2.5419e−2	1.33	1.5905e−1	0.60
1/8	2.3728e−1	0.65	8.8470e−3	1.52	1.3224e−3	1.44	2.3727e−1	0.66	8.8415e−3	1.52	1.0174e−1	0.64
1/16	1.4764e−1	0.68	3.1226e−3	1.50	5.5058e−4	1.27	1.4731e−1	0.68	3.1225e−3	1.50	6.5059e−2	0.65
1/32	9.1830e−2	0.68	1.1808e−3	1.40	2.3605e−4	1.22	9.1760e−2	0.68	1.1808e−3	1.40	4.1388e−2	0.65



(a) (b) (c)

Fig. 5. Test 1: Convergence of the velocity $\mathbf{u}$ in broken H^1- and L^2-norms and vorticity ω in L^2-norm with mesh refinement for different types of discretization.



(a) Numerical solution (b) Exact solution

Fig. 6. Test 2. Non-convex domain.

observed that the error of stream-function approximation in H^1-norm decay slightly higher order than the expected rate of convergence. Moreover, we notice that the rates of convergence predicted in Theorems 5.1 and 5.2 are attained by the postprocessed variables velocity and vorticity. Finally, the numerical and exact solutions are depicted in Fig. 6.

6.3. Test 3. Real example with the wind forcing term

In this section, we would like to study one more realistic example where the external force function is considered from the derivatives of wind stress as mentioned in [3]. The computational domain is considered as $\Omega := (0, 3) \times (0, 1) \setminus \{(0, 3/2] \times [1/2, 1)\}$ and the forcing term $f = \sin(\pi y)$. In Fig. 7, we have depicted the numerical approximations of the stream-function and velocity fields, together with the streamlines obtained with a Voronoi mesh with 18 817 degree of freedom, $\epsilon_M = 6 \times 10^{-5}$ and $\epsilon_S = 0.05$. The numerical solution have analogous behavior as mentioned in [3]. Further, the numerical solution near the corner are well captured which validate the capability of our algorithm.

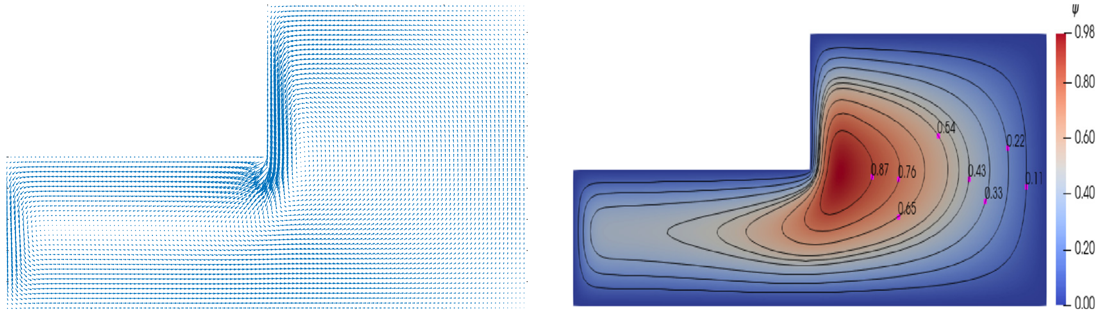


Fig. 7. Test 3. Numerical approximation of velocity field $\boldsymbol{u}_h$ and stream function ψ_h.

Data availability

Data will be made available on request.

Acknowledgments

The first author was partially supported by the National Agency for Research and Development, ANID-Chile through FONDECYT Postdoctorado project 3200242. The second author was partially supported by the National Agency for Research and Development, ANID-Chile through FONDECYT project 1220881, by project Anillo of Computational Mathematics for Desalination Processes ACT210087, and by project Centro de Modelamiento Matemático (CMM), FB210005, BASAL funds for centers of excellence. The third author was supported by the National Agency for Research and Development, ANID-Chile, Scholarship Program, Doctorado Becas Chile 2020, 21201910.

