IMA Journal of Numerical Analysis (2020) **40**, 1544–1576
doi:10.1093/imanum/drz001
Advance Access publication on 8 February 2019

A virtual element method for a nonlocal FitzHugh–Nagumo model of cardiac electrophysiology

Verónica Anaya*
GIMNAP, Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile
*Corresponding author: vanaya@ubiobio.cl

Mostafa Bendahmane
Institut de Mathématiques de Bordeaux, Université de Bordeaux, Talence, France
mostafa.bendahmane@u-bordeaux.fr

David Mora
GIMNAP, Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile and Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, Concepción, Chile
dmora@ubiobio.cl

and

Mauricio Sepúlveda
Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, Concepción, Chile and Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile
mauricio@ing-mat.udec.cl

[Received on 29 March 2018; revised on 16 October 2018]

We present a virtual element method (VEM) for a nonlocal reaction–diffusion system of the cardiac electric field. For this system, we analyze an H^1-conforming discretization by means of VEM that can make use of general polygonal meshes. Under standard assumptions on the computational domain, we establish the convergence of the discrete solution by considering a series of *a priori* estimates and by using a general L^p compactness criterion. Moreover, we obtain optimal order space-time error estimates in the L^2 norm. Finally, we report some numerical tests supporting the theoretical results.

Keywords: virtual element method; FitzHugh–Nagumo equations; convergence; error estimates.

1. Introduction

Reaction–diffusion systems appear in models of different areas such as medicine, engineering, biology, physics, etc. The study of this kind of system has attracted a lot of attention for a number of years, systems with different types of diffusion, for example, constant, nonlocal and cross. Mathematical models related to electrical activity in the heart (cardiac tissue) are becoming a powerful tool to study and understand many types of heart diseases, as for example, irregular heart rhythm.

The reaction–diffusion system of the FitzHugh–Nagumo type (FitzHugh, 1961; Nagumo *et al.*, 1962) is one of the most important and well-known generic models in physiology that describes complex wave phenomena in excitable or oscillatory media. This model is a reaction–diffusion system that is a simplification of the famous Hodgkin–Huxley model, which has been used to describe the propagation of the electrical potential in cardiac tissue (Hastings, 1975; Peskin, 1975; Sanfelici, 2002;

Coudiére & Pierre, 2006). The FitzHugh–Nagumo reaction–diffusion system consists of one nonlinear parabolic partial differential equation (PDE) that describes the dynamic of the membrane potential, coupled with an ordinary differential equation that models the ionic currents associated with the reaction term. The main difficulties associated with solving this system are related to the coupling of the equations through a nonlinear term and the regularity of the solution that is low.

In this paper, we analyze a *virtual element method* (VEM) for a nonlinear parabolic problem arising in cardiac models (electrophysiology) with nonlocal diffusion (see system (2.1) below). In our study, the self-diffusion coefficient is assumed depending on the total of electrical potential in the heart. The VEM, recently introduced in Beirão da Veiga *et al.* (2013a, 2014a), is a generalization of the finite element method that is characterized by the capability of dealing with very general polygonal/polyhedral meshes. In recent years, the interest in numerical methods that can make use of general polygonal/polyhedral meshes for the numerical solution of PDEs has undergone a significant growth; this is because of the high flexibility that this kind of mesh allows in the treatment of complex geometries. Among the large number of papers on this subject, we cite as a minimal sample Sukumar & Tabarraei (2004); Talischi *et al.* (2010); Cangiani *et al.* (2014); Beirão da Veiga *et al.* (2014b) and Di Pietro & Ern (2015).

Although the VEM is very recent, it has been applied to a large number of problems; for instance, VEM for Stokes, Brinkman, Cahn–Hilliard, plates bending, advection–diffusion, Helmholtz, parabolic and hyperbolic problems have been introduced in Brezzi & Marini (2012); Antonietti *et al.* (2014, 2016); Vacca & Beirão da Veiga (2015); Benedetto *et al.* (2016); Perugia *et al.* (2016); Brenner *et al.* (2017); Cáceres & Gatica (2017); Cáceres *et al.* (2017); Cangiani *et al.* (2017b); Vacca (2017, 2018); Beirão da Veiga *et al.* (2017b, 2019); VEM for spectral problems in Mora *et al.* (2015, 2018); Beirão da Veiga *et al.* (2017c); Gardini & Vacca (2018) and VEM for linear and nonlinear elasticity in Beirão da Veiga *et al.* (2013b, 2015); Gain *et al.* (2014); Wriggers *et al.* (2016), Artioli *et al.* (2017) whereas *a posteriori* error analyses have been developed in Beirão da Veiga & Manzini (2015); Berrone & Borio (2017); Cangiani *et al.* (2017a); Mora *et al.* (2017).

Over the past years, some papers related to numerical tools for solving this model and its variations have appeared. For example, in Chrysafinos *et al.* (2013) a continuous in space and discontinuous in time Galerkin method of arbitrary order has been developed, and under minimal regularity assumptions, space-time error estimates are established in the natural norms. In Jackson (1992) some estimates in the L^2 norm for semidiscrete Galerkin approximations for the FitzHugh–Nagumo model are derived. Moreover, Sanfelici (2002) presented the convergence analysis and *a priori* stability estimates for the semidiscrete solution given by a finite element Galerkin approximation applied to the bidomain model. In Coudiére & Pierre (2006), stability conditions and convergence results of a finite volume method for reaction–diffusion systems in electrocardiology are given. A finite difference method has been presented in Barkley (1991), Chebyshev multidomain method has been presented in Olmos & Shizgal (2009), fully space-time adaptive multiresolution methods based on the finite volume method and Barkley's method for simulating the complex dynamics of waves in excitable media in Burger *et al.* (2010). Finally, Thomée (1997) has presented other methods related to the numerical analysis of general semilinear parabolic PDE.

Numerical methods to solve these kind of models have limitations in the range of applicable meshes. In particular, finite element methods rely on triangular (simplicial) or quadrilateral meshes. Moreover, the classical finite volume method has some restriction on the admissible meshes (for instance, orthogonality constraints). However, in complex simulations like fluid–structure interaction, phase change, medical applications and many others, the geometrical complexity of the domain is a relevant issue when PDEs have to be solved on a good quality mesh; hence, it can be convenient to use more general polygonal/polyhedral meshes. Thus, in the present contribution, we are going to introduce and analyze a VEM that has the advantage of using general polygonal meshes to solve a nonlinear parabolic FitzHugh–Nagumo system, where the diffusion coefficient depends on a nonlocal quantity. The study of nonlocal diffusion problems has received considerable attention in recent years since they appear in important physical and biological applications (Chipot & Lovat, 1997; Chipot, 2000; Anaya *et al.*, 2015a,b). There are models of the FitzHugh–Nagumo type that also take into account the nonlocal diffusion phenomena; for example, Liu *et al.* (2015) considered a diffusive nonlocal term as fractional diffusion, and Oshita & Ohnishi (2003) took a nonlocal reactive term.

The aim of this paper is to introduce and analyze a conforming $H^1(\Omega)$-VEM that applies to general polygonal meshes for the two-dimensional nonlocal reaction–diffusion FitzHugh–Nagumo equations. We propose a space discretization by means of VEM, which is based on the discrete space introduced in Ahmad *et al.* (2013) for the linear reaction–diffusion equation. We construct a proper L^2-projection operator that is used to approximate the bilinear form that appears for the time derivative discretization, which is obtained by a classical backward Euler method. We also use that projection to discretize the nonlocal term presented in the system. We prove that the fully discrete scheme is well posed, and using standard space and time translates together with *a priori* estimates for the discrete solution; it is established convergence of the discrete scheme to the weak solution of the model. Due to the nonLipschitz of the nonlinear term (the ionic function) in the FitzHugh–Nagumo model, we need to relax the assumption on the nonlinearity to establish optimal order space-time error estimates in the L^2 norm.

The structure of the paper is organized as follows. In Section 2, we give some preliminaries and assumptions on the data. Moreover, we introduce the concept of weak solution. In Section 3, we propose the semidiscrete and fully discrete VEM. In Section 4, we prove the existence and convergence of the discrete solution. In Section 5, we give error estimates, and finally, in Section 6, some numerical results.

Throughout the article we will denote with c and C, with or without subscripts, tildes or hats, generic constants independent of the mesh parameter h and the time step Δt, which may take different values in different occurrences. Moreover, let $\Omega \subset \mathbb{R}^2$ be a polygonal domain; we will consider the following spaces: by $H^m(\Omega)$, we denote the usual Sobolev space of order m. Given $T > 0$ and $1 \leq p \leq \infty$, $L^p(0,T;\mathbb{R})$ denotes the space of L^p integrable functions from the interval $[0,T]$ into $\mathbb{R}$.

2. Model problem and weak solution

Fix a final time $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary Σ and outer unit normal vector $\boldsymbol{n}$. For all $(x,t) \in \Omega_T := \Omega \times (0,T)$, $v = v(x,t)$ and $w = w(x,t)$ stand for the transmembrane potential and the gating variable, respectively. The governing equations of the nonlocal reaction–diffusion FitzHugh–Nagumo system are

$$\begin{cases} \partial_t v - D\left(\int_\Omega v(x,t)\,\mathrm{d}x\right)\Delta v + I_{\mathrm{ion}}(v,w) = I_{\mathrm{app}}(x,t) & (x,t) \in \Omega_T, \\ \partial_t w - H(v,w) = 0 & (x,t) \in \Omega_T, \\ D\left(\int_\Omega v(x,t)\,\mathrm{d}x\right)\nabla v \cdot \boldsymbol{n} = 0 & (x,t) \in \Sigma_T := \Sigma \times (0,T), \\ v(x,0) = v_0(x) & x \in \Omega, \\ w(x,0) = w_0(x) & x \in \Omega. \end{cases} \tag{2.1}$$

Herein, $I_{\mathrm{app}}(x,t) \in L^2(\Omega_T)$ is the stimulus. In this work, the diffusion rate $D > 0$ is supposed to depend on the whole of the transmembrane potential in the domain rather than on the local diffusion, i.e., the diffusion of the transmembrane potential is guided by the global state of the potential in the medium. We assume that $D : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following: there exist constants $d_1,\, d_2 > 0$ such that

$$d_1 \le D \quad \text{and} \quad \left|D(I_1) - D(I_2)\right| \le d_2 \left|I_1 - I_2\right| \quad \text{for all} \quad I_1, I_2 \in \mathbb{R}. \tag{2.2}$$

Now, we make some assumptions on the data of the nonlocal FitzHugh–Nagumo model. For the ionic current $I_{\text{ion}}(v,w)$, we assume that it can be decomposed into $I_{1,\text{ion}}(v)$ and $I_{2,\text{ion}}(w)$, where $I_{\text{ion}}(v,w) = I_{1,\text{ion}}(v) + I_{2,\text{ion}}(w)$. We assume that $I_{1,\text{ion}}, I_{2,\text{ion}} : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and that there exist constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ such that

$$\begin{aligned}
&(a) \quad \frac{1}{\alpha_1} |v|^4 \le \left|I_{1,\text{ion}}(v)v\right| \le \alpha_2 \left(|v|^4 + 1\right),\\
&(b) \quad \left|I_{2,\text{ion}}(w)\right| \le \alpha_3(|w| + 1),\\
&(c) \quad \forall\, z, s \in \mathbb{R} \quad (I_{1,\text{ion}}(z) - I_{1,\text{ion}}(s))(z - s) \ge -C_h|z - s|^2,\\
&(d) \quad |H(v,w)| \le \alpha_4(|v| + |w| + 1).
\end{aligned} \tag{2.3}$$

It is well known that if the initial condition $v_0 \in L^\infty(\Omega)$ and the functions are specified as follows:

$$H(v,w) = av - bw, \tag{2.4}$$

and

$$I_{\text{ion}}(v,w) = -\lambda(w - v(1 - v)(v - \theta)), \tag{2.5}$$

where a, b, λ, θ are given parameters. Then the assumptions (2.3) are fulfilled.

The weak solution to the model (2.1) is defined as follows.

Definition 2.1 (Weak solution). A weak solution to the system (2.1) is a double function (v,w) such that $v \in L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$, $\partial_t v \in L^2(0,T;(H^1(\Omega)')) + L^{\frac{4}{3}}(\Omega_T)$, $w \in C([0,T];L^2(\Omega))$ and satisfying the following weak formulation:

$$\begin{gathered}
\iint_{\Omega_T} \partial_t v\, \varphi + \int_0^T D\left(\int_\Omega v(x,t)\,\mathrm{d}x\right) \int_\Omega \nabla v \cdot \nabla \varphi + \iint_{\Omega_T} I_{\text{ion}}(v,w)\varphi = \iint_{\Omega_T} I_{\text{app}}(x,t)\varphi,\\
\iint_{\Omega_T} \partial_t w\, \phi - \iint_{\Omega_T} H(v,w)\phi = 0,
\end{gathered} \tag{2.6}$$

for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$ and $\phi \in C([0,T];L^2(\Omega))$.

Remark 2.2 Note that, in view of the conditions stated in (2.3), we can easily check that Definition 2.1 makes sense. Furthermore, observe that Definition 2.1 implies $v \in C([0,T];L^2(\Omega))$ (see Schoenbek, 1978).

3. Virtual element scheme and main result

In this section, we recall the mesh construction and the assumptions considered to introduce the discrete virtual element space. Then we present the virtual element approximation of the FitzHugh–Nagumo model.

3.1 *The VEM semidiscrete problem*

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K\in\mathcal{T}_h} h_K$. In what follows, we denote by N_K the number of vertices of K, by e a generic edge of $\mathcal{T}_h$ and for all $e \in \partial K$, we define a unit normal vector $\boldsymbol{n}_K^e$ that points outside of K.

For the analysis, we will make the following assumptions as in Beirão da Veiga *et al.* (2013a, 2017c): there exists a positive real number $C_{\mathcal{T}}$ such that, for every h and every $K \in \mathcal{T}_h$,

A1: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$;

A2: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}} h_K$.

For any subset $S \subseteq \mathbb{R}^2$ and non-negative integer k, we indicate by $\mathbb{P}_k(S)$ the space of polynomials of degree up to k defined on S.

Now, we consider a simple polygon K (meaning open simply connected set whose boundary is a nonintersecting line made of a finite number of straight line segments), and we start by introducing a preliminary virtual element space. For all $K \in \mathcal{T}_h$, the local space $V_{k|K}$ is defined by (see Ahmad *et al.*, 2013)

$$V_{k|K} := \left\{\varphi \in H^1(K) \cap C^0(K) : \varphi_{|e} \in \mathbb{P}_k(e)\ \forall\, e \in \partial K,\ \Delta\varphi \in \mathbb{P}_k(K)\right\}.$$

Now, we introduce the following set of linear operators from $V_{k|K}$ into $\mathbb{R}$. For all $\varphi \in V_{k|K}$,

- D_1: the values of φ at the vertices of K;
- D_2: values of φ at $k-1$ distinct points in e, for all $e \in \partial K$;
- D_3: all moments $\int_K \varphi p \,\mathrm{d}x$, for all $p \in \mathbb{P}_{k-2}(K)$.