References

[1] G.K. Vallis, Atmosphere and Ocean Fluid Dynamics: Fundamentals and Large–Scale Circulation, Cambridge University Press, 2006.
[2] P.G. Myers, A.J. Weaver, A diagnostic barotropic finite–element ocean circulation model, J. Atmos. Ocean Technol. 12 (1995) 511–526.
[3] J.M. Cascón, G.C. Garcia, R. Rodríguez, A priori and a posteriori error analysis for a large–scale ocean circulation finite element model, Comput. Methods Appl. Mech. Engrg. 192 (2003) 5305–5327.
[4] E.L. Foster, T. Iliescu, Z. Wang, A finite element discretization of the stream function formulation of the stationary quasi–geostrophic equations of the ocean, Comput. Methods Appl. Mech. Engrg. 261/262 (2) (2013) 105–117.
[5] T.Y. Kim, E.J. Park, D.W. Shin, A C^0–discontinuous Galerkin method for the stationary quasi–geostrophic equations of the ocean, Comput. Methods Appl. Mech. Engrg. 300 (2015) 225–244.
[6] D. Kim, A.K. Pani, E.J. Park, Morley finite element methods for the stationary quasi–geostrophic equation, Comput. Methods Appl. Mech. Engrg. 375 (2021) 21, Paper No. 113639.
[7] J. Pedlosky, Geophysical Fluid Dynamics, Springer, 1987.
[8] J. McWilliams, Fundamentals of Geophysical Fluid Dynamics, Cambridge University Press, 2006.
[9] G.J. Fix, Finite element models for ocean circulation problems, SIAM J. Appl. Math. 29 (3) (2006) 371–387.
[10] T.Y. Kim, T. Iliescu, E. Fried, B–spline based finite–element method for the stationary quasi–geostrophic equations of the ocean, Comput. Methods Appl. Mech. Engrg. 286 (2015) 168–191.
[11] N. Rotundo, T.Y. Kim, W. Jiang, L. Heltai, E. Fried, Error analysis of a B–spline based finite–element method for modeling wind–driven ocean circulation, J. Sci. Comput. 69 (1) (2016) 430–459.
[12] L. Beirão da Veiga, A. Ern, Preface special issue – polyhedral discretization for PDE, ESAIM Math. Model. Numer. Anal. 50 (3) (2016) 633–634.
[13] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A. Russo, Basic principles of virtual element methods, Math. Models Methods Appl. Sci. 23 (2013) 199–214.
[14] P.F. Antonietti, L. Beirão da Veiga, S. Scacchi, M. Verani, A C^1 virtual element method for the Cahn–Hilliard equation with polygonal meshes, SIAM J. Numer. Anal. 54 (2016) 36–56.
[15] B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini, A. Russo, Equivalent projectors for virtual element methods, Comput. Math. Appl. 66 (2013) 376–391.
[16] A. Cangiani, G. Manzini, O.J. Sutton, Conforming and nonconforming virtual element methods for elliptic problems, IMA J. Numer. Anal. 37 (2017) 1317–1354.
[17] G. Wang, Y. Wang, Y. He, Least-squares virtual element method for the convection–diffusion-reaction problem, Internat. J. Numer. Methods Engrg. 122 (11) (2021) 2672–2693.
[18] C. Chinosi, L.D. Marini, Virtual element method for fourth order problems: L^2–estimates, Comput. Math. Appl. 72 (8) (2016) 1959–1967.
[19] D. Mora, I. Velásquez, Virtual elements for the transmission eigenvalue problem on polytopal meshes, SIAM J. Sci. Comput. 43 (1) (2021) A2425–A2447.
[20] E. Cáceres, G.N. Gatica, A mixed virtual element method for the pseudostress–velocity formulation of the Stokes problem, IMA J. Numer. Anal. 37 (2017) 296–331.
[21] L. Beirão da Veiga, C. Lovadina, G. Vacca, Divergence free virtual elements for the Stokes problem on polygonal meshes, ESAIM Math. Model. Numer. Anal. 51 (2017) 509–535.
[22] L. Beirão da Veiga, D. Mora, G. Vacca, The Stokes complex for virtual elements with application to Navier–Stokes flows, J. Sci. Comput. 81 (2) (2019) 990–1018.
[23] D. Mora, C. Reales, A. Silgado, A C^1–virtual element method of high order for the Brinkman equations in stream function formulation with pressure recovery, IMA J. Numer. Anal. 42 (4) (2022) 3632–3674.