Now, we split the bilinear form $a(\cdot,\cdot) := (\nabla\cdot, \nabla\cdot)_{0,\Omega}$,

$$a(v,\varphi) := \sum_{K\in\mathcal{T}_h} a^K(v,\varphi), \qquad \forall\, v,\varphi \in H^1(\Omega),$$

where

$$a^K(v,\varphi) := \int_K \nabla v \cdot \nabla\varphi, \qquad \forall\, v,\varphi \in H^1(\Omega).$$

For the analysis we will introduce the following broken seminorm:

$$|\varphi|_{1,h} := \left(\sum_{K\in\mathcal{T}_h} |\varphi|_{1,K}^2\right)^{1/2}.$$

Let $\Pi_{K,k} : V_{k|K} \to \mathbb{P}_k(K)$ be the projection operator defined by

$$\begin{cases} a^K(\Pi_{K,k}v, q) = a^K(v, q) & \forall\, q \in \mathbb{P}_k(K), \\ P_0(\Pi_{K,k}v) = P_0 v, \end{cases} \tag{3.1}$$

where P_0 can be taken as

$$\begin{cases} P_0 v := \frac{1}{N_K} \sum_{i=1}^{N_K} v(V_i) & k = 1, \\ P_0 v := \frac{1}{|K|} \int_K v \,\mathrm{d}x & k > 1, \end{cases}$$

with V_i the vertices of K, $1 \le i \le N_K$, where N_K is the number of vertices in K.

REMARK 3.1 The above definition of P_0 is only needed for the problem (3.1) to be well posed. We note that it is possible to consider alternative definitions for P_0. In particular, a possible computable definition, valid for any k, is to take the following average on the boundary:

$$P_0 v := \frac{1}{|\partial K|} \int_{\partial K} v,$$

which makes sense for any $v \in H^1(K)$.

Using an integration by parts, it is easy to check that, for any $\varphi \in V_{k|K}$, the values of the linear operators D_1, D_2 and D_3 given before are sufficient in order to compute $\Pi_{K,k}$. As a consequence, the projection operator $\Pi_{K,k}$ depends only on the values of the operators D_1, D_2 and D_3.

Now, we introduce our virtual local space (see Ahmad *et al.*, 2013)

$$W_{k|K} := \left\{ \varphi \in V_{k|K} : \int_K (\Pi_{K,k}\varphi) q \,\mathrm{d}x = \int_K \varphi q \,\mathrm{d}x \quad \forall\, q \in \mathbb{P}_k / \mathbb{P}_{k-2}(K) \right\},$$

where the symbol $\mathbb{P}_k/\mathbb{P}_{k-2}(K)$ denotes the polynomials of degree k living on K that are L^2-orthogonal to all polynomials of degree $k-2$ on K. We observe that, since $W_{k|K} \subset V_{k|K}$, the operator $\Pi_{K,k}$ is well defined on $W_{k|K}$ and computable only on the basis of the values of the operators D_1, D_2 and D_3.

In Ahmad *et al.* (2013) has been established that the operators D_1, D_2 and D_3 constitute a set of degrees of freedom for the space $W_{k|K}$.

The global discrete space will be

$$W_h := \left\{ \varphi \in H^1(\Omega) : \varphi|_K \in W_{k|K}, \quad \forall\, K \in \mathcal{T}_h \right\}.$$

In agreement with the local choice of the degrees of freedom, in W_h we choose the following degrees of freedom:

- DG_1: the values of φ at the vertices of $\mathcal{T}_h$;
- DG_2: values of φ at $k-1$ distinct points in e, for all $e \in \mathcal{T}_h$;
- DG_3: all moments $\int_K \varphi p \,\mathrm{d}x$, for all $p \in \mathbb{P}_{k-2}(K)$ on each element $K \in \mathcal{T}_h$.

On the other hand, let $S^K(\cdot,\cdot)$ and $S_0^K(\cdot,\cdot)$ be any symmetric positive definite bilinear forms to be chosen as to satisfy

$$c_0 a^K(\varphi_h,\varphi_h) \le S^K(\varphi_h,\varphi_h) \le c_1 a^K(\varphi_h,\varphi_h) \quad \forall\,\varphi_h \in V_{k|K} \quad \text{with} \quad \Pi_{K,k}\varphi_h = 0, \tag{3.2}$$

$$\tilde{c}_0(\varphi_h,\varphi_h)_{0,K} \le S_0^K(\varphi_h,\varphi_h) \le \tilde{c}_1(\varphi_h,\varphi_h)_{0,K} \quad \forall\,\varphi_h \in V_{k|K}, \tag{3.3}$$

for some positive constants $c_0, c_1, \tilde{c}_0$ and $\tilde{c}_1$ independent of K.

We define the local discrete bilinear and trilinear forms

$$a_h^K(\cdot,\cdot) : W_h \times W_h \to \mathbb{R}, \qquad m_h^K(\cdot,\cdot) : W_h \times W_h \to \mathbb{R},$$

$$b_h^K(\cdot,\cdot,\cdot) : W_h \times W_h \times W_h \to \mathbb{R}, \quad c_h^K(\cdot,\cdot,\cdot) : W_h \times W_h \times W_h \to \mathbb{R},$$

as follows, for all $v_h, w_h, \varphi_h \in W_{k|K}$:

$$a_h^K(v_h,\varphi_h) := a^K(\Pi_{K,k}v_h, \Pi_{K,k}\varphi_h) + S^K(v_h - \Pi_{K,k}v_h, \varphi_h - \Pi_{K,k}\varphi_h),$$

$$m_h^K(v_h,\varphi_h) := \left(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0\varphi_h\right)_{0,K} + S_0^K\left(v_h - \Pi_{K,k}^0 v_h, \varphi_h - \Pi_{K,k}^0\varphi_h\right),$$

$$b_h^K(v_h,w_h,\varphi_h) := \int_K I_{ion}\left(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h\right)\Pi_{K,k}^0\varphi_h,$$

$$c_h^K(v_h,w_h,\varphi_h) := \int_K H\left(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h\right)\Pi_{K,k}^0\varphi_h,$$

where $\Pi_{K,k}^0 : W_{k|K} \to \mathbb{P}_k(K)$ is the standard L^2-projection operator. We note that all the forms introduced above are computable on the basis of the degrees of freedom (see Ahmad *et al.*, 2013; Vacca & Beirão da Veiga, 2015).

We observe that for all $K \in \mathcal{T}_h$ it holds the following:

- k-consistency: for all $p \in \mathbb{P}_k(K)$ and for all $\varphi_h \in W_{k|K}$,

$$\begin{aligned} a_h^K(p,\varphi_h) &= a^K(p,\varphi_h),\\ m_h^K(p,\varphi_h) &= (p,\varphi_h)_{0,K}; \end{aligned} \tag{3.4}$$

- stability: there exist four positive constants, $\alpha', \alpha'', \beta'$ and β'', independent of h, such that for all $\varphi_h \in W_{k|K}$,

$$\begin{aligned} \alpha'\, a^K(\varphi_h,\varphi_h) &\le a_h^K(\varphi_h,\varphi_h) \le \alpha''\, a^K(\varphi_h,\varphi_h),\\ \beta'\, (\varphi_h,\varphi_h)_{0,K} &\le m_h^K(\varphi_h,\varphi_h) \le \beta''\, (\varphi_h,\varphi_h)_{0,K}. \end{aligned} \tag{3.5}$$

Then we set for all $v_h, w_h, \varphi_h \in W_h$,

$$a_h(v_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} a_h^K(v_h, \varphi_h), \quad m_h(v_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} m_h^K(v_h, \varphi_h),$$

$$b_h(v_h, w_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} b_h^K(v_h, w_h, \varphi_h), \quad c_h(v_h, w_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} c_h^K(v_h, w_h, \varphi_h).$$

We discretize the nonlocal diffusion term using the L^2-projection as follows:

$$J(v_h) := \int_\Omega v_h = \sum_{K \in \mathcal{T}_h} \int_K \Pi^0_{K,k} v_h, \qquad v_h \in W_h. \tag{3.6}$$

For the right-hand side, since $I_{\mathrm{app}}(x,t) \in L^2(\Omega_T)$, we set

$$I_{app,h}(t) = \Pi^0_k I_{\mathrm{app}}(\cdot, t) \quad \text{for a.e.} \quad t \in (0,T),$$

where we have introduced Π^0_k as the following operator that is defined in L^2 by

$$(\Pi^0_k g)|_K := \Pi^0_{K,k} g \qquad \text{for all } K \in \mathcal{T}_h \tag{3.7}$$

with $\Pi^0_{K,k}$ the $L^2(K)$-projection.

Now, we note that the symmetry of $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, and the stability conditions stated before, imply the continuity of a_h and m_h. In fact, for all $v_h, \varphi_h \in W_h$,

$$\begin{aligned} |a_h(v_h, \varphi_h)| &\le C \|v_h\|_{H^1(\Omega)} \|\varphi_h\|_{H^1(\Omega)}, \\ |m_h(v_h, \varphi_h)| &\le C \|v_h\|_{L^2(\Omega)} \|\varphi_h\|_{L^2(\Omega)}. \end{aligned} \tag{3.8}$$

The semidiscrete VEM formulation reads as follows. For all $t > 0$, find $v_h, w_h \in L^2(0,T;W_h)$ with $\partial_t v_h, \partial_t w_h \in L^2(0,T;W_h)$, such that

$$\begin{cases} m_h(\partial_t v_h(t), \varphi_h) + D\left(J(v_h(t))\right) a_h(v_h(t), \varphi_h) + b_h(v_h(t), w_h(t), \varphi_h) = \left(I_{app,h}(t), \varphi_h\right)_{0,\Omega} \\ m_h(\partial_t w_h(t), \phi_h) - c_h(v_h(t), w_h(t), \phi_h) = 0, \end{cases} \tag{3.9}$$

for all $\varphi_h, \phi_h \in W_h$. Additionally, we set $v_h(0) = v_h^0$ and $w_h(0) = w_h^0$. A classical backward Euler integration method is employed for the time discretization of (3.9) with time step $\Delta t = T/N$. This results in the following fully discrete method: find $v_h^n, w_h^n \in W_h$ such that

$$\begin{cases} m_h\left(\frac{v_h^n - v_h^{n-1}}{\Delta t}, \varphi_h\right) + D\left(J(v_h^n)\right) a_h\left(v_h^n, \varphi_h\right) + b_h\left(v_h^n, w_h^n, \varphi_h\right) = \left(I^n_{app,h}, \varphi_h\right)_{0,\Omega} \\ m_h\left(\frac{w_h^n - w_h^{n-1}}{\Delta t}, \phi_h\right) - c_h\left(v_h^n, w_h^n, \phi_h\right) = 0, \end{cases} \tag{3.10}$$

for all $\varphi_h, \phi_h \in W_h$, for all $n \in \{1, \ldots, N\}$; the initial condition takes the form v_h^0, w_h^0 and $I^n_{app,h} := I_{app,h}(t_n)$ with $t_n := n\Delta t$, for $n = 0, \ldots, N$. In order to simplify the notation, we denote

$$v_h := \sum_{n=1}^{N} v_h^n(x)\,\mathbb{1}_{((n-1)\Delta t, n\Delta t]}(t), \quad w_h := \sum_{n=1}^{N} w_h^n(x)\,\mathbb{1}_{((n-1)\Delta t, n\Delta t]}(t). \tag{3.11}$$

REMARK 3.2 In (3.10) we have written a conforming H^1-discretization to approximate the weak solution of the system (2.6). In particular, we have considered the virtual space W_h for the approximation of the gating variable $w \in L^2(\Omega_T)$. This choice will facilitate the presentation and the analysis of the proposed virtual method. Other discrete spaces, such as piecewise polynomial of degree k, to approximate the gating variable will be studied in a future work.

Our main result is the following theorem.

THEOREM 3.3 Assume that (2.2) and (2.3) hold. If $v_0(x) \in L^2(\Omega)$, $w_0(x) \in L^2(\Omega)$ and $I_{\text{app}}(x,t) \in L^2(\Omega_T)$ then the virtual element solution $\mathbf{u}_h^n = \left(v_h^n, w_h^n\right)$, generated by (3.10), converges along a subsequence to $\mathbf{u} = (v, w)$ as $h \to 0$, where $\mathbf{u}$ is a weak solution of (2.1). Moreover, the weak solution is unique.

In the next section, we prove Theorem 3.3 by establishing the convergence of the virtual element solution $\left(v_h^n, w_h^n\right)$, based on *a priori* estimates and the compactness method. Moreover, we provide error estimates in Section 5.

4. Existence of solution for the virtual element scheme

The existence result for the virtual element scheme is given in the following proposition.

PROPOSITION 4.1 Assume that (2.2) and (2.3) hold. Then the problem (3.10) admits a discrete solution $\mathbf{u}_h^n = \left(v_h^n, w_h^n\right)$.

Proof. The existence of $\mathbf{u}_h^n$ is shown by induction on $n = 0, \ldots, N$. For $n = 0$, the solution is given by $\mathbf{u}_h^0 = (v_h(0), w_h(0)) = (v_h^0, w_h^0)$. Assume that $\mathbf{u}_h^{n-1}$ exists. Choose $[\![\cdot\,,\cdot]\!]$ as the scalar product on $H^1(\Omega) \times L^2(\Omega)$. We define a map $L : W_h \times W_h \to W_h \times W_h$ such that for every $\mathbf{u}_h^n \in W_h \times W_h$, $L(\mathbf{u}_h^n) \in W_h \times W_h$ is the solution of following problem:

$$[\![L(\mathbf{u}_h^n), \Phi_h]\!] = m_h\left(\frac{v_h^n - v_h^{n-1}}{\Delta t}, \varphi_h\right) + D\left(J(v_h^n)\right) a_h\left(v_h^n, \varphi_h\right) + b_h\left(v_h^n, w_h^n, \varphi_n\right) - \left(I_{app,h}(t_n), \varphi_h\right)_{0,\Omega}$$
$$+ m_h\left(\frac{w_h^n - w_h^{n-1}}{\Delta t}, \phi_h\right) - c_h\left(v_h^n, w_h^n, \phi_h\right),$$

for all $\Phi_h := (\varphi_h, \phi_h) \in W_h \times W_h$. Next we are looking for a solution $\mathbf{u}_h^n$ to $[\![L(\mathbf{u}_h^n), \Phi_h]\!] = 0$. Note that the continuity of the operator L is a consequence of the continuity of m_h, a_h b_h and c_h. Moreover, the following bound holds from the discrete Hölder and Sobolev inequalities (recall that $H^1(\Omega) \subset L^q(\Omega)$ for all $1 \le q \le 6$):

$$[\![L(\mathbf{u}_h^n), \Phi_h]\!] \le C\left(\|v_h^n\|_{H^1(\Omega)} + \|w_h^n\|_{L^2(\Omega)} + 1\right)\left(\|\varphi_h\|_{H^1(\Omega)} + \|\phi_h\|_{L^2(\Omega)}\right),$$

for all $\mathbf{u}_h^n$ and Φ_h in $W_h \times W_h$. Moreover, from (2.3) and Young inequality, we get

$$\left[\!\left[L(\mathbf{u}_h^n), \mathbf{u}_h^n \right]\!\right] \geq C \left(\|v_h^n\|_{H^1(\Omega)}^2 + \|w_h^n\|_{L^2(\Omega)}^2 \right) + C'$$

for some constants $C > 0$ and C' (not necessarily positive). Finally, we conclude that $\left[\!\left[L(\mathbf{u}_h^n), \mathbf{u}_h^n \right]\!\right] \geq 0$ for $\|\mathbf{u}_h^n\|^2 := \|v_h^n\|_{H^1(\Omega)}^2 + \|w_h^n\|_{L^2(\Omega)}^2$ sufficiently large. The existence of $\mathbf{u}_h^n$ follows by the standard Brouwer fixed point argument (see Lions, 1969, Lemma 4.3). □

4.1 A priori *estimates*

In this section, we establish several *a priori* (discrete energy) estimates for the virtual element scheme, which eventually will imply the desired convergence results.

PROPOSITION 4.2 Let $\mathbf{u}_h^n = \left(v_h^n, w_h^n\right)$ be a solution of the virtual element scheme (3.10). Then there exist constants $C > 0$, depending on Ω, T, v_h^0, w_h^0, I_{app} and α_i, with $i = 1, \ldots 4$, such that

$$\begin{aligned} \|v_h\|_{L^\infty(0,T;L^2(\Omega))} + \|w_h\|_{L^\infty(0,T;L^2(\Omega))} &\leq C, \\ \| \nabla v_h\|_{L^2(\Omega_T)} &\leq C, \\ \|\Pi_k^0 v_h\|_{L^4(\Omega_T)} &\leq C, \end{aligned}$$

where Π_k^0 has been introduced in (3.7).

Proof. We use (3.10) with $\varphi_h = v_h^n$, $\phi_h = w_h^n$, and we sum over $n = 1, \ldots, \kappa$ for all $1 < \kappa \leq N$,

$$\begin{aligned} &\sum_{n=1}^{\kappa} m_h\left(v_h^n - v_h^{n-1}, v_h^n\right) + \sum_{n=1}^{\kappa} m_h\left(w_h^n - w_h^{n-1}, w_h^n\right) + \int_0^{\kappa \Delta t} D\left(J(v_h^n)\right) a_h\left(v_h^n, v_h^n\right) \\ &\quad + \int_0^{\kappa \Delta t} b_h\left(v_h^n, w_h^n, v_h^n\right) = \int_0^{\kappa \Delta t} c_h\left(v_h^n, w_h^n, w_h^n\right) + \int_0^{\kappa \Delta t} \left(I_{\text{app},h}, v_h^n\right)_{0,\Omega}. \end{aligned}$$

Observe that by an application of Hölder and Young inequalities, we get

$$\begin{aligned} \sum_{n=1}^{\kappa} m_h\left(v_h^n - v_h^{n-1}, v_h^n\right) &= \sum_{n=1}^{\kappa} m_h\left(v_h^n, v_h^n\right) - \sum_{n=1}^{\kappa} m_h\left(v_h^{n-1}, v_h^n\right) \\ &\geq \sum_{n=1}^{\kappa} m_h\left(v_h^n, v_h^n\right) - \sum_{n=1}^{\kappa} \left(m_h\left(v_h^n, v_h^n\right)\right)^{1/2} \left(m_h\left(v_h^{n-1}, v_h^{n-1}\right)\right)^{1/2} \\ &\geq \sum_{n=1}^{\kappa} m_h\left(v_h^n, v_h^n\right) - \frac{1}{2} \sum_{n=1}^{\kappa} m_h\left(v_h^n, v_h^n\right) - \sum_{n=1}^{\kappa} \frac{1}{2} m_h\left(v_h^{n-1}, v_h^{n-1}\right) \\ &= \sum_{n=1}^{\kappa} \left(\frac{1}{2} m_h\left(v_h^n, v_h^n\right) - \frac{1}{2} m_h\left(v_h^{n-1}, v_h^{n-1}\right)\right) \\ &= \frac{1}{2} m_h\left(v_h^\kappa, v_h^\kappa\right) - \frac{1}{2} m_h\left(v_h^0, v_h^0\right). \end{aligned}$$

Using the last inequality, the definition of the forms b_h, c_h, the assumptions (2.2) and (3.5), we get

$$\frac{1}{2}\beta'\big(v_h^\kappa, v_h^\kappa\big)_{0,\Omega} + \frac{1}{2}\beta'\big(w_h^\kappa, w_h^\kappa\big)_{0,\Omega} + d_1\alpha'\int_0^{\kappa\Delta t} a(v_h, v_h) + \sum_{n=1}^{\kappa}\Delta t\left(\sum_{K\in\mathcal{T}_h}\int_K I_{1,\mathrm{ion}}\big(\Pi_{K,k}^0 v_h^n\big)\Pi_{K,k}^0 v_h^n\right)$$

$$\leq \frac{1}{2}\beta''\big(v_h^0, v_h^0\big)_{0,\Omega} + \frac{1}{2}\beta''\big(w_h^0, w_h^0\big)_{0,\Omega} + \sum_{n=1}^{\kappa\Delta t}\left(\sum_{K\in\mathcal{T}_h}\int_K H\big(\Pi_{K,k}^0 v_h^n, \Pi_{K,k}^0 w_h^n\big)\Pi_{K,k}^0 w_h^n\right)$$

$$-\sum_{n=1}^{\kappa\Delta t}\left(\sum_{K\in\mathcal{T}_h}\int_K I_{2,\mathrm{ion}}\big(\Pi_{K,k}^0 w_h^n\big)\Pi_{K,k}^0 v_h^n\right) + \int_0^{\kappa\Delta t}\big(I_{\mathrm{app},h}, v_h\big)_{0,\Omega}.$$

Now, using the definition of bilinear form $a(\cdot,\cdot)$ and (2.3)(a) on the left-hand side; moreover, we use (2.3)(b), (2.3)(c), (2.3)(d) and Cauchy–Schwarz inequality, and the fact that $I_{\mathrm{app}}(x,t) \in L^2(\Omega_T)$; on the right-hand side, we obtain

$$\frac{1}{2}\beta'\|v_h^\kappa\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta'\|w_h^\kappa\|_{L^2(\Omega)}^2 + d_1\alpha'\int_0^{\kappa\Delta t}|v_h|_{1,\Omega}^2 + \sum_{n=1}^{\kappa\Delta t}\left(\sum_{K\in\mathcal{T}_h}\int_K \frac{1}{\alpha_1}|\Pi_{K,k}^0 v_h^n|^4\right)$$

$$\leq \frac{1}{2}\beta''\|v_h^0\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta''\|w_h^0\|_{L^2(\Omega)}^2 + \alpha_4\sum_{n=1}^{\kappa\Delta t}\left(\sum_{K\in\mathcal{T}_h}\int_K |\Pi_{K,k}^0 v_h^n||\Pi_{K,k}^0 w_h^n| + |\Pi_{K,k}^0 w_h^n|^2\right)$$

$$+\alpha_3\sum_{n=1}^{\kappa\Delta t}\left(\sum_{K\in\mathcal{T}_h}\int_K |\Pi_{K,k}^0 w_h^n||\Pi_{K,k}^0 v_h^n|\right) + \int_0^{\kappa\Delta t}\|v_h\|_{L^2(\Omega)}^2 + C.$$

An application of the Cauchy–Schwarz and Young inequalities, the continuity of $\Pi_{K,k}^0$ with respect to $\|\cdot\|_{0,K}$, yields

$$\frac{1}{2}\beta'\|v_h^\kappa\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta'\|w_h^\kappa\|_{L^2(\Omega)}^2 + d_1\alpha'\int_0^{\kappa\Delta t}|v_h|_{H^1(\Omega)}^2 + \sum_{n=1}^{\kappa\Delta t}\left(\sum_{K\in\mathcal{T}_h}\int_K \frac{1}{\alpha_1}|\Pi_{K,k}^0 v_h^n|^4\right)$$

$$\leq \frac{1}{2}\beta''\|v_h^0\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta''\|w_h^0\|_{L^2(\Omega)}^2$$

$$+\int_0^{\kappa\Delta t}\left(1 + \frac{\alpha_3^2+\alpha_4^2}{2}\right)\|v_h\|_{L^2(\Omega)}^2 + \int_0^{\kappa\Delta t}\left(\alpha_4 + \frac{\alpha_3^2+\alpha_4^2}{2}\right)\|w_h\|_{L^2(\Omega)}^2 + C$$

$$\leq \frac{1}{2}\beta''\|v_h^0\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta''\|w_h^0\|_{L^2(\Omega)}^2 + C_1\|v_h\|_{L^2(\Omega_T)}^2 + C_2\|w_h\|_{L^2(\Omega_T)}^2 + C, \tag{4.1}$$

for some constants $C_1, C_2 > 0$. This implies

$$\frac{1}{2}\beta' \left\|v_h^\kappa\right\|^2_{L^2(\Omega)} + \frac{1}{2}\beta' \left\|w_h^\kappa\right\|^2_{L^2(\Omega)} \le C_3 \left\|v_h\right\|^2_{L^2(\Omega_T)} + C_4 \left\|w_h\right\|^2_{L^2(\Omega_T)} + C_5, \tag{4.2}$$

for some $C_3, C_4, C_5 > 0$. Therefore, by the discrete Gronwall inequality, yields from (4.2),

$$\|v_h\|_{L^\infty(0,T;L^2(\Omega))} + \|w_h\|_{L^\infty(0,T;L^2(\Omega))} \le C_6, \tag{4.3}$$

for some constant $C_6 > 0$. Finally, using (4.3) in (4.1) and (2.3), we get

$$\|\Pi_k^0 v_h\|_{L^4(\Omega_T)} + \|\nabla v_h\|_{L^2(\Omega_T)} \le C_7, \tag{4.4}$$

for some constant $C_7 > 0$. This concludes the proof of Lemma 4.2. □

4.2 *Compactness argument and convergence*

In this section, we will use time-continuous approximation of our discrete solution to obtain compactness in $L^2(\Omega_T)$. For this, we introduce $\bar{v}_h$ and $\bar{w}_h$ the piecewise affine in t functions in $W^{1,\infty}([0,T]; W_h)$ interpolating the states $(v_h^n)_{n=0,\dots,N} \subset W_h$ and $(w_h^n)_{n=0,\dots,N} \subset W_h$ at the points $(n\Delta t)_{n=0,\dots,N}$. Then we have

$$\begin{cases} m_h(\partial_t \bar{v}_h(t), \varphi_h) + D\left(J(v_h(t))\right) a_h(v_h(t), \varphi_h) + b_h(v_h(t), w_h(t), \varphi_h) = \left(I_{app,h}(t), \varphi_h\right)_{0,\Omega}, \\ m_h(\partial_t \bar{w}_h(t), \phi_h) = c_h(v_h(t), w_h(t), \phi_h), \end{cases} \tag{4.5}$$

for all φ_h and $\phi_h \in W_h$.

LEMMA 4.3 There exists a positive constant $C > 0$ depending on Ω, T, v_0 and I_{app} such that

$$\iint_{\Omega_{\boldsymbol{r}} \times (0,T)} m_h\Big(v_h(x+\boldsymbol{r},t) - v_h(x,t), v_h(x+\boldsymbol{r},t) - v_h(x,t)\Big) \le C\,|\boldsymbol{r}|^2, \tag{4.6}$$

for all $\boldsymbol{r} \in \mathbb{R}^2$ with $\Omega_{\boldsymbol{r}} := \{x \in \Omega \mid x + \boldsymbol{r} \in \Omega\}$ and

$$\iint_{\Omega \times (0,T-\tau)} m_h\Big(v_h(x,t+\tau) - v_h(x,t), v_h(x,t+\tau) - v_h(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \le C(\tau + \Delta t), \tag{4.7}$$

for all $\tau \in (0,T)$.

Proof. In the first step, we provide the *proof of estimate* (4.6). In this regard, we start with the uniform estimate of space translate of v_h from the uniform $L^2(\Omega_T)$ estimate of ∇v_h. Observe that from $L^2(0,T;H^1(\Omega))$ estimate of v_h, we get easily the bound

$$m_h^{\boldsymbol{r}}\left(v_h(x+\boldsymbol{r},t) - v_h(x,t), v_h(x+\boldsymbol{r},t) - v_h(x,t)\right) \le C\int_0^T\int_{\Omega_{\boldsymbol{r}}} |v_h(x+\boldsymbol{r},\cdot) - v_h(x,\cdot)|^2 \le \mathcal{C}|\boldsymbol{r}|^2, \tag{4.8}$$

for some constant $\mathcal{C} > 0$, where $m_h^{\boldsymbol{r}}(\cdot,\cdot)$ is the restriction of the bilinear form $m_h(\cdot,\cdot)$ on $\Omega_{\boldsymbol{r}}$. It is clear that the right-hand side in (4.8) vanishes as $|\boldsymbol{r}| \to 0$, uniformly in h.

Now, we furnish the proof of estimate (4.7). Observe that for all $t \in [0, T-\tau]$, the function φ_h^v such that $\varphi_h^v(x,t) = v_h(x,t+\tau) - v_h(x,t)$ takes value in W_h for $(x,t) \in \Omega_T$. Therefore, we can use this function as a test function in the weak formulations (3.10). Moreover, we previously proved uniform in h bounds on v_h and ∇v_h in $L^2(\Omega_T)$ and on $\Pi_k^0 v_h$ in $L^4(\Omega_T)$. This implies the analogous bounds for the translates φ_h^v and $\nabla\varphi_h^v$ in $L^2(\Omega \times (0, T-\tau))$ and $\Pi_k^0\varphi_h^v$ in $L^4(\Omega \times (0, T-\tau))$.

We integrate the first approximation equation of (4.5) with respect to the time parameter $s \in [t, t+\tau]$ (with $0 < \tau < T$). In the resulting equations, we take the test function as the corresponding translate φ_h^v. The result is

$$\begin{aligned}
&\int_0^{T-\tau}\int_\Omega m_h\Big(v_h(x,t+\tau) - v_h(x,t), v_h(x,t+\tau) - v_h(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^{T-\tau}\int_\Omega\int_t^{t+\tau} m_h\Big(\partial_s\bar{v}_h(x,s), v_h(x,t+\tau) - v_h(x,t)\Big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{T-\tau}\int_\Omega\int_t^{t+\tau} D\left(J(v_h(x,s))\right) a_h(v_h(x,s), v_h(x,t+\tau) - v_h(x,t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^{T-\tau}\int_\Omega\int_t^{t+\tau} b_h(v_h(x,s), w_h(x,s), v_h(x,t+\tau) - v_h(x,t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{T-\tau}\int_\Omega\int_t^{t+\tau} (I_{app,h}, v_h(x,t+\tau) - v_h(x,t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now, we bound these integrals separately. For the term I_1, we have

$$\begin{aligned}
\left|I_1\right| &\le C\left[\int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau}\left|\nabla v_h(x,s)\right|\mathrm{d}s\right)^2\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}} \times \left[\int_0^{T-\tau}\int_\Omega\left|\nabla(v_h(x,t+\tau) - v_h(x,t)\right|^2\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}} \\
&\le C\tau
\end{aligned}$$

for some constant $C > 0$. Herein, we used the Fubini theorem (recall that $\int_t^{t+\tau}\,\mathrm{d}s = \tau = \int_{s-\tau}^{s}\,\mathrm{d}t$), the Hölder inequality and the bounds in L^2 of ∇v_h. Keeping in mind the growth bound of the nonlinearity I_{ion}, we apply the Hölder inequality (with $p = 4$, $p' = 4/3$ in the ionic current term and with $p = p' = 2$ in the other ones) to deduce

$$\begin{aligned}
\left|I_2\right| \le C\Bigg(&\left[\int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau}\left|\Pi_k^0 v_h(x,s)\right|^3\mathrm{d}s\right)^{\frac{4}{3}}\mathrm{d}x\,\mathrm{d}t\right]^{\frac{3}{4}} \times \left[\int_0^{T-\tau}\int_\Omega\left|\Pi_k^0\varphi_h^v(x,t)\right|^4\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{4}} \\
&+ \left[\int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau}\left|w_h(x,s)\right|\mathrm{d}s\right)^2\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}} \times \left[\int_0^{T-\tau}\int_\Omega\left|\varphi_h^v(x,t)\right|^2\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}}\Bigg) \\
\le C\tau,&
\end{aligned}$$

for some constant $C > 0$, where we have used that v_h, φ_h^v and w_h are uniformly bounded in L^2, and $\Pi_k^0 v_h, \Pi_k^0\varphi_h^v$ are bounded in L^4 and the continuity of $\Pi_{K,k}^0$ with respect to $\|\cdot\|_{L^2(K)}$.