[24] L. Beirão da Veiga, C. Lovadina, G. Vacca, Virtual elements for the Navier–Stokes problem on polygonal meshes, SIAM J. Numer. Anal. 56 (3) (2018) 1210–1242.
[25] G. Wang, Y. Wang, Y. He, Two robust virtual element methods for the Brinkman equations, Calcolo 58 (4) (2021) 27, Paper No. 49.
[26] D. Adak, D. Mora, S. Natarajan, A. Silgado, A virtual element discretization for the time dependent Navier–Stokes equations in stream-function formulation, ESAIM Math. Model. Numer. Anal. 55 (5) (2021) 2535–2566.
[27] D. Mora, A. Silgado, A C^1 virtual element method for the stationary quasi–geostrophic equations of the ocean, Comput. Math. Appl. 116 (2022) 212–228.
[28] B. Ayuso de Dios, K. Lipnikov, G. Manzini, The nonconforming virtual element method, ESAIM Math. Model. Numer. Anal. 50 (3) (2016) 879–904.
[29] A. Cangiani, V. Gyrya, G. Manzini, The nonconforming virtual element method for the Stokes equations, SIAM J. Numer. Anal. 54 (6) (2016) 3411–3435.
[30] X. Liu, Z. Chen, The nonconforming virtual element method for the Navier–Stokes equations, Adv. Comput. Math. 45 (1) (2019) 51–74.
[31] L. Mascotto, I. Perugia, A. Pichler, Non-conforming harmonic virtual element method: h- and p- versions, J. Sci. Comput. 77 (3) (2018) 1874–1908.
[32] L. Mascotto, I. Perugia, A. Pichler, A nonconforming Trefftz virtual element method for the Helmholtz problem, Math. Models Methods Appl. Sci. 29 (9) (2019) 1619–1656.
[33] J. Zhao, B. Zhang, S. Mao, S. Chen, The nonconforming virtual element method for the Darcy–Stokes problem, Comput. Methods Appl. Mech. Engrg. 370 (2020) 113251, 23 pp.
[34] J. Zhao, B. Zhang, S. Mao, S. Chen, The divergence–free nonconforming virtual element for the Stokes problem, SIAM J. Numer. Anal. 57 (6) (2019) 2730–2759.
[35] J. Zhao, S. Chen, B. Zhang, The nonconforming virtual element method for plate bending problems, Math. Models Methods Appl. Sci. 26 (9) (2016) 1671–1687.
[36] P.F. Antonietti, G. Manzini, M. Verani, The fully nonconforming virtual element method for biharmonic problems, Math. Models Methods Appl. Sci. 28 (2) (2018) 387–407.
[37] J. Zhao, B. Zhang, S. Chen, S. Mao, The Morley–type virtual element for plate bending problems, J. Sci. Comput. 76 (1) (2018) 610–629.
[38] L.S.D. Morley, The triangular equilibrium element in the solution of plate bending problems, Aeronaut. Q. 19 (1968) 149–169.
[39] C. Carstensen, R. Khot, A.K. Pani, Nonconforming virtual elements for the biharmonic equation with Morley degrees of freedom on polygonal meshes, 2022, arXiv:2205.08764 [math.NA].
[40] J. Huang, Y. Yu, A medius error analysis for nonconforming virtual element methods for Poisson and biharmonic equations, J. Comput. Appl. Math. 386 (2021) 20, Paper No. 113229.
[41] B. Zhang, J. Zhao, S. Chen, The nonconforming virtual element method for fourth–order singular perturbation problem, Adv. Comput. Math. 46 (2) (2020) 23, Paper No. 19.
[42] M. Li, J. Zhao, C. Huang, S. Chen, Conforming and nonconforming VEMs for the fourth–order reaction–subdiffusion equation: a unified framework, IMA J. Numer. Anal. 42 (3) (2022) 2238–2300.
[43] F. Wang, J. Zhao, Conforming and nonconforming virtual element methods for a Kirchhoff plate contact problem, IMA J. Numer. Anal. 41 (2) (2021) 1496–1521.
[44] A. Dedner, A. Hodson, Robust nonconforming virtual element methods for general fourth–order problems with varying coefficients, IMA J. Numer. Anal. 42 (2) (2022) 1364–1399.
[45] T.T. Medjo, Numerical simulations of a two–layer quasi–geostrophic equation of the ocean, SIAM J. Numer. Anal. 37 (2000) 2005–2022.
[46] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, second ed., Academic Press, Amsterdam, 2003.
[47] H. Blum, R. Rannacher, On the boundary value problem of the biharmonic operator on domains with angular corners, Math. Methods Appl. Sci. 2 (4) (1980) 556–581.
[48] S.C. Brenner, R.L. Scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 2008.
[49] L. Chen, J. Huang, Some error analysis on virtual element methods, Calcolo 55 (1) (2018) 5–23.
[50] P.G. Ciarlet, Interpolation error estimates for the reduced Hsieh–Clough–Tocher triangle, Math. Comp. 32 (142) (1978) 335–344.
[51] S.C. Brenner, L. Sung, H. Zhang, Y. Zhang, A Morley finite element method for the displacement obstacle problem of clamped Kirchhoff plates, J. Comput. Appl. Math. 254 (2013) 31–42.