Analogously, we obtain

$$\left|I_3\right| \leq C\,\tau,$$

for some constant $C > 0$. Collecting the previous inequalities, we readily deduce

$$\int_0^{T-\tau}\int_\Omega m_h\Big(v_h(x,t+\tau)-v_h(x,t), v_h(x,t+\tau)-v_h(x,t)\Big) \leq \mathcal{C}\,\tau.$$

Note that, it is easily seen from the definition of $(\bar{v}_h, \bar{w}_h)$ and from the discrete weak formulation (3.10) and estimates in Proposition 4.2 that

$$\|\bar{v}_h - v_h\|^2_{L^2(\Omega_T)} \leq \sum_{n=1}^{N} \Delta t \|v_h^n - v_h^{n-1}\|^2_{L^2(\Omega)} \leq \mathcal{C}(\Delta t) \to 0 \;\text{ as } \Delta t \to 0.$$

This concludes the proof of Lemma 4.3. □

4.3 *Convergence of the virtual element scheme*

For convergence of our numerical scheme we need the following estimate:

$$\left\|\Pi_k^0 u - u\right\|_{L^2(\Omega)} \leq Ch^{k+1}\,\|u\|_{H^{k+1}(\Omega)} \text{ for all } u \in H^{k+1}(\Omega), \tag{4.9}$$

for some constant $C > 0$. This result follows from standard approximation results (see Brenner & Scott, 2008).

Note that from Lemma 4.3 and the stability condition (3.5), we get

$$\iint_{\Omega_{\boldsymbol{r}}\times(0,T)} \left|v_h(x+\boldsymbol{r},t) - v_h(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \leq \frac{C}{\beta'}\,|\boldsymbol{r}|^2$$

and

$$\iint_{\Omega\times(0,T-\tau)} \left|v_h(x,t+\tau) - v_h(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \leq \frac{C}{\beta'}(\tau + \Delta t).$$

Therefore, the next lemma is a consequence of (4.9), Lemma 4.3 and Kolmogorov's compactness criterion (see, e.g., Brezis, 1983, Theorem IV.25).

LEMMA 4.4 There exists a subsequence of $\mathbf{u}_h = (v_h, w_h)$, not relabeled, such that, as $h \to 0$,

$$\begin{aligned} &v_h, \Pi_k^0 v_h \to v \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\\ &w_h, \Pi_k^0 w_h \to w \text{ weakly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\\ &v_h \rightharpoonup v \text{ weakly in } L^2(0,T;H^1(\Omega)),\\ &\Pi_k^0 v_h \rightharpoonup v \text{ weakly in } L^4(\Omega_T). \end{aligned} \tag{4.10}$$

Now, we are going to show that the limit functions $\mathbf{u} := (v, w)$ constructed in Lemma 4.4 constitute a weak solution of the nonlocal system defined in (2.6).

For that we let $\varphi \in \mathcal{D}(\Omega \times [0, T))$. We approximate φ by $\varphi_h \in C[0, T; L^2(\Omega)]$ such that $\varphi_h|_{(t^{n-1},t^n)} \in \mathcal{P}_k[t^{n-1}, t^n; W_h]$ and $\varphi_h(T) = 0$, where $\mathcal{P}_k[t^{n-1}, t^n; W_h]$ denotes the space of polynomials of degree k or less having values in W_h.

Let $\mathbf{u}_h := (v_h, w_h)$ be the unique solution of the fully discrete method (3.10). The proof is based on the convergence to zero as h goes to zero of each term of the problems.

We start with the convergence of the nonlocal diffusion term. Observe that

$$\begin{aligned}\Big|D(J(v_h))a_h(v_h, \varphi_h) - D(J(v))a(v, \varphi)\Big| &\leq \Big|D(J(v))[a_h(v_h, \varphi_h) - a(v, \varphi)]\Big| \\ &\quad + \Big|D(J(v_h)) - D(J(v))\Big||a_h(v_h, \varphi_h)| \\ &:= A_1 + A_2. \end{aligned} \tag{4.11}$$

For A_2, we have

$$\begin{aligned} A_2 = \Big|D(J(v_h)) - D(J(v))\Big||a_h(v_h, \varphi_h)| &\leq C|J(v_h) - J(v)||a_h(v_h, \varphi_h)| \\ &\leq C(\|v_h - v\|_{L^2(\Omega)} + \|v - \Pi_k^0 v\|_{L^2(\Omega)})|v_h|_{H^1(\Omega)}|\varphi_h|_{H^1(\Omega)} \\ &\leq C(\|v_h - v\|_{L^2(\Omega)} + h|v|_{H^1(\Omega)})|v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}, \end{aligned}$$

where we have used the assumption (2.2), the definition of $J(v_h)$ in (3.6), then we add and subtract an appropriate polynomial function and finally the continuity of bilinear form $a_h(\cdot, \cdot)$ in (3.8). Thus, using (4.10), we have that (recall that $\varphi \in \mathcal{D}(\Omega \times [0, T))$)

$$\lim_{h \to 0} \int_0^T A_2 \, \mathrm{d}t = 0.$$

Now, we bound the term A_1 in (4.11). Using the definition of bilinear form $a_h(\cdot, \cdot)$, the assumption (2.2), we have

$$\begin{aligned} A_1 = |D(J(v))(a_h(v_h, \varphi_h) - a(v, \varphi))| &\leq |J(v)| \sum_{K \in \mathcal{T}_h} \Big|a_h^K(v_h, \varphi_h) - a^K(v, \varphi)\Big| \\ &\leq \Big|J(v)\Big|\Bigg[\sum_{K \in \mathcal{T}_h} |a^K(\Pi_{K,k} v_h, \Pi_{K,k}\varphi_h) - a^K(v, \varphi)| + \sum_{K \in \mathcal{T}_h} |S^K(v_h - \Pi_{K,k} v_h, \varphi_h - \Pi_{K,k}\varphi_h)|\Bigg] \\ &\leq C\|v\|_{L^2(\Omega)}\Bigg[\sum_{K \in \mathcal{T}_h} |a^K(\Pi_{K,k} v_h - v, \Pi_{K,k}\varphi_h)| + \sum_{K \in \mathcal{T}_h} |a^K(v, \Pi_{K,k}\varphi_h - \varphi)| \\ &\quad + \sum_{K \in \mathcal{T}_h} \Big|a^K(v_h - \Pi_{K,k} v_h, \varphi_h - \Pi_{K,k}\varphi_h)\Big|\Bigg], \end{aligned}$$

where we have added and subtracted $a^K(v, \Pi_{K,k}\varphi_h)$ and used (3.2). Defining

$$\Theta(h) := \sum_{K\in\mathcal{T}_h} |a^K(\Pi_{K,k}v_h - v, \Pi_{K,k}\varphi_h)|.$$

Now, using this and the Cauchy–Schwarz inequality, we obtain

$$A_1 \leq C\|v\|_{L^2(\Omega)}\Big[\Theta(h) + \sum_{K\in\mathcal{T}_h} |v|_{H^1(K)}|\Pi_{K,k}\varphi_h - \varphi|_{H^1(K)} + \sum_{K\in\mathcal{T}_h} |v_h - \Pi_{K,k}v_h|_{H^1(K)}|\varphi_h - \Pi_{K,k}\varphi_h|_{H^1(K)}\Big].$$

Next we add and subtract an appropriate polynomial φ_Π in the second term, and we add and subtract φ in the last term. Thus, we have

$$\begin{aligned} A_1 &\leq C\|v\|_{L^2(\Omega)}\Big[\Theta(h) + \sum_{K\in\mathcal{T}_h} |v|_{H^1(K)}(|\Pi_{K,k}(\varphi_h - \varphi_\Pi)|_{H^1(K)} + |\varphi - \varphi_\Pi|_{H^1(K)}) \\ &\qquad + \sum_{K\in\mathcal{T}_h} |v_h - \Pi_{K,k}v_h|_{H^1(K)}(|\varphi_h - \varphi|_{H^1(K)} + |\varphi - \Pi_{K,k}\varphi_h|_{H^1(K)})\Big] \\ &\leq C\|v\|_{L^2(\Omega)}\Big[\Theta(h) + \sum_{K\in\mathcal{T}_h} |v|_{H^1(K)}(|\varphi_h - \varphi|_{H^1(K)} + |\varphi - \varphi_\Pi|_{H^1(K)}) \\ &\qquad + \sum_{K\in\mathcal{T}_h} |v_h|_{H^1(K)}(|\varphi_h - \varphi|_{H^1(K)} + |\varphi - \varphi_\Pi|_{H^1(K)})\Big]. \end{aligned}$$

Now, using (4.10), standard approximation results for polynomials and the regularity of φ, we obtain

$$\lim_{h\to 0}\int_0^T A_1\,\mathrm{d}t = 0.$$

Finally, we get

$$\int_0^T \Big|D(J(v_h))a_h(v_h, \varphi_h) - D(J(v))a(v, \varphi)\Big|\,\mathrm{d}t \to 0 \ \text{ as } h \to 0.$$

Now, we prove

$$\left|\int_0^T m_h(v_h, \partial_t\varphi_h) - (v, \partial_t\varphi)_{0,\Omega}\right| \to 0 \ \text{ as } h \to 0. \tag{4.12}$$

In fact, using the definition of the bilinear form $m_h(\cdot,\cdot)$, we obtain

$$\begin{aligned}\left|\int_0^T m_h(v_h,\partial_t\varphi_h)-(v,\partial_t\varphi)_{0,\Omega}\right| &\le \left|\sum_{K\in\mathcal{T}_h}\left(\Pi_K^0 v_h,\Pi_K^0\partial_t\varphi_h\right)_{0,K}-(v,\partial_t\varphi)_{0,K}\right| \\ &\quad+\left|S_0^K\left(v_h-\Pi_K^0 v_h,\partial_t\varphi_h-\Pi_K^0\partial_t\varphi_h\right)\right| \\ &\le \left|\sum_{K\in\mathcal{T}_h}\left(\Pi_K^0 v_h-v,\Pi_K^0\partial_t\varphi_h\right)_{0,K}\right|+\left|\left(v,\Pi_K^0\partial_t\varphi_h-\partial_t\varphi\right)_{0,K}\right| \\ &\quad+\left|S_0^K(v_h-\Pi_K^0 v_h,\partial_t\varphi_h-\Pi_K^0\partial_t\varphi_h)\right| \\ &\le \left\|v_h-v\right\|_{L^2(\Omega)}\left\|\partial_t\varphi\right\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)}\left\|\partial_t\varphi_h-\partial_t\varphi\right\|_{L^2(\Omega)} \\ &\quad+\left\|v_h\right\|_{L^2(\Omega)}\left(\left\|\partial_t\varphi_h-\partial_t\varphi_\Pi\right\|_{L^2(\Omega)}+\left\|\partial_t\varphi-\partial_t\varphi_\Pi\right\|_{L^2(\Omega)}\right).\end{aligned}$$

Using this, (4.10), standard approximation results for polynomials and the regularity of φ, we arrive to (4.12). Now, we prove

$$\int_0^T \left|b_h(v_h,w_h,\varphi_h)-(I_{\mathrm{ion}}(v,w),\varphi)_{0,\Omega}\right|\,\mathrm{d}t\to 0 \ \text{ as } h\to 0.$$

Using the definition of the form $b_h(\cdot,\cdot,\cdot)$ and the decomposition of the ionic current $I_{\mathrm{ion}}(v,w)$, we have

$$\begin{aligned}&\left|b_h(v_h,w_h,\varphi_h)-(I_{\mathrm{ion}}(v,w),\varphi)_{0,\Omega}\right|=\left|\sum_{K\in\mathcal{T}_h}(I_{\mathrm{ion}}(\Pi_K^0 v_h,\Pi_K^0 w_h),\Pi_K^0\varphi_h)_{0,K}-(I_{\mathrm{ion}}(v,w),\varphi)_{0,K}\right| \\ &=\left|\sum_{K\in\mathcal{T}_h}(I_{1,\mathrm{ion}}(\Pi_K^0 v_h),\Pi_K^0\varphi_h)_{0,K}+(I_{2,\mathrm{ion}}(\Pi_K^0 w_h),\Pi_K^0\varphi_h)_{0,K}-(I_{1,\mathrm{ion}}(v),\varphi)_{0,K}-(I_{2,\mathrm{ion}}(w),\varphi)_{0,K}\right| \\ &\le\sum_{K\in\mathcal{T}_h}|(I_{1,\mathrm{ion}}(\Pi_K^0 v_h),\Pi_K^0\varphi_h)_{0,K}-(I_{1,\mathrm{ion}}(v),\varphi)_{0,K}|+|(I_{2,\mathrm{ion}}(\Pi_K^0 w_h),\Pi_K^0\varphi_h)_{0,K}-(I_{2,\mathrm{ion}}(w),\varphi)_{0,K}| \\ &=: B_1+B_2.\end{aligned}$$

Note that since the function $I_{2,\mathrm{ion}}$ is a linear function, we get easily

$$\int_0^T B_2\,\mathrm{d}t\to 0 \text{ as } h \text{ goes to } 0.$$

Now, we turn to the term B_1, we have the following estimation:

$$B_1 \le \sum_{K\in\mathcal{T}_h} \left| \left(I_{1,\text{ion}}(\Pi_K^0 v_h), \Pi_K^0 \varphi_h\right)_{0,K} - \left(I_{1,\text{ion}}(\Pi_K^0 v_h), \varphi\right)_{0,K} \right|$$

$$+ \sum_{K\in\mathcal{T}_h} \left| \left(I_{1,\text{ion}}(\Pi_K^0 v_h), \varphi\right)_{0,K} - (I_{1,\text{ion}}(v), \varphi)_{0,K} \right|$$

$$\le \left\|\varphi_h - \varphi\right\|_{L^\infty(\Omega)} \left\|I_{1,\text{ion}}(\Pi_K^0 v_h)\right\|_{L^1(\Omega)} + Const(v, \Pi_K^0 v_h, v_h) \left\|\Pi_K^0 v_h - v\right\|_{L^2(\Omega)},$$

where $Const(v, \Pi_K^0 v_h, v_h) > 0$ is a constant. This implies that

$$\int_0^T B_1 \,\mathrm{d}t \to 0 \text{ as } h \text{ goes to } 0.$$

Similarly, we get

$$\int_0^T \left| \left(I_{app,h}, \varphi_h\right)_{0,\Omega} - (I_{\text{app}}(x,t), \varphi)_{0,\Omega} \right| \mathrm{d}t \to 0 \quad \text{as } h \to 0.$$

With the above convergences and, by density, we are ready to identify the limit $\mathbf{u} = (v, w)$ as a (weak) solution of the system (2.1). Finally, let $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$ and $\phi \in C([0,T];L^2(\Omega))$, then by passing to the limit $h \to 0$ in the following weak formulation (with the help of Lemma 4.4)

$$-\int_0^T m_h(v_h(t), \partial_t\varphi_h) + \int_0^T D\left(J(v_h(t))\right) a_h(v_h(t), \varphi_h) + \int_0^T b_h(v_h(t), w_h(t), \varphi_h) = \int_0^T (I_{\text{app},h}(t), \varphi_h)_{0,\Omega}$$

$$\int_0^T m_h(\partial_t w_h(t), \phi_h) = \int_0^T c_h(v_h(t), w_h(t), \phi_h),$$

we obtain the limit $\mathbf{u} = (v, w)$ that is a solution of system (2.1) in the sense of Definition 2.1.

5. Error estimates analysis

In this section, error estimates will be developed to our model (2.1). For technical reason (because of the nonlinearity of I_{ion}), we need to relax the assumptions (2.3). For the error estimates analysis, we will use the following assumption on I_{ion}: we assume that I_{ion} is a linear function on v and w, satisfying

$$\forall\, s_1, s_2, z_1, z_2 \in \mathbb{R} \quad |I_{\text{ion}}(s_1, z_1) - I_{\text{ion}}(s_2, z_2)| \le \alpha_7(|s_1 - s_2| + |z_1 - z_2|), \tag{5.1}$$

for some constant $\alpha_7 > 0$.

First, we introduce the projection $\mathcal{P}^h : H^1(\Omega) \to W_h$ as the solution of the following well-posed problem:

$$\begin{cases} \mathcal{P}^h u \in W_h, \\ a_h(\mathcal{P}^h u, \varphi_h) = a(u, \varphi_h) \text{ for all } \varphi_h \in W_h. \end{cases}$$

We have the following lemma; the proof can be found in Beirão da Veiga *et al.* (2016, Lemma 3.1).

LEMMA 5.1 Let $u \in H^1(\Omega)$. Then there exist $C, \tilde{C} > 0$, independent of h, such that

$$\left|\mathcal{P}^h u - u\right|_{H^1(\Omega)} \leq Ch^k \left|u\right|_{H^{k+1}(\Omega)}.$$

Moreover, if the domain is convex then

$$\left\|\mathcal{P}^h u - u\right\|_{L^2(\Omega)} \leq \tilde{C}h^{k+1} \left|u\right|_{H^{k+1}(\Omega)}.$$

Our main result in this section is the following theorem.

THEOREM 5.2 Let (v, w) be the solution of system (2.1) and let $(v_h(t), w_h(t))$ be the solution of the problem (3.9). Then for all $t \in (0, T)$, we have

$$\begin{aligned}
&\left\|v_h(\cdot,t) - v(\cdot,t)\right\|_{L^2(\Omega)} + \left\|w_h(\cdot,t) - w(\cdot,t)\right\|_{L^2(\Omega)} \\
&\quad \leq C\Bigg[\left\|v_0 - v_h^0\right\|_{L^2(\Omega)} + \left\|w_0 - w_h^0\right\|_{L^2(\Omega)} + h^{k+1}\Bigg(\left|v_0\right|_{H^{k+1}(\Omega)} + \left|w_0\right|_{H^{k+1}(\Omega)} \\
&\qquad + \int_0^t \left(\left|I_{\mathrm{app}}\right|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + \left|\partial_t v\right|_{H^{k+1}(\Omega)} + \left|\partial_t w\right|_{H^{k+1}(\Omega)}\right) \mathrm{d}t\Bigg)\Bigg] \\
&\qquad \times \exp\left(\int_0^t \left(1 + |v|_{H^2(\Omega)}\right) \mathrm{d}t\right),
\end{aligned} \tag{5.2}$$

for some constant $C > 0$. Moreover, let $\mathbf{u}_h^n = (v_h^n, w_h^n)$ be the virtual element solution generated by (3.10). Then for $n = 1, \ldots, N$,

$$\begin{aligned}
&\left\|v_h^n - v(\cdot,t_n)\right\|_{L^2(\Omega)} + \left\|w_h^n - w(\cdot,t_n)\right\|_{L^2(\Omega)} \\
&\quad \leq C\Bigg[\left\|v_0 - v_h^0\right\|_{L^2(\Omega)} + \left\|w_0 - w_h^0\right\|_{L^2(\Omega)} + \Delta t \int_0^{t_n} \left(\left|\partial_{tt}^2 v\right| + \left|\partial_{tt}^2 w\right|\right) \mathrm{d}t \\
&\qquad + h^{k+1}\Bigg(\left|v_0\right|_{H^{k+1}(\Omega)} + \left|w_0\right|_{H^{k+1}(\Omega)} \\
&\qquad + \int_0^{t_n} \left(\left|I_{\mathrm{app}}\right|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + \left|\partial_t v\right|_{H^{k+1}(\Omega)} + \left|\partial_t w\right|_{H^{k+1}(\Omega)}\right) \mathrm{d}t\Bigg)\Bigg] \\
&\qquad \times \exp\left(\int_0^{t_n} \left(1 + |v|_{H^2(\Omega)}\right) \mathrm{d}t\right).
\end{aligned} \tag{5.3}$$

Proof. We start with the proof of bound (5.2). First, note that

$$U_h(\cdot,t) - U(\cdot,t) = (U_h(\cdot,t) - \mathcal{P}^h U(\cdot,t)) + (\mathcal{P}^h U(\cdot,t) - U(\cdot,t)) \quad \text{for } U = v, w.$$

Observe that from Lemma 5.1, we get easily for $U = v, w$,

$$\begin{aligned}\|\mathcal{P}^h U(\cdot,t) - U(\cdot,t)\|_{L^2(\Omega)} &\le Ch^{k+1}\,|U|_{H^{k+1}(\Omega)} \\ &\le Ch^{k+1}\left(\left|U_0\right|_{H^{k+1}(\Omega)} + \int_0^t \left|\partial_t U(\cdot,s)\right|_{H^{k+1}(\Omega)}\,\mathrm{d}s\right) \\ &= Ch^{k+1}\left(\left|U_0\right|_{H^{k+1}(\Omega)} + \left\|\partial_t U\right\|_{L^1(0,t;H^{k+1}(\Omega))}\right),\end{aligned} \tag{5.4}$$

for all $t \in (0,T)$.

Observe that, using the continuous and semidiscrete problems (cf. (2.1) and (3.9)), the definition of the projector $\mathcal{P}^h$ and the fact that the derivative with respect to time commutes with this projector, we obtain

$$\begin{aligned}&m_h(\partial_t(v_h - \mathcal{P}^h v), \varphi_h^v) + D\left(J(v_h)\right) a_h\left((v_h - \mathcal{P}^h v), \varphi_h^v\right) \\ &\quad= \left(I_{\mathrm{app}_h}, \varphi_h^v\right)_{0,\Omega} - b_h\left(v_h, w_h, \varphi_h^v\right) - m_h\left(\partial_t \mathcal{P}^h v, \varphi_h^v\right) - D(J(v_h))a_h\left(\mathcal{P}^h v, \varphi_h^v\right) \\ &\quad= \left(I_{\mathrm{app}_h}, \varphi_h^v\right)_{0,\Omega} - b_h\left(v_h, w_h, \varphi_h^v\right) - m_h\left(\mathcal{P}^h \partial_t v, \varphi_h^v\right) - D(J(v_h))a\left(v, \varphi_h^v\right) \\ &\quad= \left[\left(I_{\mathrm{app}_h}, \varphi_h^v\right)_{0,\Omega} - \left(I_{\mathrm{app}}, \varphi_h^v\right)_{0,\Omega}\right] - \left[b_h\left(v_h, w_h, \varphi_h^v\right) - \left(I_{\mathrm{ion}}(v,w), \varphi_h^v\right)_{0,\Omega}\right] \\ &\qquad + \left[\left(\partial_t v, \varphi_h^v\right)_{0,\Omega} - m_h\left(\mathcal{P}^h \partial_t v, \varphi_h^v\right)\right] + \left[\left(D\left(J(v)\right) - D\left(J(v_h)\right)\right) a\left(v, \varphi_h^v\right)\right] \\ &\quad:= I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{5.5}$$

for all $\varphi_h^v \in W_h$. Now, we are going to bound each term $I_1, \ldots, I_4$. Regarding the first term I_1, we have

$$I_1 = (\Pi_k^0 I_{\mathrm{app}} - I_{\mathrm{app}}, \varphi_h^v)_{0,\Omega} \le Ch^{k+1}|I_{\mathrm{app}}|_{H^{k+1}(\Omega)}\left\|\varphi_h^v\right\|_{L^2(\Omega)}, \tag{5.6}$$

for some constant $C > 0$, where we have used the definition of I_{app_h}. Next for I_2, using the definition of the form $b_h(\cdot,\cdot,\cdot)$, and adding and subtracting adequate terms, we have

$$\begin{aligned}I_2 &= -\left[b_h(v_h, w_h, \varphi_h^v) - (\mathcal{P}^h I_{\mathrm{ion}}(v,w), \varphi_h^v)_{0,\Omega}\right] - \left[\left(\mathcal{P}^h I_{\mathrm{ion}}(v,w), \varphi_h^v\right)_{0,\Omega} - \left(I_{\mathrm{ion}}(v,w), \varphi_h^v\right)_{0,\Omega}\right] \\ &= -\left[\sum_{K\in\mathcal{T}_h}\left(I_{\mathrm{ion}}\left(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h\right), \Pi_{K,k}^0 \varphi_h^v\right)_{0,K} - \left(I_{\mathrm{ion}}(\mathcal{P}^h v, \mathcal{P}^h w), \varphi_h^v\right)_{0,K}\right] \\ &\quad - \left[\left(\mathcal{P}^h I_{\mathrm{ion}}(v,w) - I_{\mathrm{ion}}(v,w), \varphi_h^v\right)_{0,\Omega}\right] \\ &= -\left[\sum_{K\in\mathcal{T}_h}\left(I_{\mathrm{ion}}\left(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h\right) - I_{\mathrm{ion}}(\mathcal{P}^h v, \mathcal{P}^h w), \varphi_h^v\right)_{0,K}\right] - \left[\left(\mathcal{P}^h I_{\mathrm{ion}}(v,w) - I_{\mathrm{ion}}(v,w), \varphi_h^v\right)_{0,\Omega}\right]\end{aligned}$$

$$
\begin{aligned}
&\leq C\Big[\sum_{K\in\mathcal{T}_h}\left(\|\Pi^0_{K,k}v_h-\mathcal{P}^h v\|_{L^2(K)}+\|\Pi^0_{K,k}w_h-\mathcal{P}^h w\|_{L^2(K)}\right)\left\|\varphi^v_h\right\|_{L^2(K)}\Big]\\
&\quad + Ch^{k+1}(|v|_{H^{k+1}(\Omega)}+|w|_{H^{k+1}(\Omega)})\left\|\varphi^v_h\right\|_{L^2(\Omega)}\\
&\leq C\left(\|\Pi^0_k v_h-\mathcal{P}^h v\|_{L^2(\Omega)}+\|\Pi^0_k w_h-\mathcal{P}^h w\|_{L^2(\Omega)}\right)\|\varphi^v_h\|_{L^2(\Omega)}\\
&\quad + Ch^{k+1}\left(|v|_{H^{k+1}(\Omega)}+|w|_{H^{k+1}(\Omega)}\right)\left\|\varphi^v_h\right\|_{L^2(\Omega)},
\end{aligned}
$$

for some constant $C > 0$, where we have used that I_{ion} is a linear function, (5.1), the properties of projectors Π^0_k and $\mathcal{P}^h$, and finally Lemma 5.1.

For I_3, we use the consistency and stability properties of the bilinear for $m_h(\cdot,\cdot)$ to get

$$
\begin{aligned}
I_3 &= \sum_{K\in\mathcal{T}_h}\left[\left(\partial_t v-\Pi^0_{K,k}\partial_t v,\varphi^v_h\right)_{0,K}+m^K_h\left(\Pi^0_{K,k}\partial_t v-\mathcal{P}^h\partial_t v,\varphi^v_h\right)\right]\\
&\leq C\sum_{K\in\mathcal{T}_h}\left[\|\partial_t v-\Pi^0_{K,k}\partial_t v\|_{L^2(K)}+\|\Pi^0_{K,k}\partial_t v-\mathcal{P}^h\partial_t v\|_{L^2(K)}\right]\left\|\varphi^v_h\right\|_{L^2(K)}\\
&\leq Ch^{k+1}\left|\partial_t v\right|_{H^{k+1}(\Omega)}\left\|\varphi^v_h\right\|_{L^2(\Omega)},
\end{aligned}
$$

for some constant $C > 0$. Moreover, by using the assumption on D, an integration by parts, the Cauchy–Schwarz inequality, the continuity of projector Π^0_k and adding and subtracting $\mathcal{P}^h v$, we get

$$
I_4 \leq C\left(\|v_h-\mathcal{P}^h v\|_{L^2(\Omega)}+\|v-\mathcal{P}^h v\|_{L^2(\Omega)}+\|v-\Pi^0_k v\|_{L^2(\Omega)}\right)\|\Delta v\|_{L^2(\Omega)}\left\|\varphi^v_h\right\|_{L^2(\Omega)},
$$

for some constant $C > 0$.

On the other hand, similarly for w_h, we obtain

$$
\begin{aligned}
m_h(\partial_t(w_h-\mathcal{P}^h w),\varphi^w_h) &= \left(c_h\left(v_h,w_h,\varphi^w_h\right)-m_h\left(\partial_t\mathcal{P}^h w,\varphi^w_h\right)\right)\\
&= c_h\left(v_h,w_h,\varphi^w_h\right)-m_h\left(\mathcal{P}^h\partial_t w,\varphi^v_h\right)-\left(H(v,w),\varphi^w_h\right)_{0,\Omega}+\left(\partial_t w,\varphi^w_h\right)_{0,\Omega}\\
&\leq\left[c_h\left(v_h,w_h,\varphi^w_h\right)-\left(\mathcal{P}^h H(v,w),\varphi^w_h\right)_{0,\Omega}\right]\\
&\quad+\left[\left(\mathcal{P}^h H(v,w),\varphi^w_h\right)_{0,\Omega}-\left(H(v,w),\varphi^w_h\right)_{0,\Omega}\right]+\left[\left(\partial_t w,\varphi^w_h\right)-m_h\left(\partial_t\mathcal{P}^h w,\varphi^w_h\right)\right],
\end{aligned}
$$

for all $\varphi^w_h \in W_h$. Now, using (2.3)(d), repeating the arguments used to bound I_2, and I_3 and using once again the properties of projectors Π^0_k, and $\mathcal{P}^h$ and finally Lemma 5.1, we readily obtain

$$
\begin{aligned}
m_h(\partial_t(w_h-\mathcal{P}^h w),\varphi^w_h) &\leq C\left(\|\Pi^0_k v_h-\mathcal{P}^h v\|_{L^2(\Omega)}+\|\Pi^0_k w_h-\mathcal{P}^h w\|_{L^2(\Omega)}\right)\left\|\varphi^w_h\right\|_{L^2(\Omega)}\\
&\quad+Ch^{k+1}(|v|_{H^{k+1}(\Omega)}+|w|_{H^{k+1}(\Omega)}+\left|\partial_t w\right|_{H^{k+1}(\Omega)})\left\|\varphi^w_h\right\|_{L^2(\Omega)}, \qquad (5.7)
\end{aligned}
$$

for some constant $C > 0$.

Collecting the previous results (5.5)–(5.7) and using the approximation properties of projectors Π_k^0 and $\mathcal{P}^h$, we get

$$
\begin{aligned}
&m_h\left(\partial_t(v_h-\mathcal{P}^h v),\varphi_h^v\right)+m_h\left(\partial_t(w_h-\mathcal{P}^h w),\varphi_h^w\right)\\
&\quad\le C\Big[h^{k+1}\left(|I_{\text{app}}|_{H^{k+1}(\Omega)}+|v|_{H^{k+1}(\Omega)}+|w|_{H^{k+1}(\Omega)}+\left|\partial_t v\right|_{H^{k+1}(\Omega)}+\left|\partial_t w\right|_{H^{k+1}(\Omega)}\right)\\
&\qquad+C\left(1+\|\Delta v\|_{L^2(\Omega)}\right)\left(\|v_h-\mathcal{P}^h v\|_{L^2(\Omega)}+\|w_h-\mathcal{P}^h w\|_{L^2(\Omega)}\right)\Big]\left(\|\varphi_h^v\|_{L^2(\Omega)}+\|\varphi_h^w\|_{L^2(\Omega)}\right).
\end{aligned}
\tag{5.8}
$$

Now, we set $\varphi_h^v:=(v_h-\mathcal{P}^h v)\in W_h$ and $\varphi_h^w:=(w_h-\mathcal{P}^h w)\in W_h$ in (5.8), we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(m_h(v_h-\mathcal{P}^h v,v_h-\mathcal{P}^h v)+m_h(w_h-\mathcal{P}^h w,w_h-\mathcal{P}^h w)\Big)\\
&\quad\le C\Big[h^{k+1}\left(|I_{\text{app}}|_{H^{k+1}(\Omega)}+|v|_{H^{k+1}(\Omega)}+|w|_{H^{k+1}(\Omega)}+|\partial_t v|_{H^{k+1}(\Omega)}+|\partial_t w|_{H^{k+1}(\Omega)}\right)\\
&\qquad+C\left(1+\|\Delta v\|_{L^2(\Omega)}\right)\left(\|v_h-\mathcal{P}^h v\|_{L^2(\Omega)}+\|w_h-\mathcal{P}^h w\|_{L^2(\Omega)}\right)\Big]\\
&\qquad\times\left(\|(v_h-\mathcal{P}^h v)\|_{L^2(\Omega)}+\|(w_h-\mathcal{P}^h w)\|_{L^2(\Omega)}\right).
\end{aligned}
$$

Herein, we used the equivalence of the norm $\|\cdot\|_h:=m_h(\cdot,\cdot)$ with the L^2 norm, integrating the previous bound on $(0,t)$ and an application of Gronwall inequality, we get

$$
\begin{aligned}
&\left\|v_h-\mathcal{P}^h v\right\|_{L^2(\Omega)}+\left\|w_h-\mathcal{P}^h w\right\|_{L^2(\Omega)}\\
&\quad\le C(T)\Big[\left\|v_0-v_h^0\right\|_{L^2(\Omega)}+\left\|w_0-w_h^0\right\|_{L^2(\Omega)}+h^{k+1}\Big(\left|v_0\right|_{H^{k+1}(\Omega)}+\left|w_0\right|_{H^{k+1}(\Omega)}\\
&\qquad+\int_0^t\left(\left|I_{\text{app}}\right|_{H^{k+1}(\Omega)}+|v|_{H^{k+1}(\Omega)}+|w|_{H^{k+1}(\Omega)}+\left|\partial_t v\right|_{H^{k+1}(\Omega)}+\left|\partial_t w\right|_{H^{k+1}(\Omega)}\right)\mathrm{d}t\Big)\Big]\\
&\qquad\times\exp\left(\int_0^t\left(1+|v|_{H^2(\Omega)}\right)\mathrm{d}t\right).
\end{aligned}
$$

Using this and (5.4), we get (5.2).

Proof of (5.3) Similarly to (5.2), observe that for $n=1,\dots,N$,

$$
U_h^n-U(\cdot,t_n)=\left(U_h^n-\mathcal{P}^h U(\cdot,t_n)\right)+\left(\mathcal{P}^h U(\cdot,t_n)-U(\cdot,t_n)\right)\qquad\text{for } U=v,w
$$

and from Lemma 5.1, we get easily for $U=v,w$ and for all $t\in(0,T)$,

$$
\|\mathcal{P}^h U(\cdot,t_n)-U(\cdot,t_n)\|_{L^2(\Omega)}\le Ch^{k+1}\left(|U_0|_{H^{k+1}(\Omega)}+\|\partial_t U\|_{L^1(0,t;H^{k+1}(\Omega))}\right),
$$

for some constant $C > 0$. Next we bound the term $(U_h^n - \mathcal{P}^h U(\cdot, t_n))$ for $U = v, w$. Note that using the continuous and fully discrete problems (cf. (2.1) and (3.10)), the definition of the projector $\mathcal{P}^h$, we obtain

$$
\begin{aligned}
& m_h\left(\frac{\left(v_h^n - \mathcal{P}^h v(\cdot, t_n)\right) - \left(v_h^{n-1} - \mathcal{P}^h v(\cdot, t_{n-1})\right)}{\Delta t}, \varphi_h^v\right) + D\left(J\left(v_h^n\right)\right) a_h\left(\left(v_h^n - \mathcal{P}^h v(\cdot, t_n)\right), \varphi_h^v\right) \\
&= \left(I_{app,h}^n, \varphi_h^v\right)_{0,\Omega} - b_h\left(v_h^n, w_h^n, \varphi_h^v\right) - m_h\left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v\right) - D\left(J\left(v_h^n\right)\right) a_h\left(\mathcal{P}^h v(\cdot, t_n), \varphi_h^v\right) \\
&= \left(I_{app,h}^n, \varphi_h^v\right)_{0,\Omega} - b_h\left(v_h^n, w_h^n, \varphi_h^v\right) - \left(I_{\text{app}}(\cdot, t_n), \varphi_h^v\right)_{0,\Omega} + \left(I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v\right)_{0,\Omega} + \left(\partial_t v(\cdot, t_n), \varphi_h^v\right) \\
&\quad - m_h\left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v\right) + \left(D\left(J(v(\cdot, t_n))\right) - D\left(J\left(v_h^n\right)\right)\right) a\left(v(\cdot, t_n), \varphi_h^v\right) \\
&= \left[\left(I_{app,h}^n, \varphi_h^v\right)_{0,\Omega} - \left(I_{\text{app}}(\cdot, t_n), \varphi_h^v\right)_{0,\Omega}\right] - \left[b_h\left(v_h^n, w_h^n, \varphi_h^v\right) - \left(I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v\right)_{0,\Omega}\right] \\
&\quad + \left[\left(\partial_t v(\cdot, t_n), \varphi_h^v\right) - m_h\left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v\right)\right] \\
&\quad + \left[\left(D\left(J(v(\cdot, t_n))\right) - D\left(J\left(v_h^n\right)\right)\right) a\left(v(\cdot, t_n), \varphi_h^v\right)\right] \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned} \tag{5.9}
$$

Now, we will bound the terms $\mathcal{I}_1, \ldots, \mathcal{I}_4$. Note that the first term $\mathcal{I}_1$ can be estimated like (5.6)

$$
\mathcal{I}_1 \le C h^{k+1} |I_{\text{app}}(\cdot, t_n)|_{H^{k+1}(\Omega)} \left\|\varphi_h^v\right\|_{L^2(\Omega)},
$$

for some constant $C > 0$. Next for $\mathcal{I}_2$, using the definition of the form $b_h(\cdot, \cdot, \cdot)$, adding and subtracting adequate terms, we have

$$
\begin{aligned}
\mathcal{I}_2 &= -\left[b_h\left(v_h^n w_h^n, \varphi_h^v\right) - \left(\mathcal{P}^h I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v\right)_{0,\Omega}\right] \\
&\quad - \left[\left(\mathcal{P}^h I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v\right)_{0,\Omega} - \left(I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v\right)_{0,\Omega}\right] \\
&= -\left[\sum_{K \in \mathcal{T}_h} \left(I_{\text{ion}}\left(\Pi_{K,k}^0 v_h^n, \Pi_{K,k}^0 w_h^n\right), \Pi_{K,k}^0 \varphi_h^v\right)_{0,K} - \left(I_{\text{ion}}(\mathcal{P}^h v(\cdot, t_n), \mathcal{P}^h w(\cdot, t_n)), \varphi_h^v\right)_{0,K}\right] \\
&\quad - \left[\left(\mathcal{P}^h I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)) - I_{\text{ion}}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v\right)_{0,\Omega}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\Big[\sum_{K\in\mathcal{T}_h}\Big(I_{\mathrm{ion}}\Big(\Pi^0_{K,k}v_h^n,\Pi^0_{K,k}w_h^n\Big) - I_{\mathrm{ion}}(\mathcal{P}^h v(\cdot,t_n),\mathcal{P}^h w(\cdot,t_n)),\varphi_h^v\Big)_{0,K}\Big]\\
&\quad -\Big[\Big(\mathcal{P}^h I_{\mathrm{ion}}(v(\cdot,t_n),w(\cdot,t_n)) - I_{\mathrm{ion}}(v(\cdot,t_n),w(\cdot,t_n)),\varphi_h^v\Big)_{0,\Omega}\Big]\\
&\le C\left[\sum_{K\in\mathcal{T}_h}\Big(\Big\|\Pi^0_{K,k}v_h^n-\mathcal{P}^h v(\cdot,t_n)\Big\|_{L^2(K)}+\Big\|\Pi^0_{K,k}w_h^n-\mathcal{P}^h w(\cdot,t_n)\Big\|_{L^2(K)}\Big)\|\varphi_h^v\|_{L^2(K)}\right]\\
&\quad + Ch^{k+1}(|v(\cdot,t_n)|_{H^{k+1}(\Omega)}+|w(\cdot,t_n)|_{H^{k+1}(\Omega)})\|\varphi_h^v\|_{L^2(\Omega)}\\
&\le C\Big(\Big\|\Pi^0_k v_h^n-\mathcal{P}^h v(\cdot,t_n)\Big\|_{L^2(\Omega)}+\Big\|\Pi^0_k w_h^n-\mathcal{P}^h w(\cdot,t_n)\Big\|_{L^2(\Omega)}\Big)\|\varphi_h^v\|_{L^2(\Omega)}\\
&\quad + Ch^{k+1}(|v(\cdot,t_n)|_{H^{k+1}(\Omega)}+|w(\cdot,t_n)|_{H^{k+1}(\Omega)})\|\varphi_h^v\|_{L^2(\Omega)},
\end{aligned}
$$

for some constant $C > 0$, where we have used that I_{ion} is a linear function, (5.1), the properties of projectors Π^0_k and $\mathcal{P}^h$, and finally Lemma 5.1.

Regarding $\mathcal{I}_3$, we use the consistency and stability properties of the bilinear form m_h to get

$$
\begin{aligned}
\mathcal{I}_3 &= \sum_{K\in\mathcal{T}_h}\left[(\partial_t v(\cdot,t_n),\varphi_h^v)_{0,K} - m_h^K\left(\frac{\mathcal{P}^h v(\cdot,t_n)-\mathcal{P}^h v(\cdot,t_{n-1})}{\Delta t},\varphi_h^v\right)\right]\\
&= \sum_{K\in\mathcal{T}_h}\Bigg[\left(\partial_t v(\cdot,t_n)-\frac{v(\cdot,t_n)-v(\cdot,t_{n-1})}{\Delta t},\varphi_h^v\right)_{0,K}\\
&\quad + \left(\frac{v(\cdot,t_n)-v(\cdot,t_{n-1})}{\Delta t}-\frac{\Pi^0_{K,k}(v(\cdot,t_n)-v(\cdot,t_{n-1}))}{\Delta t},\varphi_h^v\right)_{0,K}\\
&\quad + m_h^K\left(\frac{\Pi^0_{K,k}(v(\cdot,t_n)-v(\cdot,t_{n-1}))}{\Delta t}-\frac{\mathcal{P}^h(v(\cdot,t_n)-v(\cdot,t_{n-1}))}{\Delta t},\varphi_h^v\right)\Bigg]\\
&\le \frac{C}{\Delta t}\sum_{K\in\mathcal{T}_h}\Big[\|\Delta t\partial_t v(\cdot,t_n)-(v(\cdot,t_n)-v(\cdot,t_{n-1}))\|_{L^2(K)}\\
&\quad + \Big\|(v(\cdot,t_n)-v(\cdot,t_{n-1}))-\Pi^0_{K,k}(v(\cdot,t_n)-v(\cdot,t_{n-1}))\Big\|_{L^2(K)}\\
&\quad + \Big\|\Pi^0_{K,k}(v(\cdot,t_n)-v(\cdot,t_{n-1}))-\mathcal{P}^h(v(\cdot,t_n)-v(\cdot,t_{n-1}))\Big\|_{L^2(K)}\Big]\|\varphi_h^v\|_{L^2(K)}\\
&\le \frac{C}{\Delta t}\Big[\|\Delta t\partial_t v(\cdot,t_n)-(v(\cdot,t_n)-v(\cdot,t_{n-1}))\|_{L^2(\Omega)}+h^{k+1}|v(\cdot,t_n)-v(\cdot,t_{n-1})|_{H^{k+1}(\Omega)}\Big]\|\varphi_h^v\|_{L^2(\Omega)}\\
&\le \frac{C}{\Delta t}\Big[\Delta t\int_{t_{n-1}}^{t_n}\Big\|\partial^2_{tt}v(\cdot,s)\Big\|_{L^2(\Omega)}\,\mathrm{d}s + h^{k+1}\int_{t_{n-1}}^{t_n}|v_t(\cdot,s)|_{H^{k+1}(\Omega)}\,\mathrm{d}s\Big]\|\varphi_h^v\|_{L^2(\Omega)},
\end{aligned}
$$

for some constant $C > 0$, where we have used Cauchy–Schwarz inequality and the approximation properties of Π^0_k and $\mathcal{P}^h$, and finally Lemma 5.1. Moreover, for $\mathcal{I}_4$ by using an integration by parts, the assumption on D, Cauchy–Schwarz inequality, the continuity of projector Π_k^0, and adding and subtracting $\mathcal{P}^h v$, we obtain

$$\begin{aligned}\mathcal{I}_4 \leq & C\left(\left\|\mathcal{P}^h v(\cdot,t_n) - v_h^n\right\|_{L^2(\Omega)} + \left\|v(\cdot,t_n) - \mathcal{P}^h v(\cdot,t_n)\right\|_{L^2(\Omega)}\right. \\ & \left. + \left\|v(\cdot,t_n) - \Pi_k^0 v(\cdot,t_n)\right\|_{L^2(\Omega)}\right) \|\Delta v(\cdot,t_n)\|_{L^2(\Omega)} \|\varphi_h^v\|_{L^2(\Omega)},\end{aligned}$$

for some constant $C > 0$. On the other hand, similarly for w_h, we obtain

$$\begin{aligned} m_h\left(\frac{(w_h^n(\cdot) - \mathcal{P}^h w(\cdot,t_n)) - (w_h^{n-1}(\cdot) - \mathcal{P}^h w(\cdot,t_{n-1}))}{\Delta t}, \varphi_h^w\right) &= c_h(v_h^n, w_h^n, \varphi_h^w) \\ & - m_h\left(\frac{\mathcal{P}^h w(\cdot,t_n) - \mathcal{P}^h w(\cdot,t_{n-1})}{\Delta t}, \varphi_h^w\right) \\ &= c_h(v_h^n, w_h^n, \varphi_h^w) - m_h\left(\frac{\mathcal{P}^h w(\cdot,t_n) - \mathcal{P}^h w(\cdot,t_{n-1})}{\Delta t}, \varphi_h^w\right) \\ & - (H(v(\cdot,t_n), w(\cdot,t_n)), \varphi_h^w)_{0,\Omega} + (\partial_t w(\cdot,t_n), \varphi_h^w)_{0,\Omega} \\ & \leq \left[c_h(v_h^n, w_h^n, \varphi_h^w) - (\mathcal{P}^h H(v(\cdot,t_n), w(\cdot,t_n)), \varphi_h^w)_{0,\Omega}\right] \\ & + \left[\left(\mathcal{P}^h H(v(\cdot,t_n), w(\cdot,t_n)), \varphi_h^w\right)_{0,\Omega} - \left(H(v(\cdot,t_n), w(\cdot,t_n)), \varphi_h^w\right)_{0,\Omega}\right] \\ & + \left[\left(\partial_t w(\cdot,t_n), \varphi_h^w\right)_{0,\Omega} - m_h\left(\frac{\mathcal{P}^h w(\cdot,t_n) - \mathcal{P}^h w(\cdot,t_{n-1})}{\Delta t}, \varphi_h^w\right)\right].\end{aligned}$$

Now, using (2.3)(d), repeating the arguments used to bound $\mathcal{I}_2$ and $\mathcal{I}_3$, and using once again the approximation properties of projectors Π_k^0 and $\mathcal{P}^h$, and finally Lemma 5.1, we readily obtain

$$\begin{aligned} m_h\left(\frac{(w_h^n(\cdot) - \mathcal{P}^h w(\cdot,t_n)) - (w_h^{n-1}(\cdot) - \mathcal{P}^h w(\cdot,t_{n-1}))}{\Delta t}, \varphi_h^w\right) &\leq C\Big(\|v_h^n - \mathcal{P}^h v(\cdot,t_n)\|_{L^2(\Omega)} \\ & + \|w_h^n - \mathcal{P}^h w(\cdot,t_n)\|_{L^2(\Omega)} + Ch^{k+1}\big(|v(\cdot,t_n)|_{H^{k+1}(\Omega)} + |w(\cdot,t_n)|_{H^{k+1}(\Omega)}\big)\Big)\|\varphi_h^w\|_{L^2(\Omega)} \\ & + \frac{C}{\Delta t}\left[\Delta t \int_{t_{n-1}}^{t_n} \left\|\partial_{tt}^2 w(\cdot,s)\right\|_{L^2(\Omega)} \,\mathrm{d}s + h^{k+1}\int_{t_{n-1}}^{t_n} \left|w_t(\cdot,s)\right|_{H^{k+1}(\Omega)} \mathrm{d}s\right]\left\|\varphi_h^v\right\|_{L^2(\Omega)}, \qquad (5.10)\end{aligned}$$

for some constant $C > 0$.

Collecting the previous results (5.9) and (5.10) and using the approximation properties of projectors Π_k^0 and $\mathcal{P}^h$, after substituting $\varphi_h^v = v_h^n - \mathcal{P}^h v$ and $\varphi_h^w = w_h^n - \mathcal{P}^h w$ in (5.9) and (5.10), respectively, we deduce

$$
\begin{aligned}
& m_h\left(v_h^n - \mathcal{P}^h v(\cdot,t_n), v_h^n - \mathcal{P}^h v(\cdot,t_n)\right) + m_h\left(w_h^n - \mathcal{P}^h w(\cdot,t_n), w_h^n - \mathcal{P}^h w(\cdot,t_n)\right) \\
&\quad \le \left(m_h\left(v_h^{n-1} - \mathcal{P}^h v(\cdot,t_{n-1}), v_h^n - \mathcal{P}^h v(\cdot,t_n)\right) + m_h\left(w_h^{n-1} - \mathcal{P}^h w(\cdot,t_{n-1}), w_h^n - \mathcal{P}^h w(\cdot,t_n)\right)\right) \\
&\qquad + C\Bigg[\Delta t\Bigg((1+|v(\cdot,t_n)|_{H^2(\Omega)})\left(\left\|v_h^n - \mathcal{P}^h v(\cdot,t_n)\right\|_{L^2(\Omega)} + \left\|w_h^n - \mathcal{P}^h w(\cdot,t_n)\right\|_{L^2(\Omega)}\right) \\
&\qquad + h^{k+1}\left(|v(\cdot,t_n)|_{H^{k+1}(\Omega)} + |w(\cdot,t_n)|_{H^{k+1}(\Omega)} + \left|I_{\text{app}}(\cdot,t_n)\right|_{H^{k+1}(\Omega)}\right) \\
&\qquad + \int_{t_{n-1}}^{t_n}\left(\left\|\partial_{tt}^2 v(\cdot,s)\right\|_{L^2(\Omega)} + \left\|\partial_{tt}^2 w(\cdot,s)\right\|_{L^2(\Omega)}\right)\mathrm{d}s\Bigg) \\
&\qquad + h^{k+1}\int_{t_{n-1}}^{t_n}\left(|v_t(\cdot,s)|_{H^{k+1}(\Omega)} + |w_t(\cdot,s)|_{H^{k+1}(\Omega)}\right)\mathrm{d}s\Bigg] \\
&\qquad \times\left(\left\|v_h^n - \mathcal{P}^h v(\cdot,t_n))\right\|_h + \left\|w_h^n - \mathcal{P}^h w(\cdot,t_n)\right\|_h\right).
\end{aligned}
$$

This implies

$$
\begin{aligned}
& \left\|v_h^n - \mathcal{P}^h v(\cdot,t_n)\right\|_h + \left\|w_h^n - \mathcal{P}^h w(\cdot,t_n)\right\|_h \\
&\quad \le \left(\left\|v_h^{n-1} - \mathcal{P}^h v(\cdot,t_{n-1})\right\|_h + \left\|w_h^{n-1} - \mathcal{P}^h w(\cdot,t_{n-1})\right\|_h\right) \\
&\qquad + C\Bigg[\Delta t\Bigg((1+|v(\cdot,t_n)|_{H^2(\Omega)})\left(\left\|v_h^n - \mathcal{P}^h v(\cdot,t_n)\right\|_{L^2(\Omega)} + \left\|w_h^n - \mathcal{P}^h w(\cdot,t_n)\right\|_{L^2(\Omega)}\right) \\
&\qquad + h^{k+1}\left(|v(\cdot,t_n)|_{H^{k+1}(\Omega)} + |w(\cdot,t_n)|_{H^{k+1}(\Omega)} + \left|I_{\text{app}}(\cdot,t_n)\right|_{H^{k+1}(\Omega)}\right) \\
&\qquad + \int_{t_{n-1}}^{t_n}\left(\left\|\partial_{tt}^2 v(\cdot,s)\right\|_{L^2(\Omega)} + \left\|\partial_{tt}^2 w(\cdot,s)\right\|_{L^2(\Omega)}\right)\mathrm{d}s\Bigg) \\
&\qquad + h^{k+1}\int_{t_{n-1}}^{t_n}\left(|v_t(\cdot,s)|_{H^{k+1}(\Omega)} + |w_t(\cdot,s)|_{H^{k+1}(\Omega)}\right)\mathrm{d}s\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\left\|v_h^0 - \mathcal{P}^h v(\cdot,0)\right\|_h + \left\|w_h^0 - \mathcal{P}^h w(\cdot,0)\right\|_h\right) \\
&\quad + C\sum_{\ell=1}^{n}\Bigg[\Delta t\Big((1+\left|v(\cdot,t_\ell)\right|_{H^2(\Omega)})\Big(\left\|v_h^\ell - \mathcal{P}^h v(\cdot,t_\ell)\right\|_h + \left\|w_h^\ell - \mathcal{P}^h w(\cdot,t_\ell)\right\|_h\Big) \\
&\quad + \int_{t_{\ell-1}}^{t_\ell}\Big(\left\|\partial_{tt}^2 v(\cdot,s)\right\|_{L^2(\Omega)} + \left\|\partial_{tt}^2 w(\cdot,s)\right\|_{L^2(\Omega)}\Big)\,\mathrm{d}s\Big) + \|v_0 - v_{0,h}\|_{L^2(\Omega)} + \|w_0 - w_{0,h}\|_{L^2(\Omega)} \\
&\quad + h^{k+1}\Big(\left|v_0\right|_{H^{k+1}(\Omega)} + |w_0|_{H^{k+1}(\Omega)} + \left|v(\cdot,t_\ell)\right|_{H^{k+1}(\Omega)} + |w(\cdot,t_\ell)|_{H^{k+1}(\Omega)} + |I_{\mathrm{app}}(\cdot,t_\ell)|_{H^{k+1}(\Omega)} \\
&\quad + \int_{t_{\ell-1}}^{t_\ell}\Big(|v(\cdot,s)|_{H^{k+1}(\Omega)} + |w(\cdot,s)|_{H^{k+1}(\Omega)}\Big)\,\mathrm{d}s\Big)\Bigg].
\end{aligned}
\tag{5.11}
$$

Finally, we use the equivalence of the norm $\|\cdot\|_h := m_h(\cdot,\cdot)$ with the L^2 norm and an application of discrete Gronwall inequality to (5.11) to get (5.3). This concludes the proof of Theorem 5.2. □

6. Numerical results

In the present section, we report some numerical examples of the proposed VEM. With this aim, we have implemented in a MATLAB code the lowest-order VEM ($k = 1$) on arbitrary polygonal meshes following the ideas proposed in Beirão da Veiga *et al.* (2014a). Moreover, we solve the nonlinear problem derived from (3.10) by a classical Picard-type iteration.

To complete the choice of the VEM, we have to choose the bilinear forms $S^K(\cdot,\cdot)$ and $S_0^K(\cdot,\cdot)$, satisfying (3.2) and (3.3), respectively. In this respect, we have proceeded as in Beirão da Veiga *et al.* (2013a, Section 4.6); for each polygon K with vertices $P_1,\ldots,P_{N_K}$, we have used

$$
S^K(u,v) := \sum_{r=1}^{N_K} u(P_r)v(P_r), \qquad u,v \in W_{1|K},
$$

$$
S_0^K(u,v) := h_K^2\sum_{r=1}^{N_K} u(P_r)v(P_r), \qquad u,v \in W_{1|K}.
$$

A proof of (3.2) and (3.3) for the above (standard) choices could be derived following the arguments in Ahmad *et al.* (2013) and Beirão da Veiga *et al.* (2013a, 2017a). The choices above are standard in the Virtual Element Literature and correspond to a scaled identity matrix in the space of the degrees of freedom values.

In all the numerical examples we have considered $H(v,w)$ and $I_{\mathrm{ion}}(v,w)$ as in (2.4) and (2.5), respectively. Moreover, we have tested the method by using different families of meshes (see Fig. 1).

6.1 *Test* 1

The aim of this numerical example is to test the convergence properties of the proposed VEM. With this objective, we introduce the following discrete relative L^2 norm of the difference between a reference solution u_{ref}, which is obtained on an extremely fine mesh and the numerical solution u_h at the final

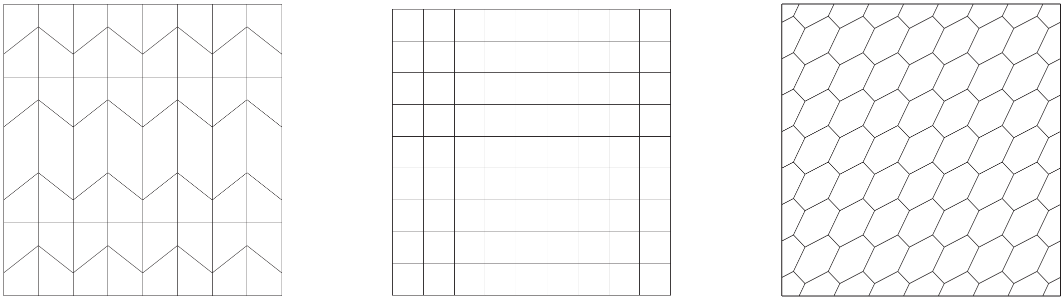

FIG. 1. Sample meshes: $\mathcal{T}_h^1$ (left), $\mathcal{T}_h^2$ (center) and $\mathcal{T}_h^3$ (right).

TABLE 1 *Test 1: $E_{h,\Delta t}$ error for v and for the meshes $\mathcal{T}_h^2$*

$h\backslash\Delta t$	$\Delta t = 1/3$	$\Delta t = 1/12$	$\Delta t = 1/48$	$\Delta t = 1/192$
1/8	0.523499772859947	0.254128190031018	0.231625702484074	0.228564582239788
1/16	0.501427757954840	0.073397686413675	0.033438153244729	0.031719551242699
1/32	0.499619638795241	0.063643322905268	0.010299560779982	0.005840961963621
1/64	0.499780908876156	0.064056553619930	0.009767337053892	0.002546001572083

TABLE 2 *Test 1: $E_{h,\Delta t}$ error for w and for the meshes $\mathcal{T}_h^2$*

$h\backslash\Delta t$	$\Delta t = 1/3$	$\Delta t = 1/12$	$\Delta t = 1/48$	$\Delta t = 1/192$
1/8	0.233922447286499	0.102194576523503	0.086875203270260	0.084789535586885
1/16	0.226571951589132	0.089790111454289	0.075921461474953	0.074408607847462
1/32	0.210582296617939	0.049672099006078	0.023584319200822	0.020543068189885
1/64	0.207657184653963	0.043302505350623	0.011225579353452	0.005588513008340

time T, that is,

$$E_{h,\Delta t}^2 := \frac{m_h(u_{ref}(\cdot,T) - u_h(\cdot,T), u_{ref}(\cdot,T) - u_h(\cdot,T))}{m_h(u_{ref}(\cdot,T), u_{ref}(\cdot,T))}.$$

For this example, the domain will be $\Omega = (0,1)^2$ and the time interval will be $[0,1]$; we will take the model constants as follows: $a = 0.2232, b = 0.9, \lambda = -1, \theta = 0.004$. We also take $I_{\text{app}} = 0$ and $D(x) = 0.01x$. Moreover, we consider the following initial data:

$$v_0(x,y) = (1 + 0.5\cos(4\pi x)\cos(4\pi y)), \qquad w_0(x,y) = (1 + 0.5\cos(8\pi x)\cos(8\pi y)).$$

Due to the lack of exact solution for this example, we compute errors using a numerical solution on an extremely fine mesh ($h = 1/512$) and time step ($\Delta t = 1/512$) as reference v_{ref}, w_{ref}.

We report in Tables 1 and 2 the relative errors $E_{h,\Delta t}$ for variables v and w, respectively, for the family of meshes $\mathcal{T}_h^2$ and different refinement levels and time steps.

It can be seen along the diagonals of Tables 1 and 2 that the error in the discrete L^2 norm reduced with a quadratic order with respect to h, which is the expected order of convergence for $k = 1$.

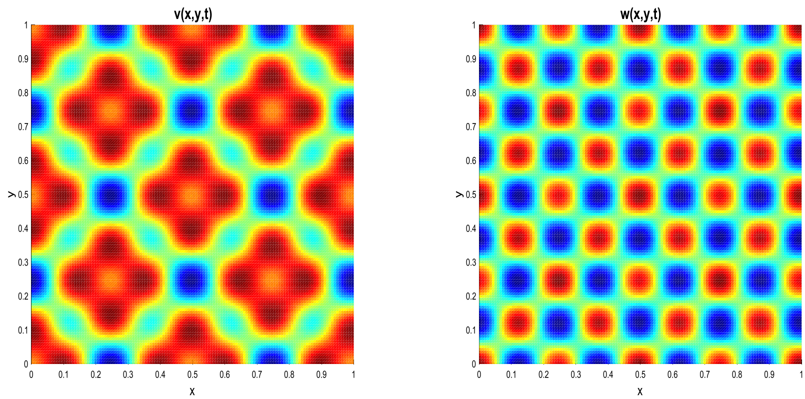

FIG. 2. Test 1: variables v (left) and w (right) for $h = 1/64$ and $\Delta t = 1/80$.

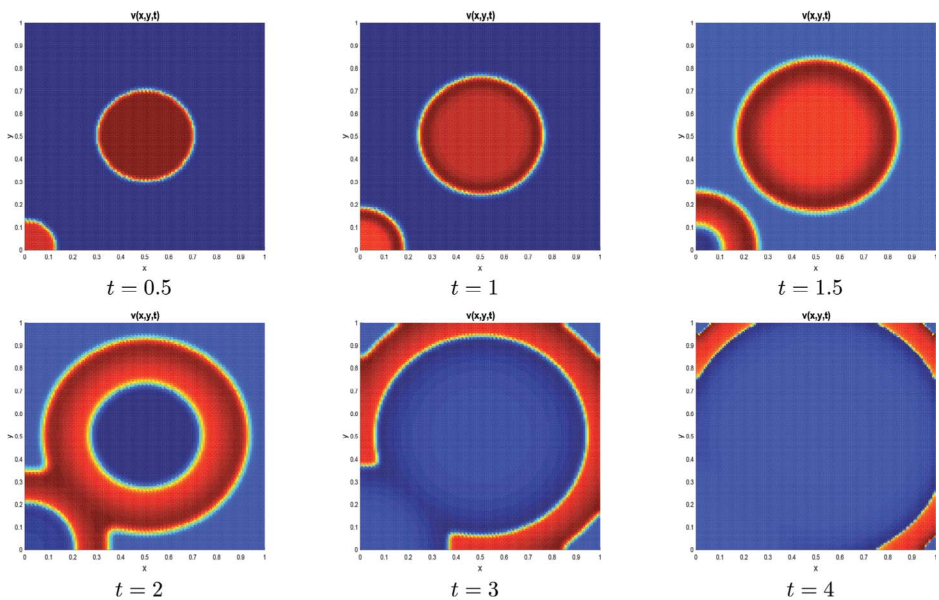

FIG. 3. Test 2: numerical solution of the transmembrane potential v for different times and $D(x) = 0.01x$.

We show in Fig. 2 the profiles of the computed quantities.

6.2 *Test* 2

We consider a benchmark example (cf. Bendahmane *et al.*, 2010). We solve the equation using meshes $\mathcal{T}_h^1$ (with $h = 1/128$) on the unit square, time interval $[0, 4]$ (with $\Delta t = 1/100$) and with the following model constants: $a = 0.16875, b = 1, \lambda = -100, \theta = 0.25$. Moreover, we consider the following initial data:

$$v_0(x, y) = \left(1 - \frac{1}{1 + e^{-50(x^2+y^2)^{1/2} - 0.1}}\right), \qquad w_0(x, y) = 0.$$

After $4ms$, an instantaneous stimulus is applied in $(x_0, y_0) = (0.5, 0.5)$ to the transmembrane potential v,

$$I_{app} = \begin{cases} 1 \quad mV \ \text{if} \quad (x - x_0)^2 + (y - y_0)^2 < 0.04 \quad cm^2, \\ 0 \quad mV \ \text{otherwise.} \end{cases}$$

We show in Fig. 3 the evolution of the numerical solution v_h (transmembrane potential) for different times and considering nonlocal diffusion, $D(x) = 0.01x$.

6.3 *Test* 3

The existence of spiral waves is an interesting phenomena in this type of model (see Liu *et al.*, 2015; Coudiére & Turpault, 2017). The aim of this test is to obtain the well-known periodic spiral wave. For this example, we use meshes $\mathcal{T}_h^3$ (with $h = 1/128$) on the domain $\Omega := (0, 1)^2$ and time interval $[0, 15]$ (with $\Delta t = 1/200$). We will take the model constants as follows: $a = 0.16875, b = 1, \lambda = -100$, $\theta = 0.25$. Moreover, we consider the following initial data:

$$v_0(x, y) = \begin{cases} 1.4 \quad \text{if} \quad x < 0.5 \ \text{and} \ y < 0.5 \\ 0 \quad \text{otherwise,} \end{cases}$$

$$w_0(x, y) = \begin{cases} 0.15 \quad \text{if} \quad x > 0.5 \ \text{and} \ y < 0.5 \\ 0 \quad \text{otherwise.} \end{cases}$$

As it is expected the initial data evolve to a spiral wave; see Fig. 4.

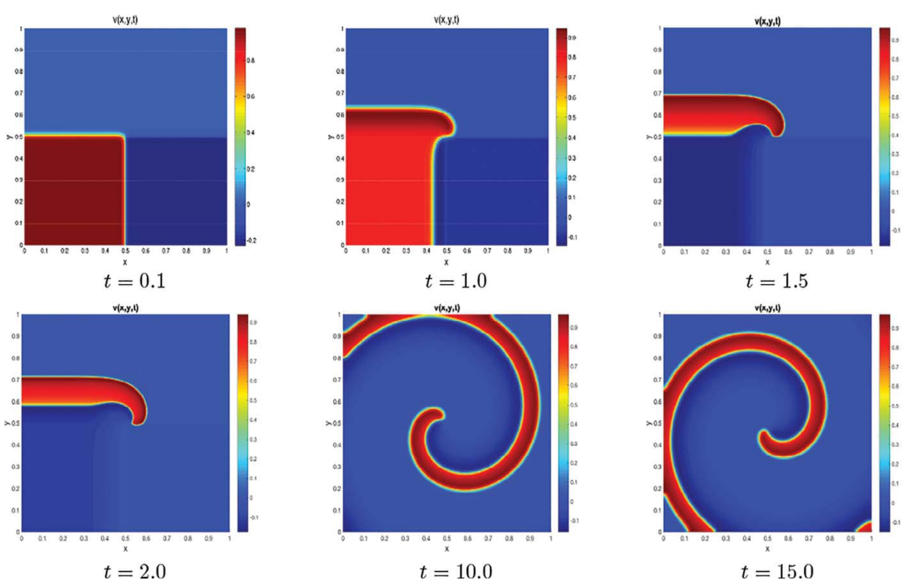

FIG. 4. Test 3: numerical solution of the transmembrane potential v for different times.

Funding

V.A. was partially supported by CONICYT-Chile (FONDECYT project 11160706); D.M. was partially supported by CONICYT-Chile (FONDECYT project 1180913); PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal (project AFB170001 to D.M. and M.S.); M.S. was partially supported by CONICYT-Chile (FONDECYT project 1180868 to M.S.).

References

Ahmad, B., Alsaedi, A., Brezzi, F., Marini, L. D. & Russo, A. (2013) Equivalent projectors for virtual element methods. *Comput. Math. Appl.*, **66**, 376–391.

Anaya, V., Bendahmane, M., Langlais, M. & Sepúlveda, M. (2015a) A convergent finite volume method for a model of indirectly transmitted diseases with nonlocal cross-diffusion. *Comput. Math. Appl.*, **70**, 132–157.

Anaya, V., Bendahmane, M. & Sepúlveda, M. (2015b) Numerical analysis for a three interacting species model with nonlocal and cross diffusion. *ESAIM Math. Model. Numer. Anal.*, **49**, 171–192.

Antonietti, P. F., Beirão da Veiga, L., Mora, D. & Verani, M. (2014) A stream virtual element formulation of the Stokes problem on polygonal meshes. *SIAM J. Numer. Anal.*, **52**, 386–404.

Antonietti, P. F., Beirão da Veiga, L., Scacchi, S. & Verani, M. (2016) A C^1 virtual element method for the Cahn–Hilliard equation with polygonal meshes. *SIAM J. Numer. Anal*, **54**, 36–56.

Artioli, E., Beirão da Veiga, L., Lovadina, C. & Sacco, E. (2017) Arbitrary order 2D virtual elements for polygonal meshes: part I, elastic problem. *Comput. Mech.*, **60**, 355–377.

Barkley, D. (1991) A model for fast computer simulation of waves in excitable media. *Phys. D*, **49**, 61–70.

Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L. D. & Russo, A. (2013a) Basic principles of virtual element methods. *Math. Models Methods Appl. Sci.*, **23**, 199–214.

Beirão da Veiga, L., Brezzi, F. & Marini, L. D. (2013b) Virtual elements for linear elasticity problems. *SIAM J. Numer. Anal.*, **51**, 794–812.

Beirão da Veiga, L., Brezzi, F., Marini, L. D. & Russo, A. (2014a) The hitchhiker's guide to the virtual element method. *Math. Models Methods Appl. Sci.*, **24**, 1541–1573.

Beirão da Veiga, L., Lipnikov, K. & Manzini, G. (2014b) *The Mimetic Finite Difference Method for Elliptic Problems*. MS&A, vol. 11. Cham: Springer.

Beirão da Veiga, L., Brezzi, F., Marini, L. D. & Russo, A. (2016) Virtual element method for general second-order elliptic problems on polygonal meshes. *Math. Models Methods Appl. Sci.*, **26**, 729–750.

Beirão da Veiga, L., Lovadina, C. & Mora, D. (2015) A virtual element method for elastic and inelastic problems on polytope meshes. *Comput. Methods Appl. Mech. Engrg.*, **295**, 327–346.

Beirão da Veiga, L., Lovadina, C. & Russo, A. (2017a) Stability analysis for the virtual element method. *Math. Models Methods Appl. Sci.*, **27**, 2527–2594.

Beirão da Veiga, L., Lovadina, C. & Vacca, G. (2017b) Divergence free virtual elements for the Stokes problem on polygonal meshes. *ESAIM Math. Model. Numer. Anal.*, **51**, 509–535.

Beirão da Veiga, L., Mora, D., Rivera, G. & Rodríguez, R. (2017c) A virtual element method for the acoustic vibration problem. *Numer. Math.*, **136**, 725–763.

Beirão da Veiga, L. & Manzini, G. (2015) Residual a posteriori error estimation for the virtual element method for elliptic problems. *ESAIM Math. Model. Numer. Anal.*, **49**, 577–599.

Beirão da Veiga, L., Mora, D. & Rivera, G. (2019) Virtual elements for a shear-deflection formulation of Reissner–Mindlin plates. *Math. Comp.*, **85**, 149–178.

Bendahmane, M., Burger, R. & Ruiz-Baier, R. (2010) A multiresolution space-time adaptive scheme for the bidomain model in electrocardiology. *Numer. Methods Partial Differential Equations*, **26**, 1377–1404.

Benedetto, M. F., Berrone, S., Borio, A., Pieraccini, S. & Scialò, S. (2016) Order preserving SUPG stabilization for the virtual element formulation of advection–diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, **311**, 18–40.

Berrone, S. & Borio, A. (2017) A residual a posteriori error estimate for the virtual element method. *Math. Models Methods Appl. Sci.*, **27**, 1423–1458.

Brenner, S. C., Guan, Q. & Sung, L.-Y. (2017) Some estimates for virtual element methods. *Comput. Methods Appl. Math.*, **17**, 553–574.

Brenner, S. C. & Scott, R. L. (2008) *The Mathematical Theory of Finite Element Methods*. New York: Springer.

Brezis, H. (1983) *Analyse Fonctionnelle, Théorie et Applications*. Paris: Masson.

Brezzi, F. & Marini, L. D. (2012) Virtual elements for plate bending problems. *Comput. Methods Appl. Mech. Engrg.*, **253**, 455–462.

Burger, R., Ruiz-Baier, R. & Schneider, K. (2010) Adaptive multiresolution methods for the simulation of waves in excitable media. *J. Sci. Comput.*, **43**, 261–290.

Cáceres, E. & Gatica, G. N. (2017) A mixed virtual element method for the pseudostress–velocity formulation of the Stokes problem. *IMA J. Numer. Anal.*, **37**, 296–331.

Cáceres, E., Gatica, G. N. & Sequeira, F. (2017) A mixed virtual element method for the Brinkman problem. *Math. Models Methods Appl. Sci.*, **27**, 707–743.

Cangiani, A., Georgoulis, E. H., Pryer, T. & Sutton, O. J. (2017a) A posteriori error estimates for the virtual element method. *Numer. Math.*, **137**, 857–893.

Cangiani, A., Manzini, G. & Sutton, O. J. (2017b) Conforming and nonconforming virtual element methods for elliptic problems. *IMA J. Numer. Anal.*, **37**, 1317–1354.

Cangiani, A., Georgoulis, E. H. & Houston, P. (2014) *hp*-Version discontinuous Galerkin methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*, **24**, 2009–2041.

Chipot, M. (2000) *Elements of Nonlinear Analysis*. Birkhäuser Advanced Texts Basler Lehrbücher. Berlin: Springer.

Chipot, M. & Lovat, B. (1997) Some remarks on nonlocal elliptic and parabolic problems. *Nonlinear Anal.*, **30**, 4619–4627.

Chrysafinos, K., Filopoulos, S. P. & Papathanasiou, T. K. (2013) Error estimates for a FitzHugh–Nagumo parameter-dependent reaction–diffusion system. *ESAIM Math. Model. Numer. Anal.*, **47**, 281–304.

Coudiére, Y. & Pierre, C. (2006) Stability and convergence of a finite volume method for two systems of reaction–diffusion equations in electro-cardiology. *Nonlinear Anal. Real World Appl.*, **7**, 916–935.

Coudiére, Y. & Turpault, R. (2017) Very high order finite volume methods for cardiac electrophysiology. *Comput. Math. Appl.*, **74**, 684–700.

Di Pietro, D. & Ern, A. (2015) A hybrid high-order locking-free method for linear elasticity on general meshes. *Comput. Methods Appl. Mech. Engrg.*, **283**, 1–21.

FitzHugh, R. (1961) Impulses and physiological states in theoretical models of nerve membrane. *J. Biophys.*, **1**, 445–466.

Gain, A. L., Talischi, C. & Paulino, G. H. (2014) On the virtual element method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes. *Comput. Methods Appl. Mech. Engrg.*, **282**, 132–160.

Gardini, F. & Vacca, G. (2018) Virtual element method for second order elliptic eigenvalue problems. *IMA J. Numer. Anal.*, **38**, 2026–2054.

Hastings, S. P. (1975) Some mathematical models from neurobiology. *Amer. Math. Monthly*, **82**, 881–895.

Jackson, D. (1992) Error estimates for the semidiscrete Galerkin approximations of the FitzHugh–Nagumo equations. *Appl. Math. Comput.*, **50**, 93–114.

Lions, J.-L. (1969) *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*. Paris: Dunod.

Liu, F., Zhuang, P., Turner, I., Anh, V. & Burrage, K. (2015) A semi-alternating direction method for a 2-D fractional FitzHugh–Nagumo monodomain model on an approximate irregular domain. *J. Comput. Phys.*, **293**, 252–263.

Mora, D., Rivera, G. & Rodríguez, R. (2015) A virtual element method for the Steklov eigenvalue problem. *Math. Models Methods Appl. Sci.*, **25**, 1421–1445.

Mora, D., Rivera, G. & Rodríguez, R. (2017) A posteriori error estimates for a virtual element method for the Steklov eigenvalue problem. *Comput. Math. Appl.*, **74**, 2172–2190.

Mora, D., Rivera, G. & Velásquez, I. (2018) A virtual element method for the vibration problem of Kirchhoff plates. *ESAIM Math. Model. Numer. Anal.*, **52**, 1437–1456.

Nagumo, J. S., Arimoto, S. & Yoshizawa, S. (1962) An active pulse transmission line simulating nerve axon. *Proc. IRE*, **50**, 2061–2070.

Olmos, D. & Shizgal, B. D. (2009) Pseudospectral method of solution of the FitzHugh–Nagumo equation. *Math. Comput. Simulation*, **79**, 2258–2278.

Oshita, Y. & Ohnishi, I. (2003) Standing pulse solutions for the FitzHugh–Nagumo equations. *Jpn. J. Ind. Appl. Math.*, **20**, 101–115.

Perugia, I., Pietra, P. & Russo, A. (2016) A plane wave virtual element method for the Helmholtz problem. *ESAIM Math. Model. Numer. Anal.*, **50**, 783–808.

Peskin, C. S. (1975) *Partial Differential Equations in Biology*. New York: Courant Institute of Mathematical Sciences.

Sanfelici, S. (2002) Convergence of the Galerkin approximation of a degenerate evolution problem in electrocardiology. *Numer. Methods Partial Differential Equations*, **18**, 218–240.

Schonbek, M. E. (1978) Boundary value problems for the FitzHugh–Nagumo equations. *J. Differential Equations*, **30**, 119–147.

Sukumar, N. & Tabarraei, A. (2004) Conforming polygonal finite elements. *Internat. J. Numer. Methods Engrg.*, **61**, 2045–2066.

Talischi, C., Paulino, G. H., Pereira, A. & Menezes, I. F. M. (2010) Polygonal finite elements for topology optimization: a unifying paradigm. *Internat. J. Numer. Methods Engrg.*, **82**, 671–698.

Thomée, V. (1997) *Galerkin Finite Element Methods for Parabolic Problems*. Berlin: Springer.

Vacca, G. (2017) Virtual element methods for hyperbolic problems on polygonal meshes. *Comput. Math. Appl.*, **74**, 882–898.

Vacca, G. (2018) An H^1-conforming virtual element for Darcy and Brinkman equations. *Math. Models Methods Appl. Sci.*, **28**, 159–194.

Vacca, G. & Beirão da Veiga, L. (2015) Virtual element methods for parabolic problems on polygonal meshes. *Numer. Methods Partial Differential Equations*, **31**, 2110–2134.

Wriggers, P., Rust, W. T. & Reddy, B. D. (2016) A virtual element method for contact. *Comput. Mech.*, **58**, 1039–1050